Math 223 VECTOR CALCULUS
PROGRESS CHECK – XV

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Line #: 32702.

- **Curvature $\kappa(s)$.**

We introduce an invariant which describes "how curved a curve is". Let $\boldsymbol{r}(s)$ be a parametrization of a curve such that

$$\left|\frac{d\boldsymbol{r}}{ds}\right| = 1, \qquad \text{independently of } s,$$

so that

$$\int_0^s \left|\frac{d\boldsymbol{r}}{ds}(s)\right| ds = s.$$

In other words, the curve is already parametrized by an arc length. Let

$$\kappa(s) = \left|\frac{d^2\boldsymbol{r}}{ds^2}(s)\right|,$$

and call it the <u>curvature</u> of the curve $\boldsymbol{r}$. Note that we have

$$\kappa(s) \geq 0.$$

- Intuitively, the curvature describes <u>the rate of change</u> of the tangent vector of $\boldsymbol{r}$, under the constraint that the <u>length</u> of the tangent vector is constantly equal to 1.

- We would like to compute the curvature of various curves. The above definition is often not practical for computational purposes. Namely, it would be ideal to have a formula to describe κ using an abitrary t which parametrizes the given same curve.

Formula. Let $\boldsymbol{r}(t)$ be an <u>arbitrary</u> parametrization of a curve. Then

$$\kappa(t) = \frac{\left|\frac{d\boldsymbol{r}}{dt}(t) \times \frac{d^2\boldsymbol{r}}{dt^2}(t)\right|}{\left|\frac{d\boldsymbol{r}}{dt}(t)\right|^3}.$$

- We note that this expression is <u>independent of</u> the choice of the parameter t.

- Evidently the line has a constantly 0 curvature. Indeed, for

$$\boldsymbol{r}(t) = \langle a + pt,\ b + qt,\ c + rt \rangle, \qquad \text{where } a,\ b,\ c,\ p,\ q,\ r \ \text{ are constant,}$$

we have $\dfrac{d\boldsymbol{r}}{dt} = \langle p,\ q,\ r \rangle$ (constant), and

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt}\right) = \langle 0,\ 0,\ 0 \rangle \qquad \text{(constant).}$$

Example. $\boldsymbol{r} = \langle t,\ t^2,\ 0 \rangle$. This is a plane parabola. To find its curvature κ (in terms of t), we simply apply the above formula, using the data;

$$\frac{d\boldsymbol{r}}{dt}(t) = \langle 1,\ 2t,\ 0 \rangle, \qquad \frac{d^2\boldsymbol{r}}{dt^2}(t) = \langle 0,\ 2,\ 0 \rangle,$$

$$\frac{d\boldsymbol{r}}{dt}(t) \times \frac{d^2\boldsymbol{r}}{dt^2}(t) = \langle 0,\ 0,\ 2 \rangle.$$

Namely, we have;

$$\kappa(t) = \frac{\left| \dfrac{d\boldsymbol{r}}{dt}(t) \times \dfrac{d^2\boldsymbol{r}}{dt^2}(t) \right|}{\left| \dfrac{d\boldsymbol{r}}{dt}(t) \right|^3} = \frac{|\langle 0,\ 0,\ 2 \rangle|}{|\langle 1,\ 2t,\ 0 \rangle|^3} = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Example. $\boldsymbol{r} = \langle a\cos t,\ a\sin t,\ 0 \rangle$. This is a circle, with the radius a. We may find its curvature κ, in the same fashion as above;

$$\frac{d\boldsymbol{r}}{dt}(t) = \langle -a\sin t,\ a\cos t,\ 0 \rangle, \qquad \frac{d^2\boldsymbol{r}}{dt^2}(t) = \langle -a\cos t,\ -a\sin t,\ 0 \rangle,$$

$$\frac{d\boldsymbol{r}}{dt}(t) \times \frac{d^2\boldsymbol{r}}{dt^2}(t) = \langle 0,\ 0,\ a^2 \rangle.$$

Hence;

$$\kappa(t) = \frac{\left| \dfrac{d\boldsymbol{r}}{dt}(t) \times \dfrac{d^2\boldsymbol{r}}{dt^2}(t) \right|}{\left| \dfrac{d\boldsymbol{r}}{dt}(t) \right|^3} = \frac{|\langle 0,\ 0,\ a^2 \rangle|}{|\langle -a\sin t,\ a\cos t,\ 0 \rangle|^3} = \frac{1}{a}$$

(independent of t).

