Math 290 ELEMENTARY LINEAR ALGEBRA

SOLUTION FOR MIDTERM EXAM – IIA (04/10)

April 17 (Thu), 2008

Instructor: Yasuyuki Kachi

Line #: 74449.

[I] (15pts) (1) For $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$, its determinant $\Delta = \det A$ is defined as

$$\Delta = aqz - ary - bpz + brx + cpy - cqx.$$

We may use this definition to calculate the determinant for

$$A = \begin{bmatrix} 1 & 4 & -4 \\ 1 & 1 & -2 \\ 1 & 2 & -3 \end{bmatrix} :$$

$$\begin{aligned} \Delta &= 1 \cdot 1 \cdot \big(-3\big) - 1 \cdot \big(-2\big) \cdot 2 - 4 \cdot 1 \cdot \big(-3\big) \\ &\quad + 4 \cdot \big(-2\big) \cdot 1 + \big(-4\big) \cdot 1 \cdot 2 - \big(-4\big) \cdot 1 \cdot 1 \\ &= \big(-3\big) - \big(-4\big) - \big(-12\big) + \big(-8\big) + \big(-8\big) - \big(-4\big) \\ &= -3 + 4 + 12 - 8 - 8 + 4 \\ &= 1. \end{aligned}$$

Since we found $\Delta \neq 0$, A is <u>non-singular</u>.

(2) For $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$, its inverse A^{-1} is found as

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} +\begin{vmatrix} q & r \\ y & z \end{vmatrix} & -\begin{vmatrix} b & c \\ y & z \end{vmatrix} & +\begin{vmatrix} b & c \\ q & r \end{vmatrix} \\ -\begin{vmatrix} p & r \\ x & z \end{vmatrix} & +\begin{vmatrix} a & c \\ x & z \end{vmatrix} & -\begin{vmatrix} a & c \\ p & r \end{vmatrix} \\ +\begin{vmatrix} p & q \\ x & y \end{vmatrix} & -\begin{vmatrix} a & b \\ x & y \end{vmatrix} & +\begin{vmatrix} a & b \\ p & q \end{vmatrix} \end{bmatrix},$$

provided $\Delta \neq 0$. For $A = \begin{bmatrix} 1 & 4 & -4 \\ 1 & 1 & -2 \\ 1 & 2 & -3 \end{bmatrix}$ (the same matrix as (1)), we found the value of Δ to be 1, which is not equal to 0. Thus

$$A^{-1} = \frac{1}{1}\begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 4 & -4 \\ 2 & -3 \end{vmatrix} & +\begin{vmatrix} 4 & -4 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix} & +\begin{vmatrix} 1 & -4 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & -4 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & -4 \\ 1 & 1 & -2 \\ 1 & 2 & -3 \end{bmatrix}.$$

[Note]: $A^{-1} = A$ in this example.

[II] (15pts) Let A, B, C be square matrices in the same size. Let I denote the identity matrix having the same size as A, B, C.

(1) True or false : " If $AB = I$, then $BA = I$. "

The answer is "true". See Formula 21 in Pg. Ch. V, page 41.

(2) True or false : " If $AB = I$ and $AC = I$, then $B = C$. Based on this, we may define A^{-1} as the unique B such that $AB = I$. "

The answer is "true". See Formula 22 in Pg. Ch. V, page 42. Or, we may prove it using (1), as follows: If $AB = I$ and $AC = I$, then by (1), $BA = I$, and hence $C = IC = BAC = BI = B$.

(3) True or false : " If $AB = C$, and C is invertible, then A, B are both invertible. Under the same assumption, we have $C^{-1} = B^{-1}A^{-1}$. "

The answer is "true". The first part of the statement is verified as follows. Assume $AB = C$ is invertible. Then C^{-1} makes sense and $CC^{-1} = I$. The same thing to say,

$$\left(AB\right)\left(C^{-1}\right) = I.$$

By the associativity law,

$$A\left(BC^{-1}\right) = I.$$

From this it follows that A is invertible, and $BC^{-1} = A^{-1}$. Note that A^{-1} is invertible. From what we have just proved, it follows that B is invertible. Multiply B^{-1} from the left to the identity $BC^{-1} = A^{-1}$ to obtain $C^{-1} = B^{-1}A^{-1}$.

[III] (15pts) (1) The number of inversions for the permutation

$$\left(3,\ 4,\ 6,\ 5,\ 7,\ 2,\ 8,\ 1\right)$$

is calculated as

$$\boxed{2} + \boxed{2} + \boxed{3} + \boxed{2}$$

contribution from $\boxed{3}$, contribution from $\boxed{4}$, contribution from $\boxed{6}$, contribution from $\boxed{5}$

$$+ \boxed{2} + \boxed{1} + \boxed{1} + \boxed{0} = \boxed{13}.$$

contribution from $\boxed{7}$, contribution from $\boxed{2}$, contribution from $\boxed{8}$, contribution from $\boxed{1}$, total

(2) Hence, the permutation in (1) is odd .

