

**Math 830 ABSTRACT ALGEBRA**  
**QUIZ – IX**

October 27 (Fri), 2006

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**Line #:** 17014.

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[I] (40pts) Let

$$f \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = x^2 + y^2,$$

regarded as a function  $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ .

(1)  $f \left( \begin{bmatrix} -y \\ x \end{bmatrix} \right) =$  \_\_\_\_\_,  $f \left( \begin{bmatrix} x \\ -y \end{bmatrix} \right) =$  \_\_\_\_\_.

(2)  $f \left( \begin{bmatrix} \frac{x-y}{\sqrt{2}} \\ \frac{x+y}{\sqrt{2}} \end{bmatrix} \right) =$  \_\_\_\_\_.

(3)  $f \left( \begin{bmatrix} (\cos \theta)x - (\sin \theta)y \\ (\sin \theta)x + (\cos \theta)y \end{bmatrix} \right) =$  \_\_\_\_\_,

$f \left( \begin{bmatrix} (\cos \theta)x + (\sin \theta)y \\ (\sin \theta)x - (\cos \theta)y \end{bmatrix} \right) =$  \_\_\_\_\_.

(4) Let  $t \in \mathbb{R}$ ,  $t > 0$ .

$$f^{-1}(t) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid \boxed{\hspace{2cm}} \right\}.$$

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([I] continued)

(5) Let

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}) \mid \text{for an arbitrary } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2, \right. \\ \left. f \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) = f \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \right\}.$$

Prove

“  $A \in G$ , if and only if  $A^T A = I$ . ”

**Proof.** The condition  $f \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) = f \left( \begin{bmatrix} x \\ y \end{bmatrix} \right)$  is

rewritten as  $(ax + by)^2 + (cx + dy)^2 = x^2 + y^2$ , that is,

$$(a^2 + c^2)x^2 + 2(ab + cd)xy + (b^2 + d^2)y^2 \\ = 1 \cdot x^2 + 0 \cdot xy + 1 \cdot y^2.$$

This holds for an arbitrary  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ . Compare the coefficients of  $x^2$ ,  $xy$ ,  $y^2$ :

(\*)  $\boxed{\phantom{000}} = 1$ ,  $\boxed{\phantom{000}} = 0$ ,  $\boxed{\phantom{000}} = 1$ .

Notice

$$A^T A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \boxed{\phantom{000}} & \boxed{\phantom{000}} \\ \boxed{\phantom{000}} & \boxed{\phantom{000}} \end{bmatrix}.$$

Thus the condition (\*) is equivalent to  $A^T A = \boxed{\phantom{000}}$ .

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([I] continued)

(6) Prove that  $G$  is a subgroup of  $GL_2(\mathbb{R})$ . Prove

$$G = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \mid \theta \in \mathbb{R} \right\} \\ \cup \left\{ \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \mid \theta \in \mathbb{R} \right\}.$$

(7) Find all the elements of  $G$  of order  $2^\ell$ , for each  $\ell = 1, 2, 3, \dots$ .

(8) List all the subgroups  $H$  of  $G$  satisfying the properties (i–ii):

- (i) “ For an arbitrary element  $A \in H$ , there exists an integer  $\ell$ ,  $\ell \geq 1$ , such that  $A^{(2^\ell)} = I$ . ”
- (ii) “ There is no subgroup  $H'$  of  $G$  satisfying (i) and  $H \subseteq H'$ , other than  $H$  itself. ”

For which pair(s) of subgroups  $H_1$  and  $H_2$  of  $G$  satisfying (i–ii), there exists  $P \in G$  such that

$$H_2 = \left\{ P A P^{-1} \mid A \in H_1 \right\} ?$$

Explain.

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([I] continued)

As for (8), use the following: For

$$P = \begin{bmatrix} \cos \frac{\tau - \theta}{2} & -\sin \frac{\tau - \theta}{2} \\ \sin \frac{\tau - \theta}{2} & \cos \frac{\tau - \theta}{2} \end{bmatrix},$$

and

$$A = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix},$$

we have

$$P A P^{-1} = \begin{bmatrix} \cos \tau & \sin \tau \\ \sin \tau & -\cos \tau \end{bmatrix}.$$

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[II] (60pts) Let

$$g \left( \begin{bmatrix} z \\ w \end{bmatrix} \right) = |z|^2 + |w|^2,$$

regarded as a function  $g: \mathbb{C}^2 \longrightarrow \mathbb{R}$ .

$$(1) \quad g \left( \begin{bmatrix} \sqrt{-1} z \\ -\sqrt{-1} w \end{bmatrix} \right) = \underline{\hspace{2cm}},$$

$$g \left( \begin{bmatrix} -w \\ z \end{bmatrix} \right) = \underline{\hspace{2cm}},$$

$$g \left( \begin{bmatrix} \sqrt{-1} w \\ \sqrt{-1} z \end{bmatrix} \right) = \underline{\hspace{2cm}}.$$

$$(2) \quad g^{-1}(0) = \left\{ \begin{bmatrix} \boxed{\phantom{0}} \\ \boxed{\phantom{0}} \end{bmatrix} \right\}.$$

