Spanning Trees of Simplicial Complexes

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Le Menu

Spanning Trees of Simplicial Complexes
1 Appetizer: Graphs
   - The incidence and Laplacian matrices
   - The matrix-tree theorem
   - The chip-firing game
   - The critical group
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   - The incidence and Laplacian matrices
   - The matrix-tree theorem
   - The chip-firing game
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2 Main Course: Simplicial Complexes
   - Crash course in algebraic topology
   - Simplicial spanning trees
   - Simplicial matrix-tree theorems
   - Simplicial critical groups

Main course is joint work with Art Duval (U. of Texas, El Paso) and Caroline Klivans (U. of Chicago)
Appetizer: Graphs
Spanning Trees

Definition  A **spanning tree** of a graph $G = (V, E)$ is a set of edges $T$ (or, equivalently, a subgraph $(V, T)$) such that:

1. $(V, T)$ is **connected**: every pair of vertices is joined by a path
2. $(V, T)$ is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

Any two of these conditions together imply the third.
Spanning Trees of Simplicial Complexes
Spanning Trees
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Spanning Trees of Simplicial Complexes
Counting Spanning Trees

$$\tau(G) = \text{number of spanning trees of } G$$

- $\tau(\text{tree}) = 1$
- $\tau(\text{n-cycle}) = n$

- Complete graph: $\tau(K_n) = n^{n-2}$ (Cayley’s formula)
- Complete bipartite graph: $\tau(K_{n,m}) = n^{m-1}m^{n-1}$
- Many other enumeration formulas for nice graphs (threshold graphs, hypercubes, ...)
The Incidence Matrix

Definition  (Signed) incidence matrix $\partial$ of $G$

- Rows indexed by vertices; columns indexed by edges
- Each column has one 1 and one $-1$ corresponding to its endpoints, and 0s elsewhere.
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$$G = \begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
1 & 2 & 3 & 4
\end{array}$$

$$\partial = \begin{pmatrix}
12 & 13 & 13 & 23 & 24 \\
1 & 1 & -1 & 0 & 0 \\
2 & -1 & 0 & 0 & -1 & 1 \\
3 & 0 & -1 & 1 & 1 & 0 \\
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\end{pmatrix}$$
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- (Exercise: Translate “cycle”, “acyclic”, “dimension”, other graph-theoretic and linear-algebraic terms across this correspondence. This amounts to describing the graphic matroid of \( G \).)
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(Exercise: Translate “cycle”, “acyclic”, “dimension”, other graph-theoretic and linear-algebraic terms across this correspondence. This amounts to describing the graphic matroid of $G$.)

If we can count column bases, we can count spanning trees.
The Laplacian Matrix

**Definition**  The Laplacian matrix of $G$ is $L = \partial \partial^T$.

Entries of $L$ are scalar products of rows of $\partial$:

$$L(i,j) = \begin{cases} 
\deg_G(i) & \text{if } i = j, \\
-(\# \text{ of edges joining } i \text{ and } j) & \text{otherwise}.
\end{cases}$$

- rank $L = \text{rank } \partial = \# \text{ vertices } - \# \text{ components}.$
The Laplacian Matrix

\[ \partial = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \]

\[ L = \begin{pmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \]
The Matrix-Tree Theorem (Kirchhoff, 1847)

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of $L$. Then the number of spanning trees of $G$ is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}.$$
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(2) Let $1 \leq i \leq n$. Form the reduced Laplacian $\tilde{L}$ by deleting the $i^{th}$ row and $i^{th}$ column of $L$. Then

$$\tau(G) = \det \tilde{L}.$$
The Matrix-Tree Theorem

*Sketch of proof:* By the Binet-Cauchy formula from linear algebra,

\[
\det \tilde{\mathbf{L}} = \det \tilde{\partial} \tilde{\partial}^T = \sum_{A \subseteq E} (\det \tilde{\partial}_A)^2 \quad (\tilde{\partial}: \text{delete a row from } \partial)
\]

\[|A| = n-1\]
Sketch of proof: By the Binet-Cauchy formula from linear algebra,

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\[ (\tilde{\partial}: \text{delete a row from } \partial) \]

\[ = \sum_A \begin{cases} 
1 & \text{if } A \text{ is a column basis for } \partial \\
0 & \text{if it isn’t} 
\end{cases} \]
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$$= \sum_A \begin{cases} 1 & \text{if } A \text{ is a column basis for } \partial \\ 0 & \text{if it isn’t} \end{cases}$$