(3) In the formula defining the determiant

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{bmatrix},$$

$- a_{13}\, a_{24}\, a_{36}\, a_{45}\, a_{57}\, a_{62}\, a_{78}\, a_{81}$ appears as a term.

[IV] (15pts) (1) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= 1 \cdot \begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - a \cdot \begin{vmatrix} 1 & 1 \\ b^2 & c^2 \end{vmatrix} + a^2 \cdot \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} \quad \Big(\text{co-factor expansion}\Big)$$

$$= 1 \cdot \left(\boxed{b\,c^2} - \boxed{c\,b^2} \right)$$

$$- a \cdot \left(\boxed{c^2} - \boxed{b^2} \right)$$

$$+ a^2 \cdot \left(\boxed{c} - \boxed{b} \right)$$

$$= bc \cdot \left(\boxed{c} - \boxed{b} \right) - a\left(c+b\right)\left(\boxed{c} - \boxed{b} \right)$$

$$+ a^2 \left(\boxed{c} - \boxed{b} \right)$$

$$= \left[bc - \left(c+b\right)a + a^2 \right] \left(\boxed{c} - \boxed{b} \right)$$

$$= \left(\boxed{b} - \boxed{a} \right) \left(\boxed{c} - \boxed{a} \right) \left(\boxed{c} - \boxed{b} \right).$$

(2) using the result of (1),

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 6 \\ 4 & 9 & 36 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 6 \\ 2^2 & 3^2 & 6^2 \end{vmatrix} = \Big(3 - 2\Big)\Big(6 - 2\Big)\Big(6 - 3\Big)$$

$$= 1 \cdot 4 \cdot 3$$

$$= 12.$$

[V] (15pts) $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$.

(1) $$A\,A^T = \begin{bmatrix} \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{a} & \boxed{b} & \boxed{c} \\ \boxed{a^2} & \boxed{b^2} & \boxed{c^2} \end{bmatrix} \begin{bmatrix} \boxed{1} & \boxed{a} & \boxed{a^2} \\ \boxed{1} & \boxed{b} & \boxed{b^2} \\ \boxed{1} & \boxed{c} & \boxed{c^2} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{1 + 1 + 1} & \boxed{a + b + c} & \boxed{a^2 + b^2 + c^2} \\ \boxed{a + b + c} & \boxed{a^2 + b^2 + c^2} & \boxed{a^3 + b^3 + c^3} \\ \boxed{a^2 + b^2 + c^2} & \boxed{a^3 + b^3 + c^3} & \boxed{a^4 + b^4 + c^4} \end{bmatrix} = B.$$

(2) To find $\det B$, use the result of problem [IV]:

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \Big(b - a\Big)\Big(c - a\Big)\Big(c - b\Big).$$

Accordingly,

$$\det B = \det\Big(A\ A^T\Big)$$

$$= \Big(\det A\Big)\Big(\det A^T\Big)$$

$$= \Big(\det A\Big)^2 = \Big(b - a\Big)^2\Big(c - a\Big)^2\Big(c - b\Big)^2.$$

(3) To find $\begin{vmatrix} 3 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{vmatrix}$, realize

$$\begin{vmatrix} 3 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{vmatrix} = \begin{vmatrix} 1 + 1 + 1 & (-1) + 1 + 2 & 1 + 1 + 4 \\ (-1) + 1 + 2 & 1 + 1 + 4 & (-1) + 1 + 8 \\ 1 + 1 + 4 & (-1) + 1 + 8 & 1 + 1 + 16 \end{vmatrix}$$

$$= \begin{vmatrix} 1 + 1 + 1 & (-1) + 1 + 2 & (-1)^2 + 1^2 + 2^2 \\ (-1) + 1 + 2 & (-1)^2 + 1^2 + 2^2 & (-1)^3 + 1^3 + 2^3 \\ (-1)^2 + 1^2 + 2^2 & (-1)^3 + 1^3 + 2^3 & (-1)^4 + 1^4 + 2^4 \end{vmatrix}.$$

The above determinant is exactly the determinant in (2), with $a = -1$, $b = 1$, and $c = 2$. Hence the above determinant equals

$$\Big(1 - (-1)\Big)^2 \Big(2 - (-1)\Big)^2 \Big(2 - 1\Big)^2 = 2^2 \cdot 3^2 \cdot 1^2$$

$$= 36.$$

[VI] (15pts) (1) The definition of an orthogonal matrix is as follows:

" A square matrix A is said to be orthogonal, when $\boxed{A^T A = I}$ holds. "

Or, alternatively to the above,

" A square matrix A is said to be orthogonal, when $\boxed{A A^T = I}$ holds. "

(2) For an arbitrary orthogonal matrix A,

$$\det A = 1, \text{ or } -1.$$

See Pg. Ch. VIII, page 33, Formula 20.