(3) Let  $t \in \mathbb{R}$ ,  $t > 0$ .

$$g^{-1}(t) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{C}^2 \mid \boxed{\phantom{0}} \right\}.$$

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([II] continued)

(4) Let

$$G = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL_2(\mathbb{C}) \mid \text{for an arbitrary } \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{C}^2, \right. \\ \left. g \left( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \right) = g \left( \begin{bmatrix} z \\ w \end{bmatrix} \right) \right\}.$$

For  $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ , define  $A^* = \begin{bmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{bmatrix}$ . Prove that for  $A \in GL_2(\mathbb{C})$ ,

“  $A \in G$ , if and only if  $A^* A = I$ . ”

First reduce the condition

$$f \left( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \right) = f \left( \begin{bmatrix} z \\ w \end{bmatrix} \right)$$

to

$$\begin{aligned} & (|\alpha|^2 + |\gamma|^2) |z|^2 + (\alpha \bar{\beta} + \gamma \bar{\delta}) z \bar{w} \\ & + (\bar{\alpha} \beta + \bar{\gamma} \delta) \bar{z} w + (|\beta|^2 + |\delta|^2) |w|^2 = |z|^2 + |w|^2. \end{aligned}$$

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([II] continued)

This holds for an arbitrary  $\begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{C}^2$ . Substitute  $\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  
to obtain

$$|\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2 = 1.$$

Substitute this back in the previous identity, and it becomes

$$(\alpha \bar{\beta} + \gamma \bar{\delta}) z \bar{w} + (\bar{\alpha} \beta + \bar{\gamma} \delta) \bar{z} w = 0.$$

This holds for an arbitrary  $\begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{C}^2$ . Substitute  $\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{-1} \end{bmatrix}$ .

Verify that what you obtain is equivalent to

$$\begin{bmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(5) Let  $G$  be as in (4). Prove that

$$H = \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \in GL_2(\mathbb{C}) \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

is a subgroup of  $G \cap SL_2(\mathbb{C})$ .

(6) **(Extra 100pts)** Let  $G$  and  $H$  be as in (4), (5). Prove

$$H = G \cap SL_2(\mathbb{C}).$$

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([II] continued)

(7) Let us introduce the following notations for  $G$  and  $H$ :

$$G = U_2, \quad H = G \cap SL_2(\mathbb{C}) = SU_2.$$

Let  $\varphi : SU_2 \longrightarrow GL_3(\mathbb{R})$  be a mapping defined as

$$\begin{aligned} & \varphi \left( \begin{bmatrix} a + \sqrt{-1}b & -c + \sqrt{-1}d \\ c + \sqrt{-1}d & a - \sqrt{-1}b \end{bmatrix} \right) \\ &= \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc + ad) & 2(bd - ac) \\ 2(bc - ad) & a^2 - b^2 + c^2 - d^2 & 2(cd + ab) \\ 2(bd + ac) & 2(cd - ab) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}. \end{aligned}$$

Quote a statement from Quiz V, problem [VI], which affirms that  $\varphi$  satisfies

$$(\text{Hom}) \quad \varphi(AB) = \varphi(A)\varphi(B) \quad (A, B \in SU_2).$$

Quote a statement from Quiz V, problem [VI], which affirms that for  $A \in SU_2$ ,  $C = \varphi(A)$  satisfies  $C^T C = I$ .

Quote a statement from Quiz V, problem [VI], which affirms that for  $A \in SU_2$ ,  $\det \varphi(A) = 1$

(8) Prove  $\varphi^{-1}(I) = \{ \pm I \}$ .

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([II] continued)

(9) Let

$$O_3 = \left\{ C = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \in GL_3(\mathbb{R}) \mid C^T C = I \right\} \subseteq GL_3(\mathbb{R}).$$

Find a function  $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$  such that

$$O_3 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \in GL_3(\mathbb{R}) \mid \text{for an arbitrary } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3, \right. \\ \left. f \left( \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = f \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \right\}.$$

(10) Let

$$SO_3 = O_3 \cap SL_3(\mathbb{R}).$$

According to (7),  $\varphi: SU_2 \longrightarrow SO_3$ . Let

$$T = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \end{bmatrix} \right\},$$

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([II] continued)

and moreover let

$$S = T \cap SL_3(\mathbb{R}).$$

Prove that  $T$  is a subgroup of  $O_3$  of order 48. Prove that  $S$  is a subgroup of  $SO_3$  of order 24.

(11) Decide whether  $S \simeq S_4$ , the symmetric group of degree 4.

(12) Prove that  $\varphi^{-1}(S)$  is a subgroup of  $SU_2$  of order 48. Decide whether  $\varphi^{-1}(S)$  is isomorphic to  $T$  above.