Math 290 ELEMENTARY LINEAR ALGEBRA

MIDTERM EXAM – IB

March 4 (Tue), 2008

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[I] (8pts) (1) The system of linear equations

$$\begin{aligned} x &+ y &= 1, \\ 2x &+ 2y &= 2 \end{aligned}$$

☐ is consistent, the (general) solution is $\left(x, \; y\right) = \left(\underline{\quad\quad}, \; \underline{\quad\quad}\right)$.

☐ is inconsistent.

(Check one).

(2) True or false:

" A system of homogeneous linear equations consisting of three equations, and three variables x, y, z, always has a non-trivial solution. "

☐ True ☐ False (Check one).

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[II] (8pts) The matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ☐ is in reduced row echelon form.
- ☐ is in row echelon form, not in reduced row echelon form.
- ☐ is not in row echelon form.

$\left(\underline{\text{ Check one }}\right)$.

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[III] (8pts)

(1)
$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(2)
$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{1} \cdot \boxed{1} & \boxed{1} \cdot \boxed{3} & \boxed{} \cdot \boxed{} \\ \boxed{3} \cdot \boxed{1} & \boxed{} \cdot \boxed{} & \boxed{} \cdot \boxed{} \\ \boxed{} \cdot \boxed{} & \boxed{} \cdot \boxed{} & \boxed{} \cdot \boxed{} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

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[IV] (12pts) Let $A = \begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 1 & -4 \end{bmatrix}$.

(1) $AB = \begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -4 \end{bmatrix}$

$$= \begin{bmatrix} \boxed{1} \cdot \boxed{4} + \boxed{4} \cdot \boxed{1} & \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{} \\ \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{} & \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(2) $BA = \begin{bmatrix} 4 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \boxed{4} \cdot \boxed{1} + \boxed{1} \cdot \boxed{4} & \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{} \\ \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{} & \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

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[V] (12pts)

(1) For $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 9 & -3 \\ 6 & 0 \end{bmatrix}$,

☐ $AB = BA$ ☐ $AB \neq BA$ (Check one).

Justification. $B = \square\, A$ (Fill in a concrete number).

(2) For $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 19 & 9 \\ 863 & 1375 \end{bmatrix}$,

☐ $AB = BA$ ☐ $AB \neq BA$ (Check one).

Justification. $A = \square\, I$ (Fill in a concrete number),

where I is the ______________ matrix.

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[VI] (12pts)

(1) $$\mathrm{Tr}\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 1 & 0 & 4 \end{bmatrix} = \boxed{} + \boxed{} + \boxed{} = \underline{}\ .$$

(2) True or false : " For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$,

we always have $\mathrm{Tr}\left(AB\right) = \mathrm{Tr}\left(BA\right)$. "

☐ True ☐ False (Check one).

Justification. From

$$AB = \begin{bmatrix} ap + br & * \\ * & cq + ds \end{bmatrix}, \quad BA = \begin{bmatrix} ap + cq & * \\ * & br + ds \end{bmatrix},$$

$$\mathrm{Tr}\left(AB\right) = \boxed{} + \boxed{} + \boxed{} + \boxed{}\ ,$$

$\|$?

$$\mathrm{Tr}\left(BA\right) = \boxed{} + \boxed{} + \boxed{} + \boxed{}\ .$$

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[VII] (10pts) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(1) $$A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}.$$

(2) $$A + A^T = \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

☐ $A + A^T$ is symmetric. ☐ $A + A^T$ is <u>not</u> symmetric.

(<u>Check one</u>).

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[VIII] (15pts)

(1) True or false : " $\Big(AB\Big)C = A\Big(BC\Big)$ always holds true, as far as A, B, and C are in right sizes so that AB and BC are defined. "

☐ True ☐ False (Check one).

(2) True or false :

" Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $A^{50} = I$. Then A is non-singular, and $A^{-1} = A^{49}$. "

☐ True ☐ False (Check one).

(3) $$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{200} = \begin{bmatrix} \boxed{1} & \boxed{} \\ \boxed{0} & \boxed{1} \end{bmatrix}$$

(no justification necessary).

