Math 223 VECTOR CALCULUS
REGULAR HW #3 – HINTS FOR SELECTED PROBLEMS
— PART 2 —

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⋆ The following problems are assigned:

Text Section 1.6 / pg. 57 # 7, 9, 10, 11

Text Section 1.4 / pg. 37 [Part 2] # 13, 20, 21, 26, 27–33

Text Section 1.7 / pg. 71 [Part 1] # 1, 5, 8, 10, 14, 17

⋆ **Due date: Wednesday, February 10th, 2010** .

⋆ **This sheet contains hints for Section 1.4 problems only.**

• **1.4. #13.** Suppose three vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ in $\mathbb{R}^3$ satisfy

$$(*) \qquad \Big(\boldsymbol{a}_1 \times \boldsymbol{a}_2\Big) \cdot \boldsymbol{a}_3 = 0 .$$

Prove the following: Let P_1, P_2 and P_3 be three points in $\mathbb{R}^3$ desgnated by the components of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, respectively. Also, let O denote the origin $\big(=$ the point designated by the components of $\mathbf{0}\big)$. Under the assumption $(*)$ as above, prove that either one of the following two scenarios occurs:

[Scenario (i)]: the four points P_1, P_2, P_3 and O are co-planar, or

[Scenario (ii)]: at least two of P_1, P_2, P_3 and O coincide.

(Alternatively, under $(*)$, prove that there exist c_1, c_2, $c_3 \in \mathbb{R}$ such that

$$c_1 \boldsymbol{a}_1 + c_2 \boldsymbol{a}_2 + c_3 \boldsymbol{a}_3 = \mathbf{0}. \quad \Big)$$

[Strategy] : Write the proof as follows:

Proof. Suppose at least one of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ equals $\mathbf{0}$, say $\boldsymbol{a}_1 = \mathbf{0}$. Then P_1 and O coincide. Hence Scenario (ii) occurs. In view of this, for the rest of proof we assume

$$\boldsymbol{a}_1 \neq \mathbf{0}, \qquad \boldsymbol{a}_2 \neq \mathbf{0}, \qquad \text{and} \qquad \boldsymbol{a}_3 \neq \mathbf{0}.$$

Next, suppose $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are mutually proportionate. Then clearly P_1, P_2 and O are co-linear. In particular, P_1, P_2, P_3 and O are co-planar. In other words, Scenario (i) occurs. Thus, for the rest of proof, we assume that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are not mutually proportionate. Or the same to say, we assume $\boldsymbol{a}_1 \times \boldsymbol{a}_2 \neq \mathbf{0}$. Now, let

$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \Big\langle\, p, \quad q, \quad r \,\Big\rangle.$$

By assumption made above, at least one of p, q, r is non-zero . Consider the following set of vectors in $\mathbb{R}^3$:

$$\begin{aligned} V &= \Big\{ \, \boldsymbol{v} \ \Big| \ \boldsymbol{v} \text{ is a vector in } \mathbb{R}^3, \ \Big(\boldsymbol{a}_1 \times \boldsymbol{a}_2\Big) \cdot \boldsymbol{v} = 0 \, \Big\} \\ &= \Big\{ \, \Big\langle x, \ y, \ z \Big\rangle \ \Big| \ p\,x + q\,y + r\,z = 0 \, \Big\}. \end{aligned}$$

Since we know

$$\Big(\boldsymbol{a}_1 \times \boldsymbol{a}_2\Big) \cdot \boldsymbol{a}_1 = 0, \qquad \text{and} \qquad \Big(\boldsymbol{a}_1 \times \boldsymbol{a}_2\Big) \cdot \boldsymbol{a}_2 = 0,$$

it follows $\boldsymbol{a}_1 \in V$ and $\boldsymbol{a}_2 \in V$. On the other hand, our assumption

$$(*) \qquad \Big(\boldsymbol{a}_1 \times \boldsymbol{a}_2\Big) \cdot \boldsymbol{a}_3 = 0$$

reads $\boldsymbol{a}_3 \in V$. These show that P_1, P_2, P_3 are all in the same plane defined by

$$p\,x + q\,y + r\,z = 0.$$

Clearly O is in the same plane. Hence we conclude that P_1, P_2, P_3 and O are co-planar. In other words, Scenario (i) occurs. $\square$

