On the Eigenvalues of Simplicial Rook Graphs

Jeremy L. Martin (University of Kansas)
Jennifer D. Wagner (Washburn University)

AMS Southeastern Sectional Meeting
Tulane University
October 13–14, 2012
Simplicial Rook Graphs

Let $d, n \in \mathbb{N}$, and let $n\Delta^{d-1}$ denote the dilated simplex
\[
\{ \mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d : \sum_{i=1}^{d} v_i = n \}.
\]

The simplicial rook graph $SR(d, n)$ is the graph with vertices
\[
V(d, n) = n\Delta^{d-1} \cap \mathbb{N}^d
\]
with two vertices adjacent iff they differ in \textbf{exactly two coordinates}. 
Simplicial Rook Graphs
Simplicial Rook Graphs
Simplicial Rook Graphs

- \(| V(d, n) | = v = \binom{n+d-1}{d-1}\)
- \(SR(d, n)\) is regular of degree \(\delta = (d - 1)n\)

Eigenspaces of adjacency matrix \(A\) and Laplacian matrix \(L\) are the same because \(AX = \lambda X \iff LX = (\delta - \lambda)X\)

- Independence number \(\alpha(SR(d, n)) = \) maximum number of nonattacking rooks on a simplicial chessboard

\[\alpha(SR(3, n)) = \lfloor (2n + 3)/3 \rfloor\]

The Adjacency and Laplacian Matrices

Adjacency matrix of a graph $G$: $A = A(G) = \text{matrix with rows and columns indexed by } V(G) \text{ with } 1\text{s for edges, } 0\text{s for non-edges}$

Laplacian matrix of $G$: $L = D - A$, where $D = \text{diagonal matrix of vertex degrees}$

- $A$ acts on the vector space $\mathbb{R}^V$ by

$$A v = \sum_{\text{neighbors } w \text{ of } v} w$$

- Eigenvalues of $A, L \implies$ connectivity, spanning trees, ...
- $G$ regular $\implies$ eigenspaces of $A, L$ are the same
The Spectrum of $A(3, n)$

Theorem (JLM/JDW, 2012)

The eigenvalues of $A(3, n) = A(SR(3, n))$ are as follows:

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>$(\binom{2m}{2})$</td>
</tr>
<tr>
<td>$-2, \ldots, m-3$</td>
<td>3</td>
</tr>
<tr>
<td>$m-1$</td>
<td>2</td>
</tr>
<tr>
<td>$m, \ldots, n-2$</td>
<td>3</td>
</tr>
<tr>
<td>$2n$</td>
<td>1</td>
</tr>
</tbody>
</table>

$n = 2m + 1$ odd

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>$(\binom{2m-1}{2})$</td>
</tr>
<tr>
<td>$-2, \ldots, m-4$</td>
<td>3</td>
</tr>
<tr>
<td>$m-3$</td>
<td>2</td>
</tr>
<tr>
<td>$m-1, \ldots, n-2$</td>
<td>3</td>
</tr>
<tr>
<td>$2n$</td>
<td>1</td>
</tr>
</tbody>
</table>

$n = 2m$ even

Method of proof: Construct explicit eigenvectors.
Corollary

The number of spanning trees of $SR(3, n)$ is

\[
\begin{cases}
32(2n + 3)^{\binom{n-1}{2}} \frac{2n+2}{\prod_{a=n+2} a^3} & \text{if } n \text{ is odd}, \\
3(n + 1)^2(n + 2)(3n + 5)^3 & \\
32(2n + 3)^{\binom{n-1}{2}} \frac{2n+2}{\prod_{a=n+2} a^3} & \text{if } n \text{ is even}, \\
3(n + 1)(n + 2)^2(3n + 4)^3
\end{cases}
\]
Conjecture

The graph \( SR(d, n) \) is integral for all \( d \) and \( n \).