$$= \text{number of column bases of } \partial$$

$$= \text{number of spanning trees!}$$
The Matrix-Tree Theorem

Example

\[ G = \begin{array}{c}
3 & -1 & -2 & 0 \\
-1 & 3 & -1 & -1 \\
-2 & -1 & 3 & 0 \\
0 & -1 & 0 & 1
\end{array} \]

\[ L = \begin{array}{cccc}
3 & -1 & -2 & 0 \\
-1 & 3 & -1 & -1 \\
-2 & -1 & 3 & 0 \\
0 & -1 & 0 & 1
\end{array} \]

\[ \tilde{L} = \begin{array}{cccc}
3 & -1 & -1 \\
-1 & 3 & 0 \\
-1 & 0 & 1
\end{array} \]

Eigenvalues: 0, 1, 4, 5

\[ \det \tilde{L} = 5 \]

\[ (1 \cdot 4 \cdot 5)/4 = 5 \]
The Matrix-Tree Theorem

Example \( G = K_n \) (complete graph on \( n \) vertices)

\[
L(K_n) = \begin{bmatrix}
n - 1 & -1 & \ldots & -1 \\
-1 & n - 1 & \ldots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \ldots & n - 1
\end{bmatrix}
\]

- Eigenvalues: 0 (multiplicity 1), \( n \) (multiplicity \( n - 1 \))
- \( \tau(K_n) = \frac{n^{n-1}}{n} = n^{n-2} \).
Example: The Hypercube

- $G = Q_n = 1$-skeleton of $n$-dimensional hypercube

- Eigenvalues of $L$: $0, 2, 4, \ldots, 2n$, with multiplicities \( \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n} \)

$$\tau(Q_n) = \prod_{k=2}^{n} (2k) \binom{n}{k}.$$
Example: The Hypercube

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- Eigenvalues of $L$: $0, 2, 4, \ldots, 2n$, with multiplicities $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$

$$\Rightarrow \tau(Q_n) = \prod_{k=2}^{n} (2k)^{\binom{n}{k}}.$$

Open Problem Find a bijective proof of this formula.
The Chip-Firing Game
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The Chip-Firing Game

- \( G \): graph with vertex set \( \{1, 2, \ldots, n\} \)
- Each vertex \( i < n \) has a finite number \( c_i \) of poker chips
- A vertex **fires** by giving one chip to each of its neighbors
- Vertex \( n \), the **bank**, only fires if no other vertex can fire
- Vertices other than the bank cannot go into debt
- **Chip configuration** = vector \( \mathbf{c} = (c_1, \ldots, c_{n-1}) \in \mathbb{N}^{n-1} \)
The Chip-Firing Game

**Theorem** (Biggs, Dhar?, Björner–Lovász–Shor)
Every initial chip configuration determines a unique critical configuration, regardless of the order of firing.
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Every initial chip configuration determines a unique critical configuration, regardless of the order of firing.

Recall that the Laplacian matrix of $G$ is $L = [\ell_{ij}]_{1 \leq i, j \leq n}$ where

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -(\# \text{ of edges joining } i \text{ and } j) & \text{otherwise.} \end{cases}$$

- Firing vertex $i \leftrightarrow$ subtracting $i^{th}$ column of $L$ from $c$. 
The Chip-Firing Game

Firing keeps $c$ in the same coset of $\text{colspace}(L) \subset \mathbb{Z}^n$. 
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I.e., each chip configuration determines an element of the quotient group $\mathbb{Z}^n / \text{colspace}(L)$. ...
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...or, if we ignore the bank, an element of $\mathbb{Z}^{n-1} / \text{colspace}(\tilde{L})$. 
Firing keeps $c$ in the same coset of $\text{colspace}(L) \subset \mathbb{Z}^n$. I.e., each chip configuration determines an element of the quotient group $\mathbb{Z}^n / \text{colspace}(L)$.

...or, if we ignore the bank, an element of $\mathbb{Z}^{n-1} / \text{colspace}(\tilde{L})$.