- **1.4. #20.** Let

$$\begin{aligned} \boldsymbol{a} &= \Big\langle\, a_1, \;\; a_2, \;\; a_3 \,\Big\rangle, \\ \boldsymbol{b} &= \Big\langle\, b_1, \;\; b_2, \;\; b_3 \,\Big\rangle, \\ \boldsymbol{c} &= \Big\langle\, c_1, \;\; c_2, \;\; c_3 \,\Big\rangle \end{aligned}$$

be three vectors in $\mathbb{R}^3$. Verify

$$\Big(\boldsymbol{a} \times \boldsymbol{b}\Big) \cdot \boldsymbol{c} \;=\; \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

[Strategy] : This is one problem that tests your understanding of the distinction between "definitions" (which do not require proofs) and "assertions" (which require proofs), along with your attentiveness as to what we adopted as definitions in class. In cass, $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ was defined as $\Big(\boldsymbol{a} \times \boldsymbol{b}\Big) \cdot \boldsymbol{c}$. So there is nothing to prove. $\square$

⋆ However: As we have already discussed in class, you may write out

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

concretely, relying on our definition, as follows:

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= \Big(\Big\langle a_1, \; a_2, \; a_3 \Big\rangle \times \Big\langle b_1, \; b_2, \; b_3 \Big\rangle \Big) \cdot \Big\langle c_1, \; c_2, \; c_3 \Big\rangle \\ &= \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \; \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \; \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \\ &\quad \cdot \Big\langle \quad c_1, \qquad c_2, \qquad c_3 \quad \Big\rangle \end{aligned}$$

$$= \Big(a_2 b_3 - a_3 b_2\Big)\, c_1 \;+\; \Big(a_3 b_1 - a_1 b_3\Big)\, c_2 \;+\; \Big(a_1 b_2 - a_2 b_1\Big)\, c_3$$

$$\begin{aligned} = \quad & a_2\, b_3\, c_1 \;-\; a_3\, b_2\, c_1 \;+\; a_3\, b_1\, c_2 \\ & -\; a_1\, b_3\, c_2 \;+\; a_1\, b_2\, c_3 \;-\; a_2\, b_1\, c_3 \end{aligned}$$

$$\begin{aligned} = \quad & a_1\, b_2\, c_3 \;+\; a_2\, b_3\, c_1 \;+\; a_3\, b_1\, c_2 \\ & -\; a_3\, b_2\, c_1 \;-\; a_1\, b_3\, c_2 \;-\; a_2\, b_1\, c_3 \end{aligned}$$

$\Big($re-ordered terms$\Big)$.

In retrospect, we <u>could have defined</u> $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ simply as

(**) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\, b_2\, c_3 + a_2\, b_3\, c_1 + a_3\, b_1\, c_2 - a_3\, b_2\, c_1 - a_1\, b_3\, c_2 - a_2\, b_1\, c_3,$$

which does not directly resort to the cross product and the dot product of vectors. With this new definition, the content of problem #13 bears a more non-trivial look:

- <u>**1.4. #13 paraphrased.**</u> Let

$$P = \Big(a_1,\ a_2,\ a_3\Big), \qquad Q = \Big(b_1,\ b_2,\ b_3\Big), \qquad R = \Big(c_1,\ c_2,\ c_3\Big)$$

be three points in $\mathbb{R}^3$. Suppose $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$, where $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is defined as in (**) above. Then either

[Scenario (i)]: the four points P_1, P_2, P_3 and O are <u>co-planar</u>, or

[Scenario (ii)]: at least two of P_1, P_2, P_3 and O coincide.

If we choose to take this standpoint, then the line

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Big(\big\langle a_1,\ a_2,\ a_3 \big\rangle \times \big\langle b_1,\ b_2,\ b_3 \big\rangle \Big) \cdot \big\langle c_1,\ c_2,\ c_3 \big\rangle,$$

which is exactly the content of problem # 20, is a formula, not a definition. Thus it requires a proof. We have actually already provided the proof in the bottom portion of page 3 to the top portion of page 4. Also, the proof of the above paraphrased version of #13 would resort to the result of problem # 20.

