Simplicial and Cellular Spanning Trees, II: Applications

Art Duval (University of Texas at El Paso)
Caroline Klivans (Brown University)
Jeremy Martin (University of Kansas)

University of California, Davis
March 2011
Definition  A simplicial complex $\Delta$ on vertices $[n]$ is **shifted** if for all $F \in \Delta$, $i \in \Delta$, $j \notin \Delta$, and $j < i$, we have $F \setminus \{i\} \cup \{j\} \in \Delta$.

Example  If $\Delta$ is shifted and $235 \in \Delta$, then $\Delta$ must also contain the faces $234$, $135$, $134$, $125$, $124$, $123$.

- Shifted complexes of dimension 1 are *threshold graphs*. 
Define the **componentwise (partial) order** on \((d + 1)\)-sets of positive integers

\[
A = \{a_1 < a_2 < \cdots < a_{d+1}\},
\]
\[
B = \{b_1 < b_2 < \cdots < b_{d+1}\}
\]

by

\[
A \preceq B \iff a_i \leq b_i \text{ for all } i.
\]

▶ The set of facets of a shifted complex is a *lower order ideal* with respect to \(\preceq\).
Proposition  Shifted complexes are shellable, hence Cohen-Macaulay, hence metaconnected.

Theorem  [Duval–Reiner 2001]  For $\Delta$ shifted, the eigenvalues of the unweighted Laplacian $L$ are given by the transpose of the vertex/facet degree sequence.

- In particular, shifted complexes are Laplacian integral.
The Combinatorial Fine Weighting

Let $\Delta^d$ be a shifted complex on vertices $[n]$. For each facet $A = \{a_1 < a_2 < \cdots < a_{d+1}\}$, define

$$x_A = \prod_{i=1}^{d+1} x_{i,a_i}.$$ 

**Example** If $\Upsilon = \langle 123, 124, 134, 135, 235 \rangle$ is a simplicial spanning tree of $\Delta$, its contribution to $h_2$ is

$$x_{1,1}^4 x_{1,2} x_{2,2}^2 x_{2,3} x_{3,3}^2 x_{3,4} x_{3,5}^2.$$
The Algebraic Fine Weighting

For faces $A \subset B \in \Delta$ with $\dim A = i - 1$, $\dim B = i$, define

$$X_{AB} = \frac{\uparrow^{d-i} x_B}{\uparrow^{d-i+1} x_A}$$

where $\uparrow x_{i,j} = x_{i+1,j}$.

- Weighted boundary maps $\partial$ satisfy $\partial \partial = 0$.
- Laplacian eigenvalues are the same as those for the combinatorial fine weighting, except for denominators.
Definition  A **critical pair** of a shifted complex $\Delta^d$ is an ordered pair $(A, B)$ of $(d + 1)$-sets of integers, where

- $A \in \Delta$ and $B \not\in \Delta$; and
- $B$ covers $A$ in componentwise order.
Shifted Simplicial Complexes
More Applications

Definitions of Shiftedness
Fine Weightings
Critical Pairs
SST Enumeration

146 236 245
136 145 235
126 135 234
125 134
124
123
Let \((A, B)\) be a critical pair of a complex \(\Delta\):

\[
A = \{a_1 < a_2 < \cdots < a_i < \cdots < a_{d+1}\},
\]

\[
B = A \setminus \{a_i\} \cup \{a_i + 1\}.
\]

**Definition**  The **signature** of \((A, B)\) is the ordered pair

\[
(\{a_1, a_2, \ldots, a_{i-1}\}, a_i).
\]
Theorem [Duval–Klivans–JLM 2007]

Let $\Delta^d$ be a shifted complex.

Then the finely weighted Laplacian eigenvalues of $\Delta$ are specified completely by the signatures of critical pairs of $\Delta$.