Partial results for least eigenvalue \( \lambda \) and corresp. eigenspace \( W \):

- Eigenvectors come from lattice permutohedra.
- If \( n \geq \binom{d}{2} \), then \( \lambda = -\binom{d}{2} \) and \( \dim W = \binom{n-(d-1)(d-2)/2}{d-1} \).

Note that

\[
\lim_{n \to \infty} \frac{\dim W}{|V(d, n)|} = 1.
\]

- If \( n < \binom{d}{2} \), then the least eigenvalue appears to be \(-n\), and \( \dim W \) is the Mahonian number \( M(d, n) \) of permutations in \( S_d \) with exactly \( n \) inversions.
Hexagon Vectors in $V(3, n)$

For each “internal” vertex $v \in V(3, n)$ (i.e., $v_i > 0$ for all $i$), the signed characteristic vector of the hexagon centered at $v$ is an eigenvector with eigenvalue $-3$. 
For each “internal” vertex $v \in V(3, n)$ (i.e., $v_i > 0$ for all $i$), the signed characteristic vector of the hexagon centered at $v$ is an eigenvector with eigenvalue $-3$. 
Hexagon Vectors in $V(3, n)$

For each “internal” vertex $v \in V(3, n)$ (i.e., $v_i > 0$ for all $i$), the signed characteristic vector of the hexagon centered at $v$ is an eigenvector with eigenvalue $-3$. 
For each “internal” vertex $v \in V(3, n)$ (i.e., $v_i > 0$ for all $i$), the signed characteristic vector of the hexagon centered at $v$ is an eigenvector with eigenvalue $-3$. 
For each “internal” vertex \( v \in V(3, n) \) (i.e., \( v_i > 0 \) for all \( i \)), the signed characteristic vector of the hexagon centered at \( v \) is an eigenvector with eigenvalue \(-3\).
For each “internal” vertex $v \in V(3, n)$ (i.e., $v_i > 0$ for all $i$), the signed characteristic vector of the hexagon centered at $v$ is an eigenvector with eigenvalue $-3$. 

\[
\begin{bmatrix}
0 & 0 & 0 & -1 & 1 & 0 \\
1 & -1 & 1 & -1 & 1 & 0 \\
-1 & 1 & -1 & 1 & 0 & 0 \\
-1 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
For each “internal” vertex $v \in V(3, n)$ (i.e., $v_i > 0$ for all $i$), the signed characteristic vector of the hexagon centered at $v$ is an eigenvector with eigenvalue $-3$. 
For each “internal” vertex \( v \in V(3, n) \) (i.e., \( v_i > 0 \) for all \( i \)), the signed characteristic vector of the hexagon centered at \( v \) is an eigenvector with eigenvalue \(-3\).
Hexagon Vectors in $V(3, n)$

- Number of possible centers for a hexagon vector $= \text{number of interior vertices of } n\Delta^{d-1} = {v - 1 \choose 2}$.

- The hexagon vectors are all linearly independent.

- The other $\left( {v + 2 \choose 2} - {v - 2 \choose 2} \right) = 3v$ eigenvectors have explicit formulas in terms of characteristic vectors of lattice lines.
Permutohedron Vectors in $G(d, n)$

**Definition**
Let $p \in \mathbb{Z}^d$ (if $d$ is odd) or $(\mathbb{Z} + \frac{1}{2})^d$ (if $d$ is even). The lattice permutohedron centered at $p$ is

$$\text{Per}(p) = \{ p + \sigma(w) : \sigma \in \mathfrak{S}_d \}$$

where $\mathfrak{S}_d$ is the symmetric group and

$$w = \left( \frac{1 - d}{2}, \frac{3 - d}{2}, \ldots, \frac{d - 3}{2}, \frac{d - 1}{2} \right).$$

“Most” eigenvectors of $SR(d, n)$ are signed characteristic vectors $\mathcal{H}_p$ of lattice permutohedra inscribed in the simplex $n \Delta^{d-1}$.