- **1.4. #21.** Let

$$\begin{aligned} \boldsymbol{a} &= \big\langle a_1,\ a_2,\ a_3 \big\rangle, \\ \boldsymbol{b} &= \big\langle b_1,\ b_2,\ b_3 \big\rangle, \\ \boldsymbol{c} &= \big\langle c_1,\ c_2,\ c_3 \big\rangle \end{aligned}$$

be three vectors in $\mathbb{R}^3$. Prove

$$\Big(\boldsymbol{a} \times \boldsymbol{b} \Big) \cdot \boldsymbol{c} = \Big(\boldsymbol{b} \times \boldsymbol{c} \Big) \cdot \boldsymbol{a}.$$

[Strategy] : Proceed as follows:

Proof. Use $(**)$ above.

$$\begin{aligned} \Big(\boldsymbol{b} \times \boldsymbol{c} \Big) \cdot \boldsymbol{a} &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= b_1\, c_2\, a_3 + b_2\, c_3\, a_1 + b_3\, c_1\, a_2 \\ &\quad - b_3\, c_2\, a_1 - b_1\, c_3\, a_2 - b_2\, c_1\, a_3 \\ &= b_2\, c_3\, a_1 + b_3\, c_1\, a_2 + b_1\, c_2\, a_3 \\ &\quad - b_1\, c_3\, a_2 - b_2\, c_1\, a_3 - b_3\, c_2\, a_1 \end{aligned}$$

(re-ordered terms)

$$= \quad a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$
$$- a_2 b_1 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2$$

(re-ordered letters in each term)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \left(\boldsymbol{a} \times \boldsymbol{b} \right) \cdot \boldsymbol{c}. \qquad \square$$

- **1.4. #27.** Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be three vectors in $\mathbb{R}^3$. Prove

$$\left(\boldsymbol{a} \times \boldsymbol{b} \right) \times \boldsymbol{c} = \left(\boldsymbol{a} \cdot \boldsymbol{c} \right) \boldsymbol{b} - \left(\boldsymbol{b} \cdot \boldsymbol{c} \right) \boldsymbol{a}.$$

No calculator or computer .

[Strategy] : Proceed as follows:

Proof. First, suppose at least one of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ equals $\boldsymbol{0}$. Then the both sides of the identity

$$\left(\boldsymbol{a} \times \boldsymbol{b} \right) \times \boldsymbol{c} = \left(\boldsymbol{a} \cdot \boldsymbol{c} \right) \boldsymbol{b} - \left(\boldsymbol{b} \cdot \boldsymbol{c} \right) \boldsymbol{a}$$

clearly equal $\boldsymbol{0}$. Hence the identity holds. In what follows, we assume that none of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ equals $\boldsymbol{0}$. Next, suppose $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{c}$ are mutually proportionate, thus $\boldsymbol{a} \times \boldsymbol{b} = t\, \boldsymbol{c}$. Then, on the one hand, we have $\left(\boldsymbol{a} \times \boldsymbol{b} \right) \times \boldsymbol{c} = \boldsymbol{0}$. On the other hand, clearly $\boldsymbol{a} \cdot \boldsymbol{c} = 0$ and $\boldsymbol{b} \cdot \boldsymbol{c} = 0$, hence

$$\left(\boldsymbol{a} \cdot \boldsymbol{c} \right) \boldsymbol{b} - \left(\boldsymbol{b} \cdot \boldsymbol{c} \right) \boldsymbol{a}$$

equals $\boldsymbol{0}$. Thus the required identity holds.