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[IX] (15pts) Let

$$A \;=\; \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad B \;=\; \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix},$$

$$C \;=\; \begin{bmatrix} \cos\Big(\theta+\phi\Big) & -\sin\Big(\theta+\phi\Big) \\ \sin\Big(\theta+\phi\Big) & \cos\Big(\theta+\phi\Big) \end{bmatrix}.$$

(1) Write out AB <u>in terms of $\sin\theta$, $\cos\theta$, $\sin\phi$ and $\cos\phi$</u>.

AB

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \square\cdot\square - \square\cdot\square & -\Big(\square\cdot\square + \square\cdot\square\Big) \\ \square\cdot\square + \square\cdot\square & \square\cdot\square - \square\cdot\square \end{bmatrix}.$$

(2) Read off the four entries of $C = AB$, and write out the consequence.

$$\cos\Big(\theta+\phi\Big) \;=\; \square\cdot\square - \square\cdot\square,$$

$$\sin\Big(\theta+\phi\Big) \;=\; \square\cdot\square + \square\cdot\square.$$

(3) <u>True or false</u> : " For the above A and B, $AB = BA$ holds. "

☐ True ☐ False (<u>Check one</u>).

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[X] (Extra 15pts) Let $\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a^2 + b^2 + c^2 = 1$.

(1) $\boldsymbol{u}^T \boldsymbol{u} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boxed{}^2 + \boxed{}^2 + \boxed{}^2 = \boxed{}$.

(2) $\boldsymbol{u}\,\boldsymbol{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} \boxed{a^2} & \boxed{} & \boxed{} \\ \boxed{ab} & \boxed{b^2} & \boxed{} \\ \boxed{ac} & \boxed{bc} & \boxed{c^2} \end{bmatrix}$.

(3) $A = I - 2\boldsymbol{u}\boldsymbol{u}^T$

$$= \begin{bmatrix} \boxed{a^2+b^2+c^2} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{a^2+b^2+c^2} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{a^2+b^2+c^2} \end{bmatrix} - \begin{bmatrix} \boxed{2a^2} & \boxed{} & \boxed{} \\ \boxed{2ab} & \boxed{2b^2} & \boxed{} \\ \boxed{2ac} & \boxed{2bc} & \boxed{2c^2} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{-a^2+b^2+c^2} & \boxed{} & \boxed{} \\ \boxed{-2ab} & \boxed{a^2-b^2+c^2} & \boxed{} \\ \boxed{-2ac} & \boxed{-2bc} & \boxed{a^2+b^2-c^2} \end{bmatrix}.$$

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([X] continued)

(4) For $\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$, write out A in (3):

$$A = \begin{bmatrix} \frac{\square}{3} & \frac{\square}{3} & \frac{\square}{3} \\ \frac{\square}{3} & \frac{\square}{3} & \frac{\square}{3} \\ \frac{\square}{3} & \frac{\square}{3} & \frac{\square}{3} \end{bmatrix}.$$

(5) Let A be as in (3). Find A^2. Use the result of (1).

$$A^2 = A\,A = \left(I - 2\,\boldsymbol{u}\,\boldsymbol{u}^T\right)\left(I - 2\,\boldsymbol{u}\,\boldsymbol{u}^T\right)$$

$$= I\,I - I\left(2\,\boldsymbol{u}\,\boldsymbol{u}^T\right) - \left(2\,\boldsymbol{u}\,\boldsymbol{u}^T\right)I + \left(2\,\boldsymbol{u}\,\boldsymbol{u}^T\right)\left(2\,\boldsymbol{u}\,\boldsymbol{u}^T\right)$$

$$= I - 2\left(\boldsymbol{u}\,\boldsymbol{u}^T\right) - 2\left(\boldsymbol{u}\,\boldsymbol{u}^T\right) + 4\left(\boldsymbol{u}\left(\boldsymbol{u}^T\,\boldsymbol{u}\right)\boldsymbol{u}^T\right)$$

$$= \square - 4\left(\square\,\square^T\right) + 4\left(\square\,(\square)\,\square^T\right)$$

$$= \square - 4\left(\square\,\square^T\right) + 4\left(\square\,\square^T\right)$$

$$= \underline{\qquad\qquad}\,.$$