(3) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal, and $ad - bc = -1$, then

$$A^2 = I.$$

Proof for (3): A is written as $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, with $a^2 + b^2 = 1$. Accordingly,

$$A^2 = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{bmatrix} \boxed{a^2 + b^2} & \boxed{0} \\ \boxed{0} & \boxed{a^2 + b^2} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} \end{bmatrix} \qquad \left[\text{by } a^2 + b^2 = 1. \right].$$

[VII] (15pts) Let a, b, c, d be real numbers, such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Let A be defined as

$$\boxed{A = \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc + ad) & 2(bd - ac) \\ 2(bc - ad) & a^2 - b^2 + c^2 - d^2 & 2(cd + ab) \\ 2(bd + ac) & 2(cd - ab) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}}.$$

(1) A is the general form of a 3×3 <u>special orthogonal</u> matrix. Hence

$$A\,A^T = I, \qquad \det A = 1.$$

(2) For $(a, b, c, d) = (1, 0, 0, 0)$, the matrix A becomes

$$A = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{1} \end{bmatrix}.$$

(3) For $(a, b, c, d) = \left(\frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$, the matrix A becomes

$$A = \begin{bmatrix} \dfrac{\boxed{-1}}{2} & \dfrac{\boxed{\sqrt{3}}}{2} & 0 \\ \dfrac{\boxed{-\sqrt{3}}}{2} & \dfrac{\boxed{-1}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(4) To find a quartuple $\left(a,\ b,\ c,\ d\right)$ such that $a^2 + b^2 + c^2 + d^2 = 1$ and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, look at the three main diagonal entries:

$$\begin{aligned} a^2 + b^2 - c^2 - d^2 &= 0, \\ a^2 - b^2 + c^2 - d^2 &= 0, \\ a^2 - b^2 - c^2 + d^2 &= 0. \end{aligned}$$

Add the first two equations, and we obtain $2\,a^2 - 2\,d^2 = 0.$ Hence

$$a^2 = d^2.$$

Similarly, add the first and the last equations, and we obtain

$$a^2 = c^2.$$

Similarly, add the middle and the last equations, and we obtain

$$a^2 = b^2.$$

In sum, we obtain $a^2 = b^2 = c^2 = d^2.$ Taking into account

$$a^2 + b^2 + c^2 + d^2 = 1,$$

we conclude

$$a^2 = b^2 = c^2 = d^2 = \frac{1}{4}.$$

Thus, each of a, b, c and d must equal either

$$\frac{1}{2} \qquad \text{or} \qquad -\frac{1}{2}.$$

Substituting

$$a = b = c = d = \frac{1}{2}$$

into A yields the given matrix. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

$\star$ $a = b = c = d = -\dfrac{1}{2}$ is an equally valid answer.

[VIII] (Extra 15pts) For $B = \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix}$,

(1) $B^2 = \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix} \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix}$

$$= \begin{bmatrix} -c^2 - d^2 & bc & bd \\ bc & -b^2 - d^2 & cd \\ bd & cd & -b^2 - c^2 \end{bmatrix},$$

$$B^3 = B^2 B$$

$$= \begin{bmatrix} -c^2 - d^2 & bc & bd \\ bc & -b^2 - d^2 & cd \\ bd & cd & -b^2 - c^2 \end{bmatrix} \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -(b^2+c^2+d^2)d & (b^2+c^2+d^2)c \\ (b^2+c^2+d^2)d & 0 & -(b^2+c^2+d^2)b \\ -(b^2+c^2+d^2)c & (b^2+c^2+d^2)b & 0 \end{bmatrix}$$

$$= \quad - \left(\boxed{b^2 + c^2 + d^2} \right) B.$$

(2) For B as above, assume $b^2 + c^2 + d^2 > 0$. Then

$$\sum_{n=0}^{\infty} \frac{1}{n!} B^n = I + B + \frac{1}{2!} B^2 + \frac{1}{3!} B^3 + \frac{1}{4!} B^4 + \frac{1}{5!} B^5 + \frac{1}{6!} B^6 + \frac{1}{7!} B^7 + \frac{1}{8!} B^8 + \cdots$$