In what follows, we assume that $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{c}$ are not mutually proportionate. Thus we assume

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} \neq \mathbf{0}.$$

In particular, we assume

$$\boldsymbol{a} \times \boldsymbol{b} \neq \mathbf{0}.$$

Now, write

$$\boldsymbol{a} \times \boldsymbol{b} = \left\langle p, \quad q, \quad r \right\rangle.$$

By virtue of $\boldsymbol{a} \times \boldsymbol{b} \neq \mathbf{0}$, at least one of p, q, r is non-zero. Consider the set

$$\begin{aligned} V &= \left\{ \boldsymbol{v} \;\middle|\; \boldsymbol{v} \text{ is a vector in } \mathbb{R}^3, \quad \left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \boldsymbol{v} = 0 \right\} \\ &= \left\{ \left\langle x, \quad y, \quad z \right\rangle \;\middle|\; p\,x + q\,y + r\,z = 0 \right\}. \end{aligned}$$

We know

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \boldsymbol{a} = 0,$$

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \boldsymbol{b} = 0,$$

and

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \left[\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} \right] = 0,$$

From this it follows that the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c}$ all belong to V:

$$\boldsymbol{a} \in V, \qquad \boldsymbol{b} \in V, \qquad \text{and} \qquad \left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} \in V.$$

In other words, the four points P, Q, R and O in $\mathbb{R}^3$, designated by the components of the vector $\boldsymbol{a}$, the vector $\boldsymbol{b}$, the vector $\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c}$, and the vector $\mathbf{0}$, respectively, reside in the same plane $px + qy + rz = 0$. In short, those four points are co-planar. Taking into account our assumption

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} \neq \mathbf{0},$$

we conclude that there exist $s, t \in \mathbb{R}$ such that

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} = s\boldsymbol{a} + t\boldsymbol{b}.$$

By taking the dot product with $\boldsymbol{c}$ of the both sides, we obtain

$$0 = s\left(\boldsymbol{a} \cdot \boldsymbol{c}\right) + t\left(\boldsymbol{b} \cdot \boldsymbol{c}\right).$$

Thus we may write

$$\begin{cases} s = k\left(\boldsymbol{b} \cdot \boldsymbol{c}\right), & \text{and} \\ t = -k\left(\boldsymbol{a} \cdot \boldsymbol{c}\right), \end{cases}$$

using a suitable $k \in \mathbb{R}$. Substitute these data back into

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} = s\boldsymbol{a} + t\boldsymbol{b},$$

and obtain

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} = k\left[\left(\boldsymbol{b} \cdot \boldsymbol{c}\right)\boldsymbol{a} - \left(\boldsymbol{a} \cdot \boldsymbol{c}\right)\boldsymbol{b}\right].$$

The value of k does not depend on the choice of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, because each of $\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c}$ and $\left(\boldsymbol{b} \cdot \boldsymbol{c}\right)\boldsymbol{a} - \left(\boldsymbol{a} \cdot \boldsymbol{c}\right)\boldsymbol{b}$ depends linearly on each of the three components of each of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. Hence we may decide the value of k by substituting concrete vectors for $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} = k\left[\left(\boldsymbol{b} \cdot \boldsymbol{c}\right)\boldsymbol{a} - \left(\boldsymbol{a} \cdot \boldsymbol{c}\right)\boldsymbol{b}\right].$$

Indeed, substitute $\boldsymbol{a}$ with $\boldsymbol{i}$, $\boldsymbol{b}$ with $\boldsymbol{j}$ and $\boldsymbol{c}$ with $\boldsymbol{i}$ into the above:

$$\left(\boldsymbol{i} \times \boldsymbol{j}\right) \times \boldsymbol{i} \;=\; k\left[\;\left(\boldsymbol{j} \cdot \boldsymbol{i}\right)\boldsymbol{i} \;-\; \left(\boldsymbol{i} \cdot \boldsymbol{i}\right)\boldsymbol{j}\;\right].$$

The left-hand side is simplified as $\boldsymbol{j}$, whereas the right-hand side is simplified as $-k\boldsymbol{j}$. From this we have $k = -1$. To conclude,

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} \;=\; -\left[\;\left(\boldsymbol{b} \cdot \boldsymbol{c}\right)\boldsymbol{a} \;-\; \left(\boldsymbol{a} \cdot \boldsymbol{c}\right)\boldsymbol{b}\;\right].$$

