Math 290 ELEMENTARY LINEAR ALGEBRA

PRACTICE EXAM – FINAL

May 8 (Thu), 2008

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Line #: 74449 / 82588 .

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[I] (20pts) In $\mathbb{C}$, let X, Y and Z be 2×2 matrices, defined as

$$X = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}.$$

True or false : " An arbitrary 2×2 matrix of form

$$\begin{bmatrix} a + \sqrt{-1}\, b & -c + \sqrt{-1}\, d \\ c + \sqrt{-1}\, d & a - \sqrt{-1}\, b \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$, is an $\mathbb{R}$-linear combination of I, X, Y, and Z. "

☐ True ☐ False (Check one).

Justification:

$$\begin{bmatrix} a + \sqrt{-1}\, b & -c + \sqrt{-1}\, d \\ c + \sqrt{-1}\, d & a - \sqrt{-1}\, b \end{bmatrix}$$

$$= \quad a\, I \quad +$$

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[II] (20pts) Let A and B be two square matrices in the same size having entries in a field k.

True or false : " If A and B are both skew-symmetric, and moreover A and B commute, then AB is symmetric. "

☐ True ☐ False (Check one).

Justification: $\left(AB\right)^T$ = ____________________

= ____________________

= ____________________

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[III] (80pts) Let $b, c, d \in \mathbb{R}$ be such that $b^2 \neq c^2$, and $d \neq 0$.

(1)

$$\begin{bmatrix} 0 & -b & -c & -d \\ b & 0 & d & -c \\ c & -d & 0 & b \\ d & c & -b & 0 \end{bmatrix} \begin{bmatrix} 0 & -c & -b & -d \\ c & 0 & -d & b \\ b & d & 0 & -c \\ d & -b & c & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2bc - d^2 & d\left(\boxed{}\right) & d\left(\boxed{}\right) & \boxed{} \\ d(b-c) & \boxed{}^{2} & -\left(\boxed{}\right) & -d\left(\boxed{}\right) \\ d(b-c) & -(b^2+c^2) & \boxed{}^{2} & -d\left(\boxed{}\right) \\ -b^2+c^2 & -d(b+c) & -d(b+c) & 2bc - d^2 \end{bmatrix}$$

$$= A.$$

(2) For the above A,

$$\mathrm{Tr}\, A = \underline{\qquad\qquad}.$$

(3) True or false : " The above A is symmetric . "

☐ True ☐ False (Check one).

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([III] continued)

(4) Let A be in the previous page $\big($page 3$\big)$. A has exactly two distinct eigenvalues λ_1 and λ_2 $\Big(\lambda_1 \neq \lambda_2\Big)$, one of which is

$$\boxed{\lambda_1 \; = \; b^2 + c^2 + d^2} \; .$$

(4a) For this λ_1, find

$$\text{rank}\,\Big(\lambda_1\, I \; - \; A\Big) \quad = \quad \underline{\hspace{3cm}} \; .$$

(4b) Find the dimension of the eigenspace V_{λ_1} of A associated with $\lambda = \lambda_1$:

$$\dim_{\mathbb{C}} V_{\lambda_1} \quad = \quad \underline{\hspace{3cm}} \; .$$

(4c) Knowing that the characteristic polynomial of A is of form

$$\chi_A\big(\lambda\big) \;\; = \;\; \lambda^4 \; + \; \Big(\boxed{\quad * \quad}\Big)\; \lambda^2 \; + \; \Big(b^2 + c^2 + d^2\Big)^4,$$

find the other eigenvalue λ_2 of A :

$$\lambda_2 \quad = \quad \underline{\hspace{5cm}} \; .$$

Give a complete justification in the next two pages .

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([III] continued)

Justification for (4a): Remember $\boxed{\lambda_1 \;=\; b^2 \,+\, c^2 \,+\, d^2}$. Write out

$\lambda_1\, I \,-\, A$:

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}.$$

Substitute $b \,=\, 0$ $\Longrightarrow$

$$\begin{bmatrix} \boxed{c^2 + 2d^2} & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \boxed{c^2 + 2d^2} \end{bmatrix}.$$

Most of the 3×3 minors have either two identical rows or two identical columns.

The only potentially non-trivial 3×3 minor:

$$\begin{vmatrix} c^2 \,+\, 2\,d^2 & cd & -\, c^2 \\ cd & c^2 & cd \\ -\, c^2 & cd & c^2 \,+\, 2\,d^2 \end{vmatrix} \;=\; \underline{\hspace{6cm}}$$

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([III] continued)

Justification for (4b): Use the formula

$$\dim_{\mathbb{C}} \operatorname{Im}\Big(\lambda_1 I - A\Big) + \dim_{\mathbb{C}} \operatorname{Ker}\Big(\lambda_1 I - A\Big) = 4.$$

Here,

$\dim_{\mathbb{C}} \operatorname{Im}\Big(\lambda_1 I - A\Big)$ is the ____________ of $\lambda_1 I - A$.

$\dim_{\mathbb{C}} \operatorname{Ker}\Big(\lambda_1 I - A\Big