Math 223 VECTOR CALCULUS

PROGRESS CHECK – II (Continued – Part 2)

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- **“Cram session” – Rudiments on matrix algebra, part 2.**

- **The inverse matrix A^{-1}.**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let us create a new matrix out of this:

$$B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Where does this come from? First, this matrix B depends on the original matrix A. Indeed, the entries of B are still written using a, b, c and d. Is B just a randomly permuted version of the original matrix A, with negative signs capriciously attached to two randomly chosen entries (out of the four)? Or are B and A mutually in a special, mathematically meaningful, relation? Why don't we just multiply the two matrices A and B and see what happens?

$$\begin{aligned} AB &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & -ab + ba \\ cd - dc & -cb + da \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}. \end{aligned}$$

Note that, from this, you cannot conclude that BA equals $\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$ also. All we know so far is that AB equals $\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$, but we have no knowledge about BA yet. This is because in general AB and BA are not equal. So, why don't we do BA next, and compare it with AB?

$$BA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} da - bc & db - bd \\ - ca + ac & - cb + ad \end{bmatrix}$$

$$= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}.$$

So, BA ended up being equal to AB. Both AB and BA are equal to

$$\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}.$$

This outcome matrix is very special. Indeed, it is of form

$$\begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}$$

where $\delta = ad - bc$. This quantity $\delta = ad - bc$ actually looks familar. Yes, it is the determinant of A:

$$\delta = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det A.$$

So far so good. But our discussion does not end here. Let μ be a scalar $\bigl(\mu \in \mathbb{R}\bigr)$. Also, let $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be a matrix. From now on, we allow ourselves to write

$$\mu \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

to mean $\begin{bmatrix} \mu p & \mu q \\ \mu r & \mu s \end{bmatrix}$. This is a universally adopted convention, and it is frequently convenient. As a special case, we can write

$$\delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

to mean $\begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}$. Now, our result above is summarized as

$$AB \;=\; \delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad BA \;=\; \delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $\delta \;=\; \det A$. Now, in each of the two identities, $\underline{\text{we want to divide the both sides by } \delta}$. We are allowed to do so, as far as we assume $\delta \neq 0$. So, under the assumption $\delta \neq 0$,

$$(*) \qquad A \left(\frac{1}{\delta} B \right) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \left(\frac{1}{\delta} B \right) A \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In the above, we slided the fraction $\dfrac{1}{\delta}$ into the B-part. $\bigl($We could have slided it into the A-part, but we didn't. This choice was purposely made, and the reason will become self-evident a few lines later.$\bigr)$ As a result, the above two identities became typographically "ugly", or, disproportionate. This can be repaired, once we adopt a new notation $\boxed{\;A^{-1}\;}$ for $\dfrac{1}{\delta}\, B$.

If you keep track of what δ and B were, you can write it as

$$\boxed{\;A^{-1} \;=\; \frac{1}{ad \,-\, bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\;}\;.$$

With this new notation A^{-1}, the above pair of identities $(*)$ reads

$$(**) \qquad \boxed{A\,A^{-1} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad A^{-1}\,A \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \;.$$

- To recap, we have just $\underline{\text{ proved }}$ the existence of a matrix A^{-1}, that satisfies

$$A\,A^{-1} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad A^{-1}\,A \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$\underline{\text{ provided } \det A \neq 0\ }$.

- Basically, what we have done is, for a given 2×2 matrix A, we have found another 2×2 matrix A^{-1} such that multiplying A and A^{-1} $\big($in either order$\big)$ will yield the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. You still wonder if there is enough reason that warrants us to regard and treat the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as something special. In other words, explanation as to how the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ occupies a special position in the geography of matrices is pending. Yes the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ indeed occupies a special position in the geography of matrices. Hard and solid facts:

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;&=\; \begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} \\ &=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \end{aligned}$$

Also,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 \cdot a + 0 \cdot c & 1 \cdot b + 0 \cdot d \\ 0 \cdot a + 1 \cdot c & 0 \cdot b + 1 \cdot d \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- These are noteworthy. For an arbitrary 2×2 matrix A, multiplying A with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (in either order) will yield A. This reminds us of the basic rule of arithmetic in the real number system:

$$a \cdot 1 = a, \qquad \text{and} \qquad 1 \cdot a = a.$$

$\underline{\text{We have just found the “matrix substitute” of the scalar (= real number) 1.}}$ In the matrix world, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ plays exactly the same role as the real number 1 does in the scalar (real number) world. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ deserves a special notation and a name:

Definition (The identity matrix I).

We denote

$$\boxed{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

and call it $\underline{\text{the identity matrix}}$.