That is,

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} = \left(\boldsymbol{a} \cdot \boldsymbol{c}\right)\boldsymbol{b} \;-\; \left(\boldsymbol{b} \cdot \boldsymbol{c}\right)\boldsymbol{a},$$

which is the required identity. $\square$

- **1.4. #28.** Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be three vectors in $\mathbb{R}^3$. Prove

(1) $$\boldsymbol{a} \cdot \left(\boldsymbol{b} \times \boldsymbol{c}\right) \;=\; \boldsymbol{b} \cdot \left(\boldsymbol{c} \times \boldsymbol{a}\right) \;=\; \boldsymbol{c} \cdot \left(\boldsymbol{a} \times \boldsymbol{b}\right),$$

(2) $$\boldsymbol{a} \cdot \left(\boldsymbol{c} \times \boldsymbol{b}\right) \;=\; \boldsymbol{c} \cdot \left(\boldsymbol{b} \times \boldsymbol{a}\right) \;=\; \boldsymbol{b} \cdot \left(\boldsymbol{a} \times \boldsymbol{c}\right),$$

and

(3) the quanitity in (1) and the quanitity in (2) are negatives of each other.

[Strategy] : Proceed as follows:

Proof. First, since $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$ holds for two vectors $\boldsymbol{v}$ and $\boldsymbol{v}$, the statement (1) and the statement (2) are rewritten as

(1)′ $$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \boldsymbol{c} \;=\; \left(\boldsymbol{b} \times \boldsymbol{c}\right) \cdot \boldsymbol{a} \;=\; \left(\boldsymbol{c} \times \boldsymbol{a}\right) \cdot \boldsymbol{b},$$

and

(2)′ $$\left(\boldsymbol{a} \times \boldsymbol{c}\right) \cdot \boldsymbol{b} \;=\; \left(\boldsymbol{c} \times \boldsymbol{b}\right) \cdot \boldsymbol{a} \;=\; \left(\boldsymbol{b} \times \boldsymbol{a}\right) \cdot \boldsymbol{c},$$

respectively.

The statement (3) is equivalent to the following:

(3)′ The quanitity in (1)′ and the quanitity in (2)′ are negatives of each other.

As for the proof of the statement (1)′, it suffices to prove

$$\Big(\boldsymbol{a} \times \boldsymbol{b}\Big) \cdot \boldsymbol{c} \;=\; \Big(\boldsymbol{b} \times \boldsymbol{c}\Big) \cdot \boldsymbol{a}.$$

This is taken care of in problem #21. Next, the statement (1)′ and the statement (2)′ are clearly equivalent (the only difference between them is that the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are interchanged). Thus it only suffices to prove the statement (3)′ above, namely,

$$\Big(\boldsymbol{a} \times \boldsymbol{b}\Big) \cdot \boldsymbol{c} \;=\; -\Big(\boldsymbol{b} \times \boldsymbol{a}\Big) \cdot \boldsymbol{c}.$$

This is an immediate consequence of $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$, and

$$\Big(-\boldsymbol{u}\Big) \cdot \boldsymbol{v} \;=\; -\boldsymbol{u} \cdot \boldsymbol{v}. \qquad \square$$

- **1.4. #29.** Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ be four vectors in $\mathbb{R}^3$. Prove

$$\Big(\boldsymbol{a} \times \boldsymbol{b}\Big) \cdot \Big(\boldsymbol{c} \times \boldsymbol{d}\Big) \;=\; \Big(\boldsymbol{a} \cdot \boldsymbol{c}\Big)\Big(\boldsymbol{b} \cdot \boldsymbol{d}\Big) - \Big(\boldsymbol{a} \cdot \boldsymbol{d}\Big)\Big(\boldsymbol{b} \cdot \boldsymbol{c}\Big)$$

$$= \begin{vmatrix} \boldsymbol{a} \cdot \boldsymbol{c} & \boldsymbol{a} \cdot \boldsymbol{d} \\ \boldsymbol{b} \cdot \boldsymbol{c} & \boldsymbol{b} \cdot \boldsymbol{d} \end{vmatrix}.$$

[Strategy] : Proceed as follows:

Proof. Use # 21:

$$\Big(\boldsymbol{a} \times \boldsymbol{b}\Big) \cdot \boldsymbol{c} \;=\; \Big(\boldsymbol{c} \times \boldsymbol{a}\Big) \cdot \boldsymbol{b}.$$

