A non-partitionable Cohen-Macaulay simplicial complex

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The focus of this talk is the following conjecture, described in Stanley’s Green Book as “a central combinatorial conjecture on Cohen-Macaulay complexes.”

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**Partitionability Conjecture** (Stanley 1979)

*Every Cohen-Macaulay simplicial complex is partitionable.*

**Theorem (DGKM ’15+)**

*The Partitionability Conjecture is false. We construct an explicit counterexample and describe a general method to construct more.*
Let \( V \) be a finite set of vertices. A **simplicial complex** on \( V \) is a family \( X \subseteq 2^V \) such that

\[
F \in X, \; G \subseteq F \implies G \in X.
\]

Equivalently, \( X \) is an order ideal in the boolean algebra on \( V \).

- **Dimension**: \( \dim F = |F| - 1 \); \( \dim X = \max\{\dim F : F \in X\} \).
- Maximal faces of \( X \) are called **facets**.
- \( X \) is **pure** if all facets have the same dimension.
- The complex **generated** by a list of face(t)s is
  \[
  \langle F_1, \ldots, F_k \rangle := \bigcup_{i=1}^k 2^{F_i}.
  \]
The Stanley-Reisner ring

Let $\mathbb{k}$ be any field, and let $X$ be a simplicial complex of dimension $d - 1$ on vertices $V = [n]$.

Associate each $S \subseteq V$ with the monomial $x_S = \prod_{i \in S} x_i$.

The Stanley-Reisner ring of $X$ over $\mathbb{k}$ is

$$\mathbb{k}[X] := \mathbb{k}[x_1, \ldots, x_n] / \langle x_S \mid S \notin X \rangle.$$

- Graded ring of Krull dimension $d$
- Algebraic properties of $\mathbb{k}[X] \iff$ combinatorial/topological properties of $X$
Let $X$ be a simplicial complex of dimension $d - 1$.

The $f$-vector is $f(X) = (f_{-1}, \ldots, f_{d-1})$, where

$$f_i = \#\{\text{faces of dimension } i\}.$$ 

The $h$-vector $h(X) = (h_0, \ldots, h_d)$ is defined by

$$\sum_{i=0}^{d} h_i x^i = \sum_{i=0}^{d} f_{i-1} x^i (1 - x)^{d-i}.$$ 

The $h$-vector has algebraic significance (it is the numerator of the Hilbert series of $\mathbb{k}[X]$), and is often positive (e.g., when $\mathbb{k}[X]$ is Cohen-Macaulay).

What is its combinatorial meaning?
A pure simplicial complex $X$ is shellable if its facets can be ordered $F_1, \ldots, F_n$ so that for each $k$, the set
\[
\langle F_1, \ldots, F_k \rangle \setminus \langle F_1, \ldots, F_{k-1} \rangle
\]
is an interval $[R_k, F_k]$ in the boolean algebra $2^V$.

**Proposition**

If $X$ is shellable, then $h_i(X) = |\{ k \in [n] : |R_k| = i \}|$.

But what if $h(X) \geq 0$ but $X$ is not shellable?
Let $X$ be a pure simplicial complex with facets $F_1, \ldots, F_n$.

**Definition**

A partitioning of $X$ is a decomposition into disjoint Boolean intervals topped by facets:

$$X = \bigsqcup_{i=1}^{n} [R_i, F_i].$$

Note that a partitioning is weaker than a shelling. Nevertheless:

**Proposition**

*If $X$ is partitionable, then $h_i(X) = |\{k \in [n] : |R_k| = i\}|$.***
Cohen-Macaulay and Constructible Complexes

- $X^d$ is Cohen-Macaulay (CM) iff $\Bbbk[X]$ is CM, i.e.,
  \[ \dim \Bbbk[X] = \text{depth } \Bbbk[X]. \]

- $X^d$ is constructible iff either it is a simplex, or the union of two constructible $d$-dimensional complexes whose intersection is constructible of dimension $d - 1$.

\[
\begin{array}{c}
\text{shellable} \implies \text{constructible} \implies \text{CM} \implies h(X) \geq 0 \\
\downarrow \\
\text{partitionable}
\end{array}
\]
The Partitionability and Constructibility Conjectures

Theorem (Reisner 1976)

\( X \) is Cohen-Macaulay iff for every face \( \sigma \in X \),

\[ \tilde{H}_i(\text{link}_X(\sigma); \mathbb{Z}) = 0 \quad \forall i < \dim \text{link}_X \sigma. \]

Theorem (Munkres 1984)

The CM condition is topological, i.e., it depends only on the geometric realization \( |X| \).