What we have calculated and confirmed earlier (in the bottom of the last page to the top of this page) is summarized as

$$\boxed{AI = A, \qquad \text{and} \qquad IA = A}\ .$$

Moreover, what we have worked out in $(**)$ two pages ago is paraphrased as follows:

Inversion Formula (2 × 2). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Assume

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\,d - b\,c \neq 0.$$

Define

$$\boxed{A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \; .$$

Then

$$\boxed{A\,A^{-1} = I, \qquad \text{and} \qquad A^{-1}\,A = I} \; .$$

- Morally, the operation that takes A into A^{-1} is the matrix substitute of the operation taking a scalar $(=$ real number$)$ a into its reciprocal $1/a$. The condition $\det A \neq 0$ substitutes the condition $a \neq 0$.

- If $\det A = 0$ for a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we do not define A^{-1}. Such a matrix A is said to be non-invertible. In this case, no 2×2 matrix multiplied to A will yield the identity matrix I.

- A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det A \neq 0$ is said to be invertible. In this case, no matrix B other than A^{-1} satisfies $A\,B = I$. Also, no matrix C other than A^{-1} satisfies $C\,A = I$.

Example 1. The matrix $A = \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}$ is invertible. Indeed,

$$\det A = \begin{vmatrix} 4 & 3 \\ 7 & 5 \end{vmatrix} = -1.$$

Hence A^{-1} exists. A^{-1} is calculated as

$$\begin{aligned} A^{-1} &= \frac{1}{-1} \begin{bmatrix} 5 & -3 \\ -7 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix}. \end{aligned}$$

Example 2. The matrix $A = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ is non-invertible. Indeed,

$$\det A = \begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0.$$

Hence A^{-1} does not exist.

- Now, regardless of whether a given matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible or not, it still makes sense to isolate the part $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ in Inversion Formula highlighted above, and give it a name.

Definition (Adjoint matrix; 2 × 2). For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define

$$\boxed{\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \, .$$

Call it $\underline{\text{ the adjoint matrix }}$ of A. Note that, $\operatorname{adj} A$ makes sense regardless of whether A is invertible or not. This is unlike A^{-1} which is only defined for an invertible A.

- With this definition, what we have observed earlier has the following paraphrase:

Adjoint Formula **(2×2).** Let A be a 2×2 matrix. Then

$$\boxed{A\left(\operatorname{adj} A\right) = \delta\, I, \qquad \text{and} \qquad \left(\operatorname{adj} A\right) A = \delta\, I} \, ,$$

where $\delta = \det A$. (This is true regardless of whether A is invertible or not.)

How to form the Inverse of A using $\operatorname{adj} A$ **(for 2×2)?**

Let A be a 2×2 matrix. If $\delta = \det A \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{\delta} \operatorname{adj} A} \, .$$

Example 3. For $A = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}$, its adjoint matrix is

$$\operatorname{adj} A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}.$$

In this case,

$$\begin{aligned} A\left(\operatorname{adj} A\right) &= \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0\, I, \end{aligned}$$

and also

$$\begin{aligned} \left(\operatorname{adj} A\right) A &= \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0\, I. \end{aligned}$$

On the other hand, for the same A as above, we have

$$\det A = \left(-1\right) \cdot 4 - 2 \cdot \left(-2\right) = 0.$$

These two results are consistent with Adjoint Formula above. Note that, in this example, A^{-1} does not exist.

Example 4. For $A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$, its adjoint matrix is

$$\operatorname{adj} A = \begin{bmatrix} 6 & -2 \\ -4 & 1 \end{bmatrix}.$$

In this case,

$$\begin{aligned} A\Big(\operatorname{adj} A\Big) &= \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2\,I, \end{aligned}$$

and also

$$\begin{aligned} \Big(\operatorname{adj} A\Big) A &= \begin{bmatrix} 6 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2\,I. \end{aligned}$$

On the other hand, for the same A as above, we have

$$\det A = 1 \cdot 6 - 2 \cdot 4 = -2.$$

These two results are consistent with Adjoint Formula above. Note that, in this example, A^{-1} exists and

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} \operatorname{adj} A \\ &= \frac{1}{-2} \begin{bmatrix} 6 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -\frac{1}{2} \end{bmatrix}. \end{aligned}$$

- **How to generalize the above picture for 3 × 3 matrices?**

Our natural question is how to generalize the above picture to 3×3 matrices. More specifically:

(i) Which 3×3 matrix is most appropriate to be called the identity matrix?

(ii) What is the right formation of the adjoint matrix $\operatorname{adj} A$ for a given 3×3 matrix A, so that the same statement as Adjoint Formula above will become true?

(iii) Would it still be appropriate to call the adjoint matrix $\operatorname{adj} A$ multiplied by $\dfrac{1}{\det A}$ as A^{-1}? Would $A\,A^{-1} = I$, and $A^{-1} A = I$ be true?