In this formula, substitute $\boldsymbol{c}$ with $\boldsymbol{c} \times \boldsymbol{d}$:

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \left(\boldsymbol{c} \times \boldsymbol{d}\right) = \left[\left(\boldsymbol{c} \times \boldsymbol{d}\right) \times \boldsymbol{a} \right] \cdot \boldsymbol{b}$$

$$= \left[\left(\boldsymbol{c} \cdot \boldsymbol{a}\right) \boldsymbol{d} - \left(\boldsymbol{d} \cdot \boldsymbol{a}\right) \boldsymbol{c} \right] \cdot \boldsymbol{b}$$

(used the formula in # 27)

$$= \cdots$$

$$= \cdots$$

- **1.4. #30.** Let

$$\boldsymbol{a} = \left\langle a_1, \ a_2, \ a_3 \right\rangle,$$
$$\boldsymbol{b} = \left\langle b_1, \ b_2, \ b_3 \right\rangle,$$
$$\boldsymbol{c} = \left\langle c_1, \ c_2, \ c_3 \right\rangle$$

be three vectors in $\mathbb{R}^3$. Prove

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \times \boldsymbol{c} + \left(\boldsymbol{b} \times \boldsymbol{c}\right) \times \boldsymbol{a} + \left(\boldsymbol{c} \times \boldsymbol{a}\right) \times \boldsymbol{b} = \boldsymbol{0}$$

(The Jacobi identity). No calculator or computer.

[Strategy] : Proceed as follows:

Proof. Using the components of $\boldsymbol{a}$ and $\boldsymbol{b}$, form the following two 3×3 matrices:

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}.$$

Note that A and B fall into the same pattern: The three main diagonal entries are 0. Also, the other six entries are made of three pairs. Each pair is a mutually negative pair of numbers. Each pair is symmetrically positioned along the main diagonal line.

Now, do the matrix multiplication of A and B $\big($with A left and B right$\big)$:

$$AB \;=\; \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

This is carried out as

$$\begin{bmatrix} 0\cdot 0 + (-a_3)b_3 + a_2(-b_2) & 0(-b_3) + (-a_3)0 + a_2b_1 & 0b_2 + (-a_3)(-b_1) + a_20 \\ a_30 + 0b_3 + (-a_1)(-b_2) & a_3(-b_3) + 0\cdot 0 + (-a_1)b_1 & a_3b_2 + 0(-b_1) + (-a_1)0 \\ (-a_2)0 + a_1b_3 + 0(-b_2) & (-a_2)(-b_3) + a_10 + 0b_1 & (-a_2)b_2 + a_1(-b_1) + 0\cdot 0 \end{bmatrix}.$$

We may simplify each of the nine entries, and obtain

$$\boxed{AB \;=\; \begin{bmatrix} -\,a_2b_2 \,-\, a_3b_3 & a_2b_1 & a_3b_1 \\ a_1b_2 & -\,a_1b_1 \,-\, a_3b_3 & a_3b_2 \\ a_1b_3 & a_2b_3 & -\,a_1b_1 \,-\, a_2b_2 \end{bmatrix}}\,.$$

Switch as and bs, and obtain

$$BA \;=\; \begin{bmatrix} -\,b_2a_2 \,-\, b_3a_3 & b_2a_1 & b_3a_1 \\ b_1a_2 & -\,b_1a_1 \,-\, b_3a_3 & b_3a_2 \\ b_1a_3 & b_2a_3 & -\,b_1a_1 \,-\, b_2a_2 \end{bmatrix}.$$

Rewtite the entries, so as comes first in each term:

$$\boxed{BA \;=\; \begin{bmatrix} -\,a_2b_2 \,-\, a_3b_3 & a_1b_2 & a_1b_3 \\ a_2b_1 & -\,a_1b_1 \,-\, a_3b_3 & a_2b_3 \\ a_3b_1 & a_3b_2 & -\,a_1b_1 \,-\, a_2b_2 \end{bmatrix}}\,.$$