- To iterate,

" the curvature of a circle with radius a is a^{-1} . "

- **Normal, and Binormal vectors.**

The derivatives; $\dfrac{d\boldsymbol{r}}{ds}(s)$, $\dfrac{d^2\boldsymbol{r}}{ds^2}(s)$, with respect to the arc length s as parameter, play fundamental roles in characterizing the shape of the curve. We denote

$$\mathbb{T}(s) = \frac{d\boldsymbol{r}}{ds}(s).$$

Provided that $\kappa(s) \neq 0$, we denote;

$$\mathbb{N}(s) = \frac{1}{\kappa(s)} \frac{d\mathbb{T}}{ds}(s) = \frac{1}{\kappa(s)} \frac{d^2\boldsymbol{r}}{ds^2}(s).$$

Properties. $\quad \big|\mathbb{T}(s)\big| = \big|\mathbb{N}(s)\big| = 1, \qquad \mathbb{T}(s) \cdot \mathbb{N}(s) = 0.$

Indeed, $\big|\mathbb{T}(s)\big| = 1$ is equivalent to the assumption that s is the arc length. $\big|\mathbb{N}(s)\big| = 1$ follows immediately from the definition of $\mathbb{N}(s)$. Finally, as to obtaining $\mathbb{T}(s) \cdot \mathbb{N}(s) = 0$ we simply apply $\dfrac{d}{ds}$ to the identity;

$$\big|\mathbb{T}(s)\big|^2 = 1.$$

- Furthermore, under the same assumption $\kappa(s) \neq 0$, we define

$$\mathbb{B}(s) = \mathbb{T}(s) \times \mathbb{N}(s).$$

Properties. $\quad \big|\mathbb{B}(s)\big| = 1, \qquad \mathbb{B}(s) \cdot \mathbb{T}(s) = \mathbb{B}(s) \cdot \mathbb{N}(s) = 0.$

In sum, we have defined three vector functions

$$\mathbb{T}(s), \quad \mathbb{N}(s), \quad \mathbb{B}(s),$$

arising from a vector function $\boldsymbol{r}(s)$, with $\left|\dfrac{d\boldsymbol{r}}{ds}\right| = 1$, which form an orthonormal basis of $\mathbb{R}^3$, at each s.

- **Frenet–Serret Equation.**

For a parametrization of a curve; $\boldsymbol{r}(s)$, where s is the arc length, we have defined three vectors

$$\mathbb{T}(s), \quad \mathbb{N}(s), \quad \mathbb{B}(s).$$

By definition, these are related through;

$$\frac{d\mathbb{T}}{ds}(s) = \kappa(s)\mathbb{N}(s), \qquad \big|\,\mathbb{N}(s)\,\big| = 1, \qquad \mathbb{B}(s) = \mathbb{T}(s) \times \mathbb{N}(s).$$

- In below we will deduce a fundamental system of equations which these three vector functions are to satisfy, called the <u>Frenet–Serret equation</u>.

