Math 830 ABSTRACT ALGEBRA

QUIZ – V

August 30 (Wed), 2006

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[I] (8pts)

(1)
$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2p_2 + a_3p_3} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(2)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix},$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

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([I] continued)

(3) For $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, define

$$A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Find

$A\,A^T$

$$= \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

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[II] (10pts) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Define

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

(1) $$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}$$

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([II] continued)

(2) $$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 11 & 23 & 37 \end{vmatrix} = \underline{\qquad\qquad} .$$

(3) $$\begin{vmatrix} 1 & 0 & 5 \\ 6 & 0 & 8 \\ -4 & 0 & 9 \end{vmatrix} = \underline{\qquad\qquad} .$$

(4) $$\begin{vmatrix} 2 & 18 & 27 \\ 0 & 2 & 15 \\ 0 & 0 & -3 \end{vmatrix} = \underline{\qquad\qquad} .$$

(5) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \underline{\qquad\qquad\qquad\qquad\qquad} .$$

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[III] (8pts) For $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 6 \\ 5 & 7 & 8 \end{bmatrix}$, let

$$C = \begin{bmatrix} +\begin{vmatrix} 4 & 6 \\ 7 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} \\ -\begin{vmatrix} 3 & 6 \\ 5 & 8 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 5 & 8 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 6 \end{vmatrix} \\ +\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 5 & 7 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}.$$

Then

$$AC = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 6 \\ 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix},$$

$\det A = \square$, and

$$A^{-1} = \frac{1}{\square} \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \quad \left(\text{if exists}\right).$$

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[IV] (16pts) For $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, define

$$\operatorname{Tr} A = a_1 + b_2 + c_3.$$

(1) $\operatorname{Tr} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & -4 \\ -3 & 4 & 5 \end{bmatrix} =$ ________ .

(2) Use the result of [I], (1) to find

$$\operatorname{Tr} \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix} \right)$$

= ____ .

Also find

$$\operatorname{Tr} \left(\begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right)$$

= ____ .

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([IV] continued)

(3) For $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, write

$$\det \Big(t\, I - A \Big) = t^3 - \xi_2\, t^2 + \xi_1\, t - \xi_0 \,,$$

where ξ_2, ξ_1 and ξ_0 are functions on a_i, b_i, c_i. Prove

$$\xi_2 = \operatorname{Tr} A, \qquad \text{and} \qquad \xi_0 = \det A.$$

In particular, if we write

$$\det \Big(t\, I - A \Big) = \Big(t - \lambda_1 \Big) \Big(t - \lambda_2 \Big) \Big(t - \lambda_3 \Big) \,,$$

then

$$\operatorname{Tr} A = \lambda_1 + \lambda_2 + \lambda_3, \qquad \text{and} \qquad \det A = \lambda_1\, \lambda_2\, \lambda_3.$$

(4) Let I be as in (3). Use (2) to prove $AB - BA \neq I$ for arbitrary

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix}.$$

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[V] (18pts) Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(1) $A^2 = \begin{bmatrix} \Box & \Box & \Box \\ \Box & \Box & \Box \\ \Box & \Box & \Box \end{bmatrix}$. $B^2 = \begin{bmatrix} \Box & \Box & \Box \\ \Box & \Box & \Box \\ \Box & \Box & \Box \end{bmatrix}$.

☐ $\{I, \ A, \ B\}$ forms a group.

☐ $\{I, \ A, \ B\}$ <u>does not</u> form a group.

(2) $B^3 = \begin{bmatrix} \Box & \Box & \Box \\ \Box & \Box & \Box \\ \Box & \Box & \Box \end{bmatrix}$. $AB = \begin{bmatrix} \Box & \Box & \Box \\ \Box & \Box & \Box \\ \Box & \Box & \Box \end{bmatrix}$.

☐ $\{I, \ A, \ B, \ B^2\}$ forms a group.

☐ $\{I, \ A, \ B, \ B^2\}$ <u>does not</u> form a group.

(3) $BA = \begin{bmatrix} \Box & \Box & \Box \\ \Box & \Box & \Box \\ \Box & \Box & \Box \end{bmatrix}$.

☐ $AB = BA$.