$\text{signature } (S, a) \implies \text{eigenvalue } \frac{1}{X_S} \sum_{j=1}^{a} X_{S \cup j}$
Examples of Finely Weighted Eigenvalues

- Critical pair (135,145); signature (1,3):
  \[
  \frac{X_{11}X_{21} + X_{11}X_{22} + X_{11}X_{23}}{X_{21}}
  \]

- Critical pair (235,236); signature (23,5):
  \[
  \frac{X_{11}X_{22}X_{33} + X_{12}X_{22}X_{33} + X_{12}X_{23}X_{33} + X_{12}X_{23}X_{34} + X_{12}X_{23}X_{35}}{X_{22}X_{33}}
  \]
Sketch of Proof

- Calculate eigenvalues of $\Delta$ in terms of eigenvalues of the deletion and link:

  \[
  \text{del}_1 \Delta = \{ F \in \Delta \mid 1 \notin F \},
  \]
  \[
  \text{link}_1 \Delta = \{ F \in \Delta \mid 1 \notin F, \, F \cup \{1\} \in \Delta \}.
  \]

- If $\Delta$ is shifted, then so are $\text{del}_1 \Delta$ and $\text{link}_1 \Delta$. 
Sketch of Proof

- Calculate eigenvalues of $\Delta$ in terms of eigenvalues of the deletion and link:
  
  \[
  \text{del}_1 \Delta = \{ F \in \Delta \mid 1 \notin F \},
  \]
  \[
  \text{link}_1 \Delta = \{ F \in \Delta \mid 1 \notin F, \ F \cup \{1\} \in \Delta \}.  
  \]

- If $\Delta$ is shifted, then so are $\text{del}_1 \Delta$ and $\text{link}_1 \Delta$.

- Establish a recurrence for critical pairs of $\Delta$ in terms of those of $\text{del}_1 \Delta$ and $\text{link}_1 \Delta$. 

Sketch of Proof

▷ Calculate eigenvalues of $\Delta$ in terms of eigenvalues of the deletion and link:

$$\text{del}_1 \Delta = \{ F \in \Delta \mid 1 \not\in F \},$$
$$\text{link}_1 \Delta = \{ F \in \Delta \mid 1 \not\in F, \ F \cup \{1\} \in \Delta \}.$$

▷ If $\Delta$ is shifted, then so are $\text{del}_1 \Delta$ and $\text{link}_1 \Delta$.

▷ Establish a recurrence for critical pairs of $\Delta$ in terms of those of $\text{del}_1 \Delta$ and $\text{link}_1 \Delta$

▷ “Here see ye two recurrences, and lo! they are the same.”
Consequences of the Main Theorem

- Passing to the unweighted version (by setting $x_{i,j} = 1$ for all $i, j$) recovers the Duval–Reiner theorem.


- A shifted complex is determined by its set of signatures, so we can “hear the shape of a shifted complex” from its Laplacian spectrum.
A **Ferrers graph** is a bipartite graph whose vertices correspond to the rows and columns of a Ferrers diagram.
A **Ferrers graph** is a bipartite graph whose vertices correspond to the rows and columns of a Ferrers diagram.
A **Ferrers graph** is a bipartite graph whose vertices correspond to the rows and columns of a Ferrers diagram.
A **Ferrers graph** is a bipartite graph whose vertices correspond to the rows and columns of a Ferrers diagram.
A **Ferrers graph** is a bipartite graph whose vertices correspond to the rows and columns of a Ferrers diagram.
Ferrers Graphs

Ferrers graphs are bipartite analogues of threshold graphs.


- Formula can also be derived from our finely weighted spanning tree enumerator for a threshold graph

- Higher-dimensional analogues?
Let $\Delta$ be a complex on $V = \bigcup_i V_i$, where

$$V_1 = \{v_{11}, \ldots, v_{1r_1}\}, \ldots, V_n = \{v_{n1}, \ldots, v_{nr_n}\}.$$

are disjoint vertex sets ("color classes").

**Definition**  $\Delta$ is **color-shifted** if

- no face contains more than one vertex of the same color; and
- if $\{v_{1b_1}, \ldots, v_{nb_n}\} \in \Delta$ and $a_i \leq b_i$ for all $i$, then $\{v_{1a_1}, \ldots, v_{na_n}\} \in \Delta$. 
Color-Shifted Complexes

- Color-shifted complexes generalize Ferrers graphs (Ehrenborg–van Willigenburg) and complete colorful complexes (Adin).

- Not in general Laplacian integral...

- ...but they do seem to have nice degree-weighted spanning tree enumerators.
**Definition**  A pure simplicial complex $\Delta$ is a **matroid complex** if its facets form a matroid basis system:

- if $F, G$ are facets and $i \in F \setminus G$,
- then there exists $j \in G \setminus F$ such that $F \setminus \{i\} \cup \{j\}$ is a facet.


- Experimentally, degree-weighted spanning tree enumerators seem to have nice factorizations.