

Math 830 ABSTRACT ALGEBRA

HOMEWORK – IV

October 18 (Wed), 2006

Revised: October 27 (Fri), 2006

Due Date: October 27 (Fri), 2006

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[I] (50pts) For $n \in \mathbb{Z}$, $n \geq 0$, define $\chi_n \in \mathbb{Q}[t]$ as

$$\chi_n(t) = \frac{(t+1)(t+2)\cdots(t+n)}{n!}.$$

Agree that χ_n has degree n . Agree also that the substitution of χ_n at a positive integer $t = d$ is

$$\chi_n(d) = \binom{d+n}{n} = \binom{d+n}{d} = \chi_d(n).$$

(1) Let $d \in \mathbb{Z}$. Explain that $\chi_n(d)$ is the exact number of monomials

$$x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$$

in $(n+1)$ variables $\{x_0, x_1, \dots, x_n\}$ having degree d : $d = i_0 + i_1 + \cdots + i_n$.

(2) Prove the identity:

$$\begin{aligned} \chi_n(d) &= \chi_n(d-1) + \chi_{n-1}(d-1) + \chi_{n-2}(d-1) + \cdots \\ &\quad + \cdots + \chi_1(d-1) + \chi_0(d-1), \end{aligned}$$

that is,

$$\begin{aligned} &\binom{d+n}{n} \\ &= \binom{d+n-1}{n} + \binom{d+n-2}{n-1} + \binom{d+n-3}{n-2} + \cdots \\ &\quad + \cdots + \binom{d}{1} + \binom{d-1}{0}. \end{aligned}$$

(3) Prove the identity

$$\begin{aligned}\chi_n(t) &= \chi_n(t-1) + \chi_{n-1}(t-1) + \chi_{n-2}(t-1) + \cdots \\ &\quad + \cdots + \chi_1(t-1) + \chi_0(t-1)\end{aligned}$$

as elements of the polynomial ring $\mathbb{Q}[t]$. Use the fact that for two elements $\eta, \eta' \in \mathbb{Q}[t]$, if $\eta(d) = \eta'(d)$ for infinitely many $d \in \mathbb{Q}$, then $\eta = \eta'$ as elements of $\mathbb{Q}[t]$.

(4) Let $\xi \in \mathbb{Q}[t]$ be a numerical polynomial of degree at most n , namely,

$$\xi = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0,$$

where $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{Q}$, such that

$$\xi(d) \in \mathbb{Z} \quad \text{for an arbitrary } d \in \mathbb{Z}.$$

Prove that ξ is expressed as a \mathbb{Z} -linear combination of

$$\left\{ \chi_n(t-1), \chi_{n-1}(t-1), \chi_{n-2}(t-1), \dots, \chi_1(t-1), \chi_0(t-1) \right\}.$$

That is,

$$\begin{aligned}\xi_n(t) &= \lambda_n \cdot \chi_n(t-1) + \lambda_{n-1} \cdot \chi_{n-1}(t-1) + \lambda_{n-2} \cdot \chi_{n-2}(t-1) + \cdots \\ &\quad + \cdots + \lambda_1 \cdot \chi_1(t-1) + \lambda_0 \cdot \chi_0(t-1),\end{aligned}$$

for some integers $\lambda_n, \lambda_{n-1}, \lambda_{n-2}, \dots, \lambda_1, \lambda_0$.

(5) Verify

$$\begin{aligned}\frac{1}{\chi_2(t)} &= \frac{2}{(t+1)(t+2)} = 2 \int_{x=0}^1 x^t (1-x) dx \\ &\quad (t > -1), \\ \frac{1}{\chi_3(t)} &= \frac{6}{(t+1)(t+2)(t+3)} = 3 \int_{x=0}^1 x^t (1-x)^2 dx \\ &\quad (t > -1).\end{aligned}$$

(6) (Extra 20pts) Make a conjecture about the formula expressing $\chi_n(t)$ in terms of

$$\int_{x=0}^1 x^t (1-x)^{n-1} dx.$$

Give what you think of as the most natural definition of $\chi_s(t)$ for s a real number, not necessarily a positive integer, so that, when $s = n$ is a positive integer, your $\chi_s(t)$ equals $\chi_n(t)$ defined above. Generalize the identity (3) for $\chi_s(t)$.