The \textit{co}critical group of a cell complex

Art M. Duval (U. Texas, El Paso)
Caroline J. Klivans (Brown)
Jeremy L. Martin (University of Kansas)

AMS Central Sectional Meeting
Washington University, St. Louis
October 20, 2013
Throughout, $X^d$ is a finite cell (CW) complex of dimension $d$.

**Acyclization**\(^1\) of $X$: $(d + 1)$-dimensional complex $\Omega$ such that $\tilde{H}_{d+1}(\Omega; \mathbb{Q}) = \tilde{H}_d(\Omega; \mathbb{Q}) = 0$ and $X = d$-skeleton of $\Omega$.

**Augmented cellular chain complex of $\Omega$ (over $\mathbb{Z}$):**

$$\cdots \xleftarrow{\partial} C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xleftarrow{\partial_i} C_{i-1} \xrightarrow{\partial_i} \cdots$$

(Identifying each $i$-cell with its characteristic function in $C^i$).

**Combinatorial Laplacians** (updown and downup):

$$L_{i}^{ud} = \partial_i \partial_i^* : C_{i-1} \rightarrow C_{i-1} \quad L_{i}^{du} = \partial_{i+1} \partial_{i+1}^* : C_{i+1} \rightarrow C_{i+1}$$

\(^1\)Not every complex has an acyclization, but many interesting ones do.
Critical and cocritical groups

Notation: \( T(G) = \) torsion summand of a f.g. abelian group \( G \).

**Critical groups of** \( X \):

\[
K_{i-1}(X) := T(\text{coker} L_i^{ud}: C_{i-1} \to C_{i-1})
\]

**Cocritical groups of** \( X \):

\[
K^*_{i+1}(X) := T(\text{coker} L_{i+1}^{du}: C_{i+1} \to C_{i+1})
\]

- Shorthand: \( K(X) = K^{d-1}(X) \) and \( K^*(X) = K^*_{d+1}(X) \)
- \( K_{i+1}(X) \) is independent of the choice of acyclization \( \Omega \).
- To compute \( K \) and \( K^* \), find Smith normal forms of Laplacians.
- \( X \) connected graph \( \implies K(X) = \) usual critical group (cardinality = number of spanning trees).
Critical groups and cut and flow lattices

Let $n = \text{number of } i\text{-cells}$, so $C_i(X, \mathbb{Z}) \cong \mathbb{Z}^n$.

**Cut lattice:** $C_i = \text{Im } \partial_i^* \subseteq \mathbb{Z}^n$

**Flow lattice:** $\mathcal{F}_i = \ker \partial_i \subseteq \mathbb{Z}^n$

**Dual of a lattice** $\mathcal{L} \subseteq \mathbb{Z}^n$:

$$\mathcal{L}^\# := \{v \in \mathcal{L} \otimes \mathbb{R}^n : \langle v, w \rangle \in \mathbb{Z} \ \forall w \in \mathcal{L}\} \cong \text{Hom}_\mathbb{Z}(\mathcal{L}, \mathbb{Z}).$$

**Theorem (DKM 12)**

$K(X) \cong C^\#/C$ and $K^*(X) \cong \mathcal{F}^\#/\mathcal{F}$.

Moreover, there are short exact sequences

$$0 \rightarrow \mathbb{Z}^n/(C \oplus \mathcal{F}) \rightarrow K(X) \rightarrow T(\tilde{H}_{d-1}(X; \mathbb{Z})) \rightarrow 0,$$

$$0 \rightarrow T(\tilde{H}_{d-1}(X; \mathbb{Z})) \rightarrow \mathbb{Z}^n/(C \oplus \mathcal{F}) \rightarrow K^*(X) \rightarrow 0.$$
Critical groups and cut and flow lattices

Theorem (DKM 12)

\[ K(X) \cong \mathcal{C}^\# / \mathcal{C} \quad \text{and} \quad K^*(X) \cong \mathcal{F}^\# / \mathcal{F}. \]

Moreover, there are short exact sequences

\[ 0 \to \mathbb{Z}^n / (\mathcal{C} \oplus \mathcal{F}) \to K(X) \to T(\tilde{H}_{d-1}(X; \mathbb{Z})) \to 0, \]

\[ 0 \to T(\tilde{H}_{d-1}(X; \mathbb{Z})) \to \mathbb{Z}^n / (\mathcal{C} \oplus \mathcal{F}) \to K^*(X) \to 0. \]

- If \( \tilde{H}_{d-1}(X; \mathbb{Z}) \) is torsion-free (for example, if \( X \) is a graph) then \( K(X) \cong K^*(X) \).
- Graph case (and motivation for present work):
  Bacher–de La Harpe–Nagnibeda 1997
- “Torsion makes \( K(X) \) bigger and \( K^*(X) \) smaller.”
Example 1

\[ \partial_2(\Omega) = \begin{pmatrix}
123 & 124 \\
12 & 1 & 1 \\
13 & -1 & 0 \\
23 & 1 & 0 \\
14 & 0 & -1 \\
24 & 0 & 1 \\
\end{pmatrix} \]

\[ L^2_{du} = \partial_2^* \partial_2 = \begin{pmatrix}
3 & 1 \\
1 & 3 \\
\end{pmatrix} \]