**Definition** The critical group of $G$ is

$$K(G) = \mathbb{Z}^{n-1} / \text{colspace}(\tilde{L}).$$

- $|K(G)| = \tau(G)$ by Matrix-Tree Theorem
- Critical configurations are a system of coset representatives
Cuts and Flows

\[ G = \begin{tabular}{c|c|c|c}
1 & 2 \\
3 & 4
\end{tabular} \]

\[ \partial = \begin{pmatrix}
12 & 13 & 13 & 23 & 24 \\
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Cuts and Flows

\[ G = \begin{tikzpicture} % Graph edges here \end{tikzpicture} \]

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Cut space \[ C = \text{colspace}(\partial^T) \] (generated by edge cuts)
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Cut space \[ C = \text{colspace}(\partial^T) \] (generated by edge cuts)

Flow space \[ \mathcal{F} = \ker(\partial) = C^\perp \] (generated by cycles)
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Cuts and Flows

**Cut space** \( C = \text{colspace}(\partial^T) \)

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**Theorem** [Bacher, de la Harpe, Nagnibeda 1997]

\[
K(G) = \mathbb{Z}^{n-1} / \text{colspace } \tilde{L} \cong \mathbb{Z}^E / (C \oplus F).
\]
Main Course: Simplicial Complexes
Definition  A simplicial complex is a family $\Delta \subseteq \text{powerset}(\{1, 2, \ldots, n\})$ such that

$$\text{if } \sigma \in \Delta \text{ and } \sigma' \subseteq \sigma, \text{ then } \sigma' \in \Delta.$$ 

- Think of a simplicial complex as a higher-dimensional generalization of a graph.

- Elements of $\Delta$ are called faces or simplices.

- $\dim \sigma = |\sigma| - 1$

- $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$

- $f_i(\Delta) =$ number of $i$-dimensional faces of $\Delta$
Definition A **simplicial complex** is a family $\Delta \subseteq \text{powerset}(\{1, 2, \ldots, n\})$ such that

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- Simplicial polytopes (minus geometry)
- Every “reasonable” topological space can be represented as a simplicial complex
- Graphs = 1-dimensional simplicial complexes
- Simplicial complexes arise frequently in combinatorics: e.g., order complexes of posets
For $i \in \mathbb{N}$, the **i-dimensional boundary matrix** $\partial_i$ of $\Delta$ records which $(i - 1)$-simplices are contained in which $i$-simplices.
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(E.g., $\partial_1 = \text{signed incidence matrix of 1-skeleton of } \Delta$ — records which vertices are contained in which edges.)
For $i \in \mathbb{N}$, the i-dimensional boundary matrix $\partial_i$ of $\Delta$ records which $(i - 1)$-simplices are contained in which $i$-simplices.

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**Fact** $\partial_i \partial_{i+1} = 0$. Equivalently, $\text{im}(\partial_{i+1}) \subseteq \ker(\partial_i)$. 
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**Definition** The **$i$th (reduced) homology group** of $\Delta$ is

$$\tilde{H}_i(\Delta) = \ker(\partial_i) / \text{im}(\partial_{i+1})$$

$$\cong \mathbb{Z}^\beta_i(\Delta) \oplus \text{finite “torsion” group}$$
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\]

(If you’re new at this: Don’t worry about the twiddles!)
Why Should You Care About Homology?

- \( \tilde{H}_i(\Delta) \) measures holes (\( \tilde{\beta}_i \)) and nonorientability (torsion)
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- For any complex $\Delta$, $\tilde{H}_0(\Delta) = \mathbb{Z} \# \text{ connected cpts} - 1$
  \[ \tilde{H}_0(\Delta) = 0 \iff \Delta \text{ is connected.} \]
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- If $\Delta$ is a connected graph, then $\tilde{H}_1(\Delta) = \mathbb{Z}^{e-v+1}$
  
  $\tilde{H}_1(\Delta) = 0 \iff \Delta \text{ is acyclic.}$
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- If \( \Delta \) is a connected graph, then \( \tilde{H}_1(\Delta) = \mathbb{Z}^{e-v+1} \)
  \[ \tilde{H}_1(\Delta) = 0 \iff \Delta \text{ is acyclic}. \]

- If \( \Delta \) is a \( d \)-sphere, then
  \[ \tilde{H}_i(\Delta) = \begin{cases} \mathbb{Z} & \text{for } i = d, \\ 0 & \text{for } i < d. \end{cases} \]
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What happens to the homology of $\Delta$ when you delete a $d$-dimensional facet?
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- **Case 1:** Pop a $d$-dimensional bubble: $\tilde{\beta}_d$ drops by 1
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What happens to the homology of \( \Delta \) when you delete a \( d \)-dimensional facet?