First, $\big|\,\mathbb{N}(s)\,\big|^2 = 1$, $\big|\,\mathbb{B}(s)\,\big|^2 = 1$. We may apply $\dfrac{d}{ds}$ to these two identities, and obtain

$$\mathbb{N}(s) \cdot \frac{d\mathbb{N}}{ds}(s) = \mathbf{0}, \qquad \mathbb{B}(s) \cdot \frac{d\mathbb{B}}{ds}(s) = \mathbf{0}.$$

Moreover, we may apply $\dfrac{d}{ds}$ to the identity $\mathbb{B}(s) = \mathbb{T}(s) \times \mathbb{N}(s)$ and obtain;

$$\begin{aligned} \frac{d\mathbb{B}}{ds}(s) &= \frac{d\mathbb{T}}{ds}(s) \times \mathbb{N}(s) + \mathbb{T}(s) \times \frac{d\mathbb{N}}{ds}(s) \\ &= \mathbb{T}(s) \times \frac{d\mathbb{N}}{ds}(s). \end{aligned}$$

From this it follows

$$\mathbb{T}(s) \cdot \frac{d\mathbb{B}}{ds}(s) = 0.$$

Hence $\dfrac{d\mathbb{B}}{ds}(s)$ is orthogonal to both $\mathbb{T}(s)$ and $\mathbb{B}(s)$. Since $\mathbb{T}(s)$, $\mathbb{N}(s)$, $\mathbb{B}(s)$ form an <u>orthonormal basis</u> of $\mathbb{R}^3$, there exists a function $\tau(s)$ such that

$$\frac{d\mathbb{B}}{ds}(s) = -\tau(s)\,\mathbb{N}(s).$$

(The negative sign is for convenience.)

- We have thus far obtained two main differential equations;

$$\frac{d\mathbb{T}}{ds}(s) \;=\; \kappa(s)\mathbb{N}(s), \qquad \frac{d\mathbb{B}}{ds}(s) \;=\; -\tau(s)\mathbb{N}(s).$$

Finally, from $\mathbb{B}(s) = \mathbb{T}(s) \times \mathbb{N}(s)$ we have

$$\mathbb{N}(s) \;=\; \mathbb{B}(s) \times \mathbb{T}(s), \quad \text{and}$$

$$\begin{aligned} \frac{d\mathbb{N}}{ds}(s) &= \mathbb{B}(s) \times \frac{d\mathbb{T}}{ds}(s) + \frac{d\mathbb{B}}{ds}(s) \times \mathbb{T}(s) \\ &= \kappa(s)\,\mathbb{B}(s) \times \mathbb{N}(s) - \tau(s)\,\mathbb{N}(s) \times \mathbb{T}(s) \\ &= -\kappa(s)\,\mathbb{T}(s) + \tau(s)\,\mathbb{B}(s). \end{aligned}$$

In sum, we have deduced the following <u>system of (ordinary) differential equations</u>

$$\begin{cases} \dfrac{d\mathbb{T}}{ds}(s) = \kappa(s)\,\mathbb{N}(s), \\[2ex] \dfrac{d\mathbb{N}}{ds}(s) = -\kappa(s)\,\mathbb{T}(s) + \tau(s)\,\mathbb{B}(s), \\[2ex] \dfrac{d\mathbb{B}}{ds}(s) = -\tau(s)\,\mathbb{N}(s), \end{cases}$$

or

$$\frac{d}{ds}\begin{bmatrix} \mathbb{T}(s) \\ \mathbb{N}(s) \\ \mathbb{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbb{T}(s) \\ \mathbb{N}(s) \\ \mathbb{B}(s) \end{bmatrix}$$

(<u>Frenet–Serret Equation</u>).

- **Torsion** $\tau(s)$**.**

We call the function $\tau(s)$ the <u>torsion</u> of the curve. This, along with the curvature $\kappa(s)$, are the two fundamental invariants of the curve $\boldsymbol{r}(s)$. Indeed, these two invariants <u>characterize the shape of the curve</u>.

- **Formula.** The torsion τ, using an arbitrary parameter t, is calculated as;

$$\tau(t) = \frac{\left(\dfrac{d\boldsymbol{r}}{dt}(t) \times \dfrac{d^2\boldsymbol{r}}{dt^2}(t)\right) \cdot \dfrac{d^3\boldsymbol{r}}{dt^3}(t)}{\left|\dfrac{d\boldsymbol{r}}{dt}(t) \times \dfrac{d^2\boldsymbol{r}}{dt^2}(t)\right|^2}.$$