Now, subtract BA from AB $\big($the two highlighted matrices, where the subtraction is the entry-wise subtraction$\big)$,

$$AB - BA$$

$$= \begin{bmatrix} 0 & a_2b_1 - a_1b_2 & a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 & 0 & a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\big(a_1b_2 - a_2b_1\big) & a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 & 0 & -\big(a_2b_3 - a_3b_2\big) \\ -\big(a_3b_1 - a_1b_3\big) & a_2b_3 - a_3b_2 & 0 \end{bmatrix}.$$

Observe that, this matrix holds exactly the same pattern as A and B:

$$\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix},$$

where

$$p_1 = a_2\,b_3 - a_3\,b_2, \qquad p_2 = a_3\,b_1 - a_1\,b_3, \qquad p_3 = a_1\,b_2 - a_2\,b_1,$$

that is,

$$p_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad p_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \qquad p_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Agree that, these three numbers are exactly the three components of $\boldsymbol{a} \times \boldsymbol{b}$:

$$\boldsymbol{a} \times \boldsymbol{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle.$$

Thus we draw the following conclusion: The operation called the cross product of two vectors $\boldsymbol{a} \times \boldsymbol{b}$ is competely absorbed in the following matrix operation:

$$\big[A,\ B\big] = AB - BA.$$

Under the correspondence

$$\boldsymbol{a} = \Big\langle a_1, \ a_2, \ a_3 \Big\rangle \qquad \longleftrightarrow \qquad A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

the two operations $\boldsymbol{a} \times \boldsymbol{b}$ and $[A, \ B]$ are entirely parallel.

Now we are able to prove

$$\Big(\boldsymbol{a} \times \boldsymbol{b}\Big) \times \boldsymbol{c} + \Big(\boldsymbol{b} \times \boldsymbol{c}\Big) \times \boldsymbol{a} + \Big(\boldsymbol{c} \times \boldsymbol{a}\Big) \times \boldsymbol{b} = \mathbf{0}$$

as follows. Due to the above observation, it suffices to prove

$$(*) \qquad \Big[[A, \ B], \ C\Big] + \Big[[B, \ C], \ A\Big] + \Big[[C, \ A], \ B\Big] = O,$$

for arbitrary three matrices $A, \ B, \ C,$ with entries in $\mathbb{R}$. Here O denotes the zero matrix: $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We may proceed the proof of $(*)$ as follows:

$$\begin{aligned} &\Big[[A, \ B], \ C\Big] + \Big[[B, \ C], \ A\Big] + \Big[[C, \ A], \ B\Big] \\ &\quad = \Big[AB - BA, \ C\Big] + \Big[BC - CB, \ A\Big] + \Big[CA - AC, \ B\Big] \\ &\quad = \Big(AB - BA\Big)C - C\Big(AB - BA\Big) \\ &\qquad + \Big(BC - CB\Big)A - A\Big(BC - CB\Big) \\ &\qquad + \Big(CA - AC\Big)B - B\Big(CA - AC\Big) \\ &\quad = ABC - BAC - CAB + CBA \\ &\qquad + BCA - CBA - ABC + ACB \\ &\qquad + CAB - ACB - BCA + BAC = O. \quad \square \end{aligned}$$

⋆ [Note] : In the above proof, we have freely used

$$\big(AB\big)C = A\big(BC\big)$$

(the associativity of matrix multiplication), which is a common knowledge in "Linear Algebra".

- **1.4. #31.** Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ be four vectors in $\mathbb{R}^3$. Prove

$$\big(\boldsymbol{a} \times \boldsymbol{b}\big) \times \big(\boldsymbol{c} \times \boldsymbol{d}\big) = \Big[\boldsymbol{a} \cdot \big(\boldsymbol{c} \times \boldsymbol{d}\big)\Big]\boldsymbol{b} - \Big[\boldsymbol{b} \cdot \big(\boldsymbol{c} \times \boldsymbol{d}\big)\Big]\boldsymbol{a}.$$

No calculator or computer .