- **Case 1**: Pop a \( d \)-dimensional bubble: \( \tilde{\beta}_d \) drops by 1
- **Case 2**: Tear a \((d - 1)\)-dimensional hole: \( \tilde{\beta}_{d-1} \) increases by 1
Why Should You Care About Homology?

What happens to the homology of $\Delta$ when you delete a $d$-dimensional facet?

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- **Case 2**: Tear a $(d - 1)$-dimensional hole: $\tilde{\beta}_{d-1}$ increases by 1

**Fact** The (reduced) Euler characteristic of $\Delta$ is

$$\tilde{\chi}(\Delta) = \sum_i (-1)^i f_i(\Delta) = \sum_i (-1)^i \tilde{\beta}_i(\Delta).$$
Definition  Let $\Delta$ be a simplicial complex of dimension $d$.

A simplicial spanning tree (SST) is a subcomplex $\Upsilon \subset \Delta$, with $\Upsilon_{(d-1)} = \Delta_{(d-1)}$, such that

1. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$;
2. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ is a finite group;
3. $f_d(\Upsilon) = f_{d-1}(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$. 

Spanning Trees of Simplicial Complexes
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- When $d = 1$, this is just the usual graph-theoretic definition of a spanning tree.
- Any two of conditions 1,2,3 together imply the third (just as for graphs).
Examples of SSTs

What if $\Delta$ is a simplicial $d$-sphere?

- Recall that $\tilde{H}_d(\Delta) = \mathbb{Z}$. To make $\tilde{H}_d(\Upsilon) = 0$, “pop the bubble” by deleting a single facet from $\Delta$. (But don’t delete more than one or $\tilde{H}_{d-1}$ will become nonzero.)

- In particular, # of SSTs = # facets = $f_d(\Delta)$. (Analogous to the statement that the spanning trees of a cycle graph are formed by deleting a single edge.)
Let $K_n^d$ be the $d$-skeleton of the $n$-vertex simplex, i.e.,

$$K_n^d = \left\{ F \subseteq \{1, 2, \ldots, n\} \mid \dim F \leq d \right\}$$

and let $\mathcal{T}(\Delta)$ denote the set of SSTs of $\Delta$. 
Kalai’s Theorem

Let $K^d_n$ be the $d$-skeleton of the $n$-vertex simplex, i.e.,

$$K^d_n = \left\{ F \subseteq \{1, 2, \ldots, n\} \mid \dim F \leq d \right\}$$

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**Theorem**  [Kalai 1983]

$$\sum_{\gamma \in \mathcal{T}(K^d_n)} |\tilde{H}_{d-1}(\gamma; \mathbb{Z})|^2 = n^{\binom{n-2}{d}}.$$

- Setting $d = 1$ recovers Cayley’s formula $\tau(K_n) = n^{n-2}$. 
Counting Simplicial Spanning Trees

\[ \Delta = d\text{-dim’l simplicial complex with } |\tilde{H}_i(\Delta)| < \infty \ \forall \ i < d \]

\[ L = \partial_d \partial_d^T \text{ (simplicial Laplacian)} \]

\[ \tau_k(\Delta) = \sum_{\gamma \in T(\Delta_{(k)})} |\tilde{H}_{k-1}(\gamma)|^2 \text{ ("number" of } k\text{-dim’l trees")} \]

**Simplicial Matrix-Tree Theorem I [Duval–Klivans–JLM 2007]**

\[ \tau_d(\Delta) = |\tilde{H}_{d-2}(\Delta)|^2 \cdot \frac{\text{product of nonzero eigenvalues of } L}{\tau_{d-1}(\Delta)}. \]
Counting Simplicial Spanning Trees

\[ \tau_k(\Delta) = \sum_{\gamma \in T(\Delta_{(k)})} |\tilde{H}_{k-1}(\gamma)|^2 \]

\( \Gamma \) = simplicial spanning tree of \( \Delta_{(d-1)} \)

\( L_{\Gamma} \) = reduced Laplacian obtained from \( L = \partial_d \partial_d^T \) by deleting \( \Gamma \)

**Simplicial Matrix-Tree Theorem II**

\[ \tau_d(\Delta) = \frac{|\tilde{H}_{d-2}(\Delta)|^2}{|\tilde{H}_{d-2}(\Gamma)|^2} \det L. \]
The Punchline: You can count the spanning trees of a simplicial complex using Laplacians, just as you can for a graph...
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...but some trees may be more equal than others.
An Example: The Equatorial Bipyramid $B$

Facets: 123 ("equator")
124, 134, 234 ("northern")
125, 135, 235 ("southern")

$f(\Delta) = (5, 9, 7)$

$\tilde{H}_0(\Delta) = 0$
$\tilde{H}_1(\Delta) = 0$
$\tilde{H}_2(\Delta) = \mathbb{Z}^2$
Example 1: The Equatorial Bipyramid

To make an SST of $B$, we need to pop two bubbles.
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To make an SST of $B$, we need to pop two bubbles.