$$\begin{aligned} = \quad & I + B + \frac{1}{2!} B^2 \\ & - \frac{1}{3!} \left(\boxed{b^2 + c^2 + d^2} \right) B \\ & - \frac{1}{4!} \left(\boxed{b^2 + c^2 + d^2} \right) B^2 \\ & + \frac{1}{5!} \left(\boxed{b^2 + c^2 + d^2} \right)^2 B \\ & + \frac{1}{6!} \left(\boxed{b^2 + c^2 + d^2} \right)^2 B^2 \\ & - \frac{1}{7!} \left(\boxed{b^2 + c^2 + d^2} \right)^3 B \\ & - \frac{1}{8!} \left(\boxed{b^2 + c^2 + d^2} \right)^3 B^2 + \cdots \end{aligned}$$

$$\begin{aligned} = \quad & I + \frac{\sin \sqrt{\boxed{b^2 + c^2 + d^2}}}{\sqrt{\boxed{b^2 + c^2 + d^2}}} B \\ & + \frac{1 - \cos \sqrt{\boxed{b^2 + c^2 + d^2}}}{\boxed{b^2 + c^2 + d^2}} B^2 . \end{aligned}$$

(3) Let $t > 0$. In the result of (2), substitute $\boxed{B}$ with $\boxed{t\,B}$:

$$\sum_{n=0}^{\infty} \frac{1}{n\,!} \left(t\,B \right)^n$$

$$= \quad I \; + \; \frac{\sin\left(t \sqrt{\boxed{b^2 + c^2 + d^2}} \right)}{t \cdot \sqrt{\boxed{b^2 + c^2 + d^2}}} \left(t\,B \right)$$

$$+ \; \frac{1 - \cos\left(t \sqrt{\boxed{b^2 + c^2 + d^2}} \right)}{t^2 \cdot \left(\boxed{b^2 + c^2 + d^2} \right)} \left(t\,B \right)^2$$

$$= \quad I \; + \; \frac{\sin\left(t \sqrt{\boxed{b^2 + c^2 + d^2}} \right)}{\sqrt{\boxed{b^2 + c^2 + d^2}}} \, B$$

$$+ \; \frac{1 - \cos\left(t \sqrt{\boxed{b^2 + c^2 + d^2}} \right)}{\boxed{b^2 + c^2 + d^2}} \, B^2 \, .$$

(4) <u>Equate</u> the result of (3) with $I \; + \; \left(2\,a \right) B \; + \; 2\,B^2$:

(a) $\sin\left(t \sqrt{\boxed{b^2 + c^2 + d^2}} \right) \quad = \quad 2\,a \sqrt{b^2 + c^2 + d^2}$,

(b) $\cos\left(t \sqrt{\boxed{b^2 + c^2 + d^2}} \right) \quad = \quad -\,2 \left(b^2 + c^2 + d^2 \right) \; + \; 1$.

Do " $(\mathrm{a})^2 + (\mathrm{b})^2$ " side by side:

$$1 = \left[2\,a \sqrt{\,b^2 + c^2 + d^2\,} \right]^2 + \left[-\,2\left(b^2 + c^2 + d^2\right) + 1 \right]^2$$

$$= 4\,a^2 \left(b^2+c^2+d^2\right) + 4\left(b^2+c^2+d^2\right)^2 - 4\left(b^2+c^2+d^2\right) + 1.$$

This is simplified as

$$0 = 4\left(b^2 + c^2 + d^2\right)\left(a^2 + \left(b^2 + c^2 + d^2\right) - 1\right).$$

Since $b^2 + c^2 + d^2 \neq 0$ by assumption, we have

$$a^2 + b^2 + c^2 + d^2 = 1.$$

• The above argument proves the following: For given b, c, d with

$$0 < b^2 + c^2 + d^2 \leq 1,$$

there is a suitable parameter $t > 0$, such that

$$\sum_{n=0}^{\infty} \frac{1}{n\,!}\left(t\,B\right)^n = I + \left(2\,a\right)B + 2\,B^2,$$

where $a^2 + b^2 + c^2 + d^2 = 1.$

⋆ We may calculate $I + \left(2\,a\right)B + 2\,B^2$ as

$$\begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc + ad) & 2(bd - ac) \\ 2(bc - ad) & a^2 - b^2 + c^2 - d^2 & 2(cd + ab) \\ 2(bd + ac) & 2(cd - ab) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}.$$

Note that this matrix is exactly the matrix A in problem [VII].

Accordingly, we may paraphrase the above statement as follows. For four real numbers a, b, c, d satisfying

$$a^2 + b^2 + c^2 + d^2 = 1,$$

there is a suitable real number t such that

$$\sum_{n=0}^{\infty} \frac{1}{n\,!} \begin{bmatrix} 0 & td & -tc \\ -td & 0 & tb \\ tc & -tb & 0 \end{bmatrix}^n$$

$$= \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2\big(bc + ad\big) & 2\big(bd - ac\big) \\ 2\big(bc - ad\big) & a^2 - b^2 + c^2 - d^2 & 2\big(cd + ab\big) \\ 2\big(bd + ac\big) & 2\big(cd - ab\big) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}.$$