Partitionability Conjecture (Stanley 1979)

Every Cohen-Macaulay simplicial complex is partitionable.

Constructibility Conjecture (Hachimori 2000)

Every constructible simplicial complex is partitionable.
Theorem (DGKM 2015+)

The Partitionability and Constructibility Conjectures are false.

We exhibit an explicit simplicial complex $\Omega$ that is constructible, hence Cohen-Macaulay, but not partitionable.

$\Omega$ is a contractible 3-dimensional complex (but not a ball) with

\[ f(\Omega) = (1, 16, 71, 98, 42), \quad h(\Omega) = (1, 12, 29). \]
Definition

Let $S = \mathbb{k}[x_1, \ldots, x_n]$; $\mu \in S$ a monomial; and $A \subseteq \{x_1, \ldots, x_n\}$. The corresponding Stanley space in $S$ is the vector space

$$\mu \cdot \mathbb{k}[A] = \mathbb{k}\text{-span}\{\mu \nu \mid \text{supp}(\nu) \subseteq A\}.$$ 

Let $I \subseteq S$ be a monomial ideal. A Stanley decomposition of $S/I$ is a family of Stanley spaces

$$D = \{\mu_1 \cdot \mathbb{k}[A_1], \ldots, \mu_r \cdot \mathbb{k}[A_r]\}$$

such that

$$S/I = \bigoplus_{i=1}^{r} \mu_i \cdot \mathbb{k}[A_i].$$
Two Stanley decompositions of $R = \mathbb{k}[x, y]/\langle x^2 y \rangle$:
Stanley Decompositions and Stanley Depth

Definition

The **Stanley depth** of $S/I$ is

$$\text{sdepth } S/I = \max_{D} \min\{|A_i|\}.$$ 

where $D$ runs over all Stanley decompositions of $S/I$.

Stanley’s Depth Conjecture

**Depth Conjecture (Stanley 1982)**

Let $S = \mathbb{k}[x_1, \ldots, x_n]$ and $I \subset S$ be any monomial ideal. Then

$$\text{sdepth } S/I \geq \text{depth } S/I.$$

**Theorem (Herzog, Jahan and Tassemi '08)**

The Depth Conjecture implies the Partitionability Conjecture

**Corollary (DGKM '15+)**

The Depth Conjecture is false.
**Definition**

A *relative simplicial complex* $Q$ on vertex set $V$ is a convex subset of the Boolean algebra $2^V$. That is,

$$F, H \in Q, \ F \subseteq G \subseteq H \implies G \in Q.$$ 

Every relative complex can be written as $(X, A) = X \setminus A$, where $A \subseteq X$ are simplicial complexes.
Relative Simplicial Complexes

Simplicial combinatorics (f- and h-vector, pure, shellable, CM, partitionable, etc.) carries over nicely to the relative setting.

A pure relative simplicial complex \( Q \) is Cohen-Macaulay (CM) if a relative version of Reisner’s criterion holds, and \( Q \) is partitionable if

\[
Q = \bigsqcap_{k=1}^{n} [R_k, F_k]
\]

where the \( F_k \) are the facets of \( Q \).

- Shellable relative complexes are partitionable.
- If \( A \subseteq X \) are CM of the same dimension, then so is \( (X, A) \).
Reducing to the Relative Case

\[ X = \text{CM complex} \quad A \subset X: \text{induced, CM, codim 0 or 1} \]
\[ Q = (X, A) \quad N > \# \text{ faces of } A \]

Idea: Construct \( \Omega \) by gluing \( N \) copies of \( X \) together along \( A \).

