Math 830 ABSTRACT ALGEBRA
HOMEWORK – II

September 22 (Fri), 2006

Due Date: September 29 (Fri), 2006

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[I] (10pts) $\big($= Pg. Ch. – VI, problem [I]$\big)$ (1) Let

$$L \;=\; \left\{ \begin{bmatrix} m \\ n \end{bmatrix} \;\middle|\; m \in \mathbb{Z},\;\; n \in \mathbb{Z} \right\}.$$

This L forms an additive group under the usual addition:

$$\begin{bmatrix} m_1 \\ n_1 \end{bmatrix} + \begin{bmatrix} m_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 \\ n_1 + n_2 \end{bmatrix}.$$

$L \simeq \mathbb{Z}^2$. Consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$. Prove that the following two conditions are equivalent :

(a) For $m,\; n \in \mathbb{R}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \in L$ if and only if $\begin{bmatrix} m \\ n \end{bmatrix} \in L$.

(b) $a \in \mathbb{Z},\; b \in \mathbb{Z},\; c \in \mathbb{Z},\; d \in \mathbb{Z}$, and $ad - bc \in \{1,\; -1\}$.

[II] (10pts) $\big($= Pg. Ch. – VI, problem [II]$\big)$ (1) Prove that

$$GL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}) \;\middle|\; a,\; b,\; c,\; d \in \mathbb{Z},\;\; ad - bc \in \{1,\; -1\} \right\}$$

is a subgroup of $GL_2(\mathbb{R})$.

(2) Let

$$\mathrm{Aut}\,(L) = \left\{ \sigma : L \xrightarrow{\;\sim\;} L \;\middle|\; \sigma \text{ is an additive group isomorphism} \right\}.$$

Explain that $\mathrm{Aut}\,(L)$ forms a group, under composition of mappings.

(3) Use the result of [I] above to prove that $GL_2(\mathbb{Z})$ is isomorphic to $\mathrm{Aut}\,(L)$.

[III] (10pts) (= Pg. Ch. – VI, problem [III])

(1) Establish the formula

$$(*) \qquad \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b + b' \\ 0 & 1 \end{bmatrix} \qquad \left(b,\ b' \in \mathbb{R}\right).$$

Also, use $(*)$ to prove

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & mb \\ 0 & 1 \end{bmatrix} \quad \left(m \in \mathbb{Z}; \quad b \in \mathbb{R}\right).$$

(2) Use $(*)$ to prove that

$$\mathbb{G}_a(\mathbb{Z}) = \left\{ \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R}) \;\middle|\; m \in \mathbb{Z} \right\}$$

is a subgroup of $GL_2(\mathbb{Z})$.

(3) Use the results of (1), (2) to prove that the group $\mathbb{G}_a(\mathbb{Z})$ is isomorphic to the additive group $\mathbb{Z}$.

[IV] (10pts) (= Pg. Ch. – VI, problem [IV])

Mimic the above definition of $\mathbb{G}_a(\mathbb{Z})$, and also

$$\mathbb{G}_a(\mathbb{Q}) = \left\{ \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R}) \;\middle|\; r \in \mathbb{Q} \right\},$$

$$\mathbb{G}_a(\mathbb{R}) = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R}) \;\middle|\; x \in \mathbb{R} \right\},$$

to define $\mathbb{G}_a(\mathbb{C})$ as a subgroup of $GL_2(\mathbb{C})$.

[V] (10pts) (1) Prove or disprove

$$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \simeq \boldsymbol{\mu}_4.$$

Note that

$$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

(2) Prove or disprove

$$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 \simeq \boldsymbol{\mu}_6.$$

Note that

$$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \zeta_3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & {\zeta_3}^2 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & \zeta_3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & {\zeta_3}^2 \end{bmatrix} \right\}.$$

[VI] (10pts) Let G be the subgroup of $GL_2(\mathbb{C})$ generated by

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \zeta_3 & 0 \\ 0 & {\zeta_3}^2 \end{bmatrix}.$$

(1) Find $\operatorname{ord} A$, and $\operatorname{ord} B$.

(2) Find the order of G.

(3) Decide whether G is isomorphic to $\boldsymbol{\mu}_6$.

(4) Decide whether G is isomorphic to S_3, the symmetric group.

[VII] (Extra 20pts) Let F be the set

$$F = \{0, 1\},$$

where $0 \neq 1$. We introduce the additive structure of F as

$$0 + 0 = 0, \qquad 0 + 1 = 1, \qquad 1 + 0 = 1, \qquad 1 + 1 = 0.$$

(1) Prove that F is the additive group $\mathbb{Z}/2\mathbb{Z}$.

(2) We further introduce the multiplicative structure of the above $F = \mathbb{Z}/2\mathbb{Z}$ as

$$0 \cdot 0 = 0, \qquad 0 \cdot 1 = 0, \qquad 1 \cdot 0 = 0, \qquad 1 \cdot 1 = 1.$$

Prove that, the above additive and multiplicative structrures defined on F make F into a field. Namely, F satisfies the field axioms :

(F1) $\alpha + \beta = \beta + \alpha, \qquad \alpha\beta = \beta\alpha,$

(F2) $\alpha + \left(\beta + \gamma\right) = \left(\alpha + \beta\right) + \gamma, \qquad \alpha\left(\beta\gamma\right) = \left(\alpha\beta\right)\gamma,$

(F3) $\left(\alpha + \beta\right)\gamma = \alpha\gamma + \beta\gamma,$

(F4) $0 + \alpha = \alpha, \qquad 1 \cdot \alpha = \alpha,$

(F5) $\alpha + \left(-\alpha\right) = 0, \qquad \alpha\alpha^{-1} = 1 \qquad \left(\text{the latter assumes } \alpha \neq 0\right),$

for a suitable $\alpha^{-1} \in F$.

(2) Explain that a matrix multiplication on 2×2 matrices with entries in $F = \left\{0, 1\right\}$ makes sense:

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix},$$

where $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2 \in F$. Note that there are exactly 16 matrices with entries in F. You do not have to write out all the possible $\left(16^2 = 256\right)$ multiplications. Rather, you explain verbally how you multiply out matrices under the given arithmetic rules of F, and show a couple of concrete examples.

(3) Define

$$GL_2(F) = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \;\middle|\; \alpha, \beta, \gamma, \delta \in F, \quad \alpha\delta - \beta\gamma \neq 0 \right\}.$$

List all the elements that belong to $GL_2(F)$.

(4) Prove that $GL_2(F)$ is a group of order 6 under the multiplication rule that you have defined in (2).

(5) Prove $GL_2(F) \simeq S_3$.