☐ $AB \neq BA$.

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([V] continued)

(4) Let $G = \left\{ I, \;\; A, \;\; B, \;\; B^2, \;\; AB, \;\; BA \right\}$.

- ☐ G forms a group.
- ☐ G does not form a group.

(5) If your answer in (4) is: " G forms a group " .

- ☐ G is abelian.
- ☐ G is non-abelian.

- ☐ G is isomorphic to the cyclic group $\boldsymbol{\mu}_6$.
- ☐ G is not isomorphic to the cyclic group $\boldsymbol{\mu}_6$.

- ☐ G is isomorphic to the dihedral group of order 6.
- ☐ G is not isomorphic to the dihedral group of order 6.

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([V] continued)

(6) Let A be as above. Find $\lambda_1, \ \lambda_2, \ \lambda_3 \ \in \ \mathbb{C}$ such that

$$\det \left(t\, I - A \right) \ = \ \left(t - \lambda_1 \right) \left(t - \lambda_2 \right) \left(t - \lambda_3 \right) .$$

For each $i = 1, 2, 3,$ find all $\boldsymbol{x}_i = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in \mathbb{C}^3$ such that $A\, \boldsymbol{x}_i = \lambda_i\, \boldsymbol{x}_i$.

Note that some of λ_i's may coincide.

(7) Let B be as above. Find $\lambda_1, \ \lambda_2, \ \lambda_3 \ \in \ \mathbb{C}$ such that

$$\det \left(t\, I - B \right) \ = \ \left(t - \lambda_1 \right) \left(t - \lambda_2 \right) \left(t - \lambda_3 \right) .$$

For each $i = 1, 2, 3,$ find all $\boldsymbol{x}_i = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in \mathbb{C}^3$ such that $B\, \boldsymbol{x}_i = \lambda_i\, \boldsymbol{x}_i$.

(8) Use (6), (7) to find all pairs of $\lambda \in \mathbb{C}$ and $\boldsymbol{x} \in \mathbb{C}^3$ such that

$$C\, \boldsymbol{x} \ = \ \lambda\, \boldsymbol{x}$$

holds for an <u>arbitrary</u> $C \in G$.

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[VI] (40pts) For $a, b, c, d \in \mathbb{R}$, define

$$A_{(a,b,c,d)} = \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2\big(bc + ad\big) & 2\big(bd - ac\big) \\ 2\big(bc - ad\big) & a^2 - b^2 + c^2 - d^2 & 2\big(cd + ab\big) \\ 2\big(bd + ac\big) & 2\big(cd - ab\big) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}.$$

(1) Find $\operatorname{Tr} A_{(a,b,c,d)}$.

(2) Verify $\Big(A_{(a,b,c,d)}\Big)\Big(A_{(a,b,c,d)}\Big)^T = \Big(a^2 + b^2 + c^2 + d^2\Big)^2 I.$

(3) Use (2) to find $\det A_{(a,b,c,d)}$.

(4) Find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that

$$\det\Big(t\,I - A\Big) = \Big(t - \lambda_1\Big)\Big(t - \lambda_2\Big)\Big(t - \lambda_3\Big).$$

Note that $\lambda_1 \in \mathbb{R}$, and $\lambda_2 = \overline{\lambda_3}$. First guess λ_1. Keep the results of (1), (3) above in mind, and use the formula in [IV] (3) to deduce a quadratic equation in t that has two complex number roots $t = \lambda_2, \lambda_3$.

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([VI] continued)

(5) Let

$$a,\ b,\ c,\ d,\ a',\ b',\ c',\ d' \in \mathbb{R},$$

and let

$$\begin{aligned} a'' &= a\,a' - b\,b' - c\,c' - d\,d', \\ b'' &= a\,b' + b\,a' - c\,d' + d\,c', \\ c'' &= a\,c' + b\,d' + c\,a' - d\,b', \\ d'' &= a\,d' - b\,c' + c\,b' + d\,a'. \end{aligned}$$

Prove

$$A_{(a,b,c,d)}\,A_{(a',b',c',d')} = A_{(a'',b'',c'',d'')}.$$

Compare this result with the identity $(*)$ in Quiz IV, problem [IV] (7).