Cokernel: \( \mathbb{Z}/8\mathbb{Z} \cong K(X) \)
Example 2

\[
L_2^{du}(\Omega) = \begin{pmatrix}
R & S & T \\
R & 3 & -1 & 0 \\
S & -1 & 4 & -1 \\
T & 0 & -1 & 3
\end{pmatrix}
\]
Example 2 and Planar Duality

\[
L^\text{du}(\Omega) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3 \end{pmatrix} = \text{reduced Laplacian of planar dual } X^* 
\]

**Corollary [Cori–Rossin 2000]**: If \( X \) is a planar graph and \( X^* \) is any planar dual then \( K(X) \cong K^*(X) \cong K(X^*) \).
Enumerating Cellular Spanning Trees

Recall that when $X$ is a connected graph, $|K(X)| =$ number of spanning trees. More generally

$$|K(X)| = \tau_d(X) := \sum_{\Upsilon} |T(\tilde{H}_{d-1}(\Upsilon; \mathbb{Z}))|^2$$

where $\Upsilon$ ranges over all cellular spanning forests in $X$: subcomplexes with complete $(d - 1)$-skeleton such that

- $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$ ("acyclic") and
- $|\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = \tilde{H}_{d-1}(X; \mathbb{Q})$ ("connected").

(Lyons, DKM, Catanzaro–Chernyak–Klein)
Theorem (Lyons 09, DKM 11, Catanzaro–Chernyak–Klein 12)

The critical group counts forests by torsion homology:

\[ |K(X)| = \tau_d(X) := \sum_{\text{forests } \Upsilon \subseteq X} |T(\tilde{H}_{d-1}(\Upsilon; \mathbb{Z}))|^2 \]

Theorem (DKM 12)

The cocritical group counts forests by relative torsion homology:

\[ |K^*(X)| = \tau_d^*(X) := \sum_{\text{forests } \Upsilon \subseteq X} |\tilde{H}_d(X, \Upsilon; \mathbb{Z})|^2 \]
Theorem (DKM 11)

Let $X$ be a cellular sphere with $n$ facets (e.g., the boundary of a convex polytope). Then $K(X) \cong \mathbb{Z}/n\mathbb{Z}$.

Our original proof: Blah blah blah.

New proof: $K(X) \cong K^*(X)$ (since $\tilde{H}_{d-1}(X; \mathbb{Z}) = 0$). Form an acyclization $\Omega$ by attaching one $(d + 1)$-cell whose boundary is a signed sum of the $d$-cells. Therefore

$$K^*(X) \cong \text{coker } L_{d+1}^{\text{du}}(\Omega) = \text{coker } [n] = \mathbb{Z}/n\mathbb{Z}.$$
Question: Are there other complexes for which it is easier to compute the cocritical group than the critical group, or at least to count spanning trees?
Example 1: $X =$ octahedron subdivided into eight tetrahedra; $f(X) = (1, 7, 18, 20, 8)$.

How many spanning 2-trees does $X$ have?

- $L_{1d}^{ud}(X) = \partial_2 \partial_2^* =$ some $18 \times 18$ matrix
- $L_{3d}^{du}(X) = \partial_3^* \partial_3 = I + L(Q_3)$ $(Q_3 =$ cube graph$)$
- Eigenvalues of $L(Q_3)$: $0, 2, 2, 2, 4, 4, 4, 6$
- Eigenvalues of $I + L(Q_3)$: $1, 3, 3, 3, 5, 5, 5, 7$

$$\tau_2(X) = 3^3 \cdot 5^3 \cdot 7.$$  

(Note: $L_{1d}^{ud}$ has integer eigenvalues.)
More Applications

Example 1: \( X = \) octahedron subdivided into eight tetrahedra

Example 2: \( Y = \) polyhedral cell complex from \( X \) obtained by “puffing up” each tetrahedron into a bipyramid.

\[
L^\text{du}_3(Y) = \partial_3^* \partial_3 = 3I + L(Q_3)
\]

- Eigenvalues of \( L(Q_3) \): \( 0, 2, 2, 2, 4, 4, 4, 6 \)
- Eigenvalues of \( 3I + L(Q_3) \): \( 3, 5, 5, 5, 7, 7, 7, 9 \)

\[
\tau_2(Y) = 3 \cdot 5^3 \cdot 7^3 \cdot 9.
\]

- \( L^\text{ud}_1(Y) \) does not have integer eigenvalues.
Some Questions

1. For some small complexes, $L_{i-1}^{ud}$ and $L_{i+1}^{du}$ are simultaneously Laplacian integral. Is this a coincidence or is there some connection between their spectra?

2. Are there (families of) complexes other than spheres for which the structure of $K^*(X)$ can easily be determined?

3. Generalization: (co)critical groups of arbitrary chain complexes — it is still the case that $K_{i-1} = K_{i+1}^*$ if there is no torsion homology.
Some Questions

1. For some small complexes, $L_{i-1}^{ud}$ and $L_{i+1}^{du}$ are simultaneously Laplacian integral. Is this a coincidence or is there some connection between their spectra?

2. Are there (families of) complexes other than spheres for which the structure of $K^*(X)$ can easily be determined?

3. Generalization: (co)critical groups of arbitrary chain complexes — it is still the case that $K_{i-1} = K_{i+1}^*$ if there is no torsion homology

Thanks for listening!