- \( \Omega \) is CM by Mayer-Vietoris. On the level of face posets,
  \[ \Omega = Q_1 \cup \cdots \cup Q_N \cup A, \quad Q_i \cong Q \quad \forall i. \]

- If \( \Omega \) has a partitioning \( \mathcal{P} \), then by pigeonhole \( \exists i \) such that
  \[ \exists i \in [n]: \quad \forall \text{ facets } F_k \in Q_i: \quad R_k \not\in A. \]

- Therefore, the partitioning of \( \Omega \) induces a partitioning of \( Q \).

Problem: Find a suitable \( Q \).
Mary Ellen Rudin (1958) constructed a simplicial 3-ball that is not shellable, with $f$-vector $(1, 14, 66, 94, 41)$ and $h$-vector $(1, 10, 30)$.

Günter Ziegler (1998) constructed a smaller non-shellable simplicial 3-ball with $f$-vector $(1, 10, 38, 50, 21)$ and $h$-vector $(1, 6, 14)$. Its facets are

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0123 0125 0237 0256 0267 1234 1249
1256 1269 1347 1457 1458 1489 1569
1589 2348 2367 2368 3478 3678 4578
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Theorem (DGKM 2015+)

Let $Z$ be Ziegler’s ball, and let $B = Z|_{0,2,3,4,6,7,8}$.

1. $B$ is a shellable, hence CM, simplicial 3-ball.
2. $Q = (Z, B)$ is not partitionable. Its minimal faces are the three vertices 1, 5, 9.
3. Therefore, the simplicial complex obtained by gluing $|B| + 1 = 53$ copies of $Z$ together along $B$ is not partitionable.

Assertion (2) can be proved by elementary methods.
Let $X$ be the smallest simplicial complex containing $Q$. Then $Q = (Z, B) = (X, A)$, where

$$f(X) = (1, 10, 31, 36, 14), \quad f(A) = (1, 7, 11, 5).$$

So a much smaller counterexample can be constructed by gluing together $(1 + 7 + 11 + 5) + 1 = 25$ copies of $X$ along $A$. In fact, gluing three copies of $X$ along $A$ produces a CM nonpartitionable complex $\Omega$, with

$$f(\Omega) = 3f(X) - 2f(A) = (1, 16, 71, 98, 42).$$

This is the smallest such complex we know, but there may well be smaller ones.
There is a much smaller non-partitionable CM relative complex $Q'$ inside Ziegler's ball $Z$, with face poset

A partitioning of $Q'$ would correspond to a decomposition of this poset into five pairwise-disjoint diamonds. It is not hard to check by hand that no such decomposition exists.
A Much Smaller Relative Counterexample

Construction: $Q' = (X', A')$, where

$$X' = \langle 1589, 1489, 1458, 1457, 4578 \rangle = Z|_{145789},$$

$$A' = \langle 489, 589, 578, 157 \rangle.$$ 

- $Q'$ is CM (since $X', A'$ are shellable and $A' \subset \partial X'$)
- $f(Q') = (0, 0, 5, 10, 5)$
- Minimal faces are edges rather than vertices, so $Q'$ cannot be expressed as $(X, A)$ where $A$ is an induced subcomplex.
- $k[Q']$ is a small counterexample to the Depth Conjecture
  [computation by Lukas Katthän]
Open Questions

▶ Is there a smaller counterexample, perhaps in dimension 2?

▶ What is the “right” strengthening of constructibility that implies partitionability? (“Strongly constructible” complexes, as studied by Hachimori, are partitionable.)

▶ Is there a better combinatorial interpretation of the $h$-vectors of Cohen-Macaulay complexes? (Duval–Zhang)

▶ Are all simplicial balls partitionable? (Yes if convex.)

▶ Does the Partitionability Conjecture hold for balanced simplicial complexes, as conjectured by Garsia? (Bennet Goeckner is working on this.)

▶ What are the consequences for Stanley depth? Does $\text{sdepth } M \geq \text{depth } M - 1$ (as conjectured by Lukas Katthän)?
Thanks for listening!