[SHOW THE NIFTY SAGE PICTURE]
Simplicial Rook Graphs
Permutohedron Eigenvectors

- Each \( \mathcal{H}_p \) is an eigenvalue of \( A(d, n) \) with eigenvalue \(-\binom{d}{2}\).

- The \( \mathcal{H}_p \) are linearly independent.

- Permutohedron vectors account for “most” eigenvectors:

\[
\frac{\#\{p : \text{Per}(p) \subset V(d, n)\}}{|V(d, n)|} = \frac{n-\binom{d-1}{2}}{\binom{n+d-1}{d-1}} \to 1 \quad \text{as} \quad n \to \infty.
\]
When \( n < \binom{d}{2} \), the simplex \( n\Delta^{d-1} \) is too small to contain any lattice permutohedra.
The Case $n < \binom{d}{2}$

When $n < \binom{d}{2}$, the simplex $n\Delta^{d-1}$ is too small to contain any lattice permutohedra.

On the other hand, characteristic vectors of partial permutohedra

$$\text{Per}(p) \cap n\Delta^{d-1}$$

are eigenvectors with eigenvalue $-n$. 
The Case $n < \binom{d}{2}$

When $n < \binom{d}{2}$, the simplex $n\Delta^{d-1}$ is too small to contain any lattice permutohedra.

On the other hand, characteristic vectors of partial permutohedra

$$\text{Per}(p) \cap n\Delta^{d-1}$$

are eigenvectors with eigenvalue $-n$.

Number of partial permutohedra = Mahonian number $M(d, n)$

$= \text{number of permutations in } S_d \text{ with } n \text{ inversions}$

$= \text{coefficient of } q^n \text{ in } (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{d-1})$
The Case \( n < \binom{d}{2} \)

When \( n < \binom{d}{2} \), the simplex \( n\Delta^{d-1} \) is too small to contain any lattice permutohedra.

On the other hand, characteristic vectors of partial permutohedra

\[
\text{Per}(p) \cap n\Delta^{d-1}
\]

are eigenvectors with eigenvalue \(-n\).

Number of partial permutohedra = Mahonian number \( M(d, n) \)

\[
= \text{number of permutations in } \mathfrak{S}_d \text{ with } n \text{ inversions}
\]

\[
= \text{coefficient of } q^n \text{ in } (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{d-1})
\]

Construction uses (ordinary, non-simplicial) rook theory!
The Case $n < \binom{d}{2}$

- Permutation $\pi \in S_d$ with $n$ inversions $\rightarrow$ “inversion word” $(a_1, \ldots, a_d)$, where $a_i = \# \{ j \in [d] : \pi_i > \pi_j \}$ (note that $\sum a_i = n$)

- Rook placement $\sigma$ on skyline Ferrers board $(a_1, \ldots, a_d)$ $\rightarrow$ lattice point $x(\sigma) = (a_i + i - \sigma_i) \in n\Delta^{d-1}$

- Eigenvector $X_\pi = \sum_\sigma \varepsilon(\sigma)x(\sigma)$

- Proof that $X_\pi$ is an eigenvector: sign-reversing involution moving rooks around

Simplicial Rook Graphs
Open Problems

- (The big one.) Prove that $A(d, n)$ (equivalently, $L(d, n)$) has integral spectrum for all $d, n$. (Verified for lots of $d, n$.)

- The induced subgraphs

$$SR(d, n)|_{V(d, n) \cap \text{Per}(p)}$$

also appear to be Laplacian integral for all $d, n, p$. (Verified for $d \leq 6$.)

- Is $A(d, n)$ determined up to isomorphism by its spectrum? (We don’t know.)
Acknowledgements

Thanks to...

- The MathOverflow crowd (in particular Cristi Stoica and Noam Elkies)
- Sage (sagemath.org)
- Mahir, Michael and Jeff for organizing
- You for listening!

Preprint: arxiv:1209.3493