[Strategy] : This is immediate. Indeed, realize that, in the formula in #27:

$$\big(\boldsymbol{a} \times \boldsymbol{b}\big) \times \boldsymbol{c} = \big(\boldsymbol{a} \cdot \boldsymbol{c}\big)\boldsymbol{b} - \big(\boldsymbol{b} \cdot \boldsymbol{c}\big)\boldsymbol{a},$$

you may substitute $\boldsymbol{c}$ with $\boldsymbol{c} \times \boldsymbol{d}$. □

- **1.4. #32.** Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be three vectors in $\mathbb{R}^3$. Prove

$$\big(\boldsymbol{a} \times \boldsymbol{b}\big) \cdot \Big[\big(\boldsymbol{b} \times \boldsymbol{c}\big) \times \big(\boldsymbol{c} \times \boldsymbol{a}\big)\Big] = \Big[\boldsymbol{a} \cdot \big(\boldsymbol{b} \times \boldsymbol{c}\big)\Big]^2.$$

No calculator or computer .

[Strategy] : This is immediate. First, use formula in #31:

$$\big(\boldsymbol{a} \times \boldsymbol{b}\big) \times \big(\boldsymbol{c} \times \boldsymbol{d}\big) = \Big[\boldsymbol{a} \cdot \big(\boldsymbol{c} \times \boldsymbol{d}\big)\Big]\boldsymbol{b} - \Big[\boldsymbol{b} \cdot \big(\boldsymbol{c} \times \boldsymbol{d}\big)\Big]\boldsymbol{a}.$$

Thus you may rewrite $\left(\boldsymbol{b} \times \boldsymbol{c}\right) \times \left(\boldsymbol{c} \times \boldsymbol{a}\right)$ as

$$\left[\boldsymbol{b} \cdot \left(\boldsymbol{c} \times \boldsymbol{a}\right)\right]\boldsymbol{c} - \left[\boldsymbol{c} \cdot \left(\boldsymbol{c} \times \boldsymbol{a}\right)\right]\boldsymbol{b},$$

which is simplified as $\left[\boldsymbol{b} \cdot \left(\boldsymbol{c} \times \boldsymbol{a}\right)\right]\boldsymbol{c}$, by virtue of $\boldsymbol{c} \cdot \left(\boldsymbol{c} \times \boldsymbol{a}\right) = 0$. In short,

$$\left(\boldsymbol{b} \times \boldsymbol{c}\right) \times \left(\boldsymbol{c} \times \boldsymbol{a}\right) = \left[\boldsymbol{b} \cdot \left(\boldsymbol{c} \times \boldsymbol{a}\right)\right]\boldsymbol{c}.$$

Furthermore, by the result of problem #28, we may rewrite $\boldsymbol{b} \cdot \left(\boldsymbol{c} \times \boldsymbol{a}\right)$ as $\boldsymbol{a} \cdot \left(\boldsymbol{b} \times \boldsymbol{c}\right)$. Thus

$$\left(\boldsymbol{b} \times \boldsymbol{c}\right) \times \left(\boldsymbol{c} \times \boldsymbol{a}\right) = \left[\boldsymbol{a} \cdot \left(\boldsymbol{b} \times \boldsymbol{c}\right)\right]\boldsymbol{c}.$$

Do the cross product with the vector $\left(\boldsymbol{a} \times \boldsymbol{b}\right)$ from the left, to the both sides of the above identity:

$$\begin{aligned}\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \left[\left(\boldsymbol{b} \times \boldsymbol{c}\right) \times \left(\boldsymbol{c} \times \boldsymbol{a}\right)\right] \\ &= \left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \left[\boldsymbol{a} \cdot \left(\boldsymbol{b} \times \boldsymbol{c}\right)\right]\boldsymbol{c} \\ &= \left[\boldsymbol{a} \cdot \left(\boldsymbol{b} \times \boldsymbol{c}\right)\right]\left[\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \boldsymbol{c}\right].\end{aligned}$$

Here, use

$$\left(\boldsymbol{a} \times \boldsymbol{b}\right) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \left(\boldsymbol{b} \times \boldsymbol{c}\right),$$

which is a combination of the formula $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$ and the formula in # 28, part (1). □

- **1.4. #33.** Identical to #29 (do not have to do it twice).