- Delete equator and any other triangle: 6 SSTs
- Delete one northern and one southern triangle: $3 \times 3 = 9$ SSTs
- Total: $\tau_2(B) = 15$. 
Example 1: The Equatorial Bipyramid

To make an SST of $B$, we need to pop two bubbles.

- Delete equator and any other triangle: 6 SSTs
- Delete one northern and one southern triangle: $3 \times 3 = 9$ SSTs
- Total: $\tau_2(B) = 15$.
- Meanwhile, $\tau_1(B) = \tau_1(K_5 \text{ minus an edge}) = 75$. 
Example 1: The Equatorial Bipyramid

To make an SST of $B$, we need to pop two bubbles.

- Delete equator and any other triangle: 6 SSTs
- Delete one northern and one southern triangle: $3 \times 3 = 9$ SSTs
- Total: $\tau_2(B) = 15$.
- Meanwhile, $\tau_1(B) = \tau_1(K_5 \text{ minus an edge}) = 75$.

**SMTT-I:** Eigenvalues of $L$ are 5, 5, 5, 3, 3, 0, 0, 0, 0
$\tau_2 = \frac{(\text{product of NZEs})}{\tau_1} = \frac{5^33^2}{75} = 15$. 
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**SMTT-II:** Take $\Gamma = \{12, 13, 14, 15\}$; then $\det L_\Gamma = 15$. 
Some Open Problems

Pick your favorite simplicial (or even cell) complex and count its spanning trees!

It helps if the complex is *Laplacian integral* (i.e., the Laplacian matrix has integer eigenvalues).

- Complete colorful complexes: Adin ’92
- Shifted complexes: Duval–Reiner ’03, weighted DKM ’07
- Skeletons of cubes: DKM ’10
- Matroid complexes: Kook–Reiner–Stanton ’01; *weighted*?
- Matching and chessboard complexes?
Critical Groups of Simplicial Complexes

Critical group of a graph $G$:

$$K(G) = \text{coker } \tilde{L} = \text{coker}(\tilde{\partial} \tilde{\partial}^T) = \mathbb{Z}^{|E|}/(C \oplus F)$$

where $\partial$ = incidence matrix; $C = \text{colspace } \partial^T$; $F = \text{ker } \partial$. 
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**Definition**  The $(i - 1)^{\text{th}}$ critical group of a complex $\Delta$ is

$$K_{i-1}(\Delta) = \text{coker } \tilde{L}_{i-1}^{ud} = \text{coker}(\tilde{\partial}_i\tilde{\partial}_i^T) = \mathbb{Z}^{f_i(\Delta)}/(C_i \oplus F_i)$$

where $C_i = \text{colspace}(\partial_i^T)$, $F_i = \ker(\partial_i)$.
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\[
K_{i-1}(\Delta) = \text{coker } \tilde{L}_{i-1}^\text{ud} = \text{coker}(\tilde{\partial}_i \tilde{\partial}_i^T) = \mathbb{Z}^f_i(\Delta)/(\mathcal{C}_i \oplus \mathcal{F}_i)
\]

where \(\mathcal{C}_i = \text{colspace}(\tilde{\partial}_i^T), \mathcal{F}_i = \text{ker}(\partial_i)\).
Critical Groups of Simplicial Complexes

Definition  The \((i - 1)\)th critical group of a complex \(\Delta\) is

\[ K_{i-1}(\Delta) = \text{coker} \, \tilde{L}_{i-1}^{ud} = \text{coker} (\tilde{\partial}_i \tilde{\partial}_i^T) = \mathbb{Z}^{f_i(\Delta)} / (C_i \oplus F_i) \]

where \(C_i = \text{colspace} (\partial_i^T)\), \(F_i = \ker (\partial_i)\).

Theorem  [DKM’10]  \(|K_{i-1}(\Delta)| = \tau_i(\Delta)\) for all \(i\).
Open Problem

Develop a simplicial analogue of the chip-firing game whose critical configurations correspond to elements of the simplicial critical group.