- These three questions all have affirmative answers. First, answer to question (i):

Definition (The identity matrix I).

We denote

$$\boxed{I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

and call it $\underline{\text{the } (3 \times 3) \text{ identity matrix}}$.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ be an arbitrary 3×3 matrix. Then

$$\boxed{A\,I = A, \qquad \text{and} \qquad I\,A = A}$$

The next is the most non-trivial part.

Definition (Adjoint matrix; 3 × 3).

For $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, define

$$\boxed{\operatorname{adj} A = \begin{bmatrix} + \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & - \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & - \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{bmatrix}} .$$

Call it <u>the adjoint matrix</u> of A.

- The above formation of $\operatorname{adj} A$ looks somewhat complicated. However, this is not a randomly chosen formation. Indeed, with this formation of $\operatorname{adj} A$, the following 'Adjoint Formula', whose formulation is exactly identical to the 2×2 case which we have covered (three pages ago), becomes true:

<u>Adjoint Formula</u> (3 × 3). Let A be a 3×3 matrix. Then

$$\boxed{A \left(\operatorname{adj} A \right) = \delta\, I, \qquad \text{and} \qquad \left(\operatorname{adj} A \right) A = \delta\, I} ,$$

where $\delta = \det A$.

This answers the question (ii) above. Finally, as for the question (iii):

Inversion Formula **(3×3).** Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Assume

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$$

Define

$$\boxed{A^{-1} = \frac{1}{\det A} \begin{bmatrix} +\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & +\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & +\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ +\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & +\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{bmatrix}}\ .$$

Then

$$\boxed{A\,A^{-1} = I, \qquad \text{and} \qquad A^{-1}\,A = I}\ .$$

- With the adjoint matrix $\operatorname{adj} A$ of A, the above has the following simple paraphrase:

How to form the Inverse of A using $\operatorname{adj} A$ (for 3×3)?

Let A be a 3×3 matrix. If $\delta = \det A \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{\delta} \operatorname{adj} A}\ .$$

- This formula for A^{-1} looks exactly the same as in the 2×2 case.

Example 5. For $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -5 & 2 \\ 6 & -6 & 2 \end{bmatrix}$, its adjointmatrix $\operatorname{adj} A$ is

$$\operatorname{adj} A = \begin{bmatrix} +\begin{vmatrix} -5 & 2 \\ -6 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 2 \\ -6 & 2 \end{vmatrix} & +\begin{vmatrix} -3 & 2 \\ -5 & 2 \end{vmatrix} \\ \\ -\begin{vmatrix} 3 & 2 \\ 6 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \\ +\begin{vmatrix} 3 & -5 \\ 6 & -6 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 6 & -6 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 3 & -5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -6 & 4 \\ 6 & -10 & 4 \\ 12 & -12 & 4 \end{bmatrix}.$$

In this case,

$$A\Big(\operatorname{adj} A\Big) = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -5 & 2 \\ 6 & -6 & 2 \end{bmatrix} \begin{bmatrix} 2 & -6 & 4 \\ 6 & -10 & 4 \\ 12 & -12 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\,I,$$

and also

$$\Big(\operatorname{adj} A\Big)A = \begin{bmatrix} 2 & -6 & 4 \\ 6 & -10 & 4 \\ 12 & -12 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ 3 & -5 & 2 \\ 6 & -6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\,I.$$

On the other hand, for the same A as above, we have

$$\begin{aligned}
\det A &= \begin{vmatrix} 1 & -3 & 2 \\ 3 & -5 & 2 \\ 6 & -6 & 2 \end{vmatrix} \\
&= \quad 1 \cdot \big(-5\big) \cdot 2 + \big(-3\big) \cdot 2 \cdot 6 + 2 \cdot 3 \cdot \big(-6\big) \\
&\quad - 1 \cdot 2 \cdot \big(-6\big) - \big(-3\big) \cdot 3 \cdot 2 - 2 \cdot \big(-5\big) \cdot 6 \\
&= \quad 8.
\end{aligned}$$

These two results are consistent with Adjoint Formula above. Note that, in this example, A^{-1} exists and

$$\begin{aligned}
A^{-1} &= \frac{1}{\det A} \operatorname{adj} A \\
&= \frac{1}{8} \begin{bmatrix} 2 & -6 & 4 \\ 6 & -10 & 4 \\ 12 & -12 & 4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} \\ \frac{3}{4} & -\frac{5}{4} & \frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.
\end{aligned}$$

- Another concrete example of how to form the adjoint matrix adj A for a given concrete 3×3 matrix A is given in “Quiz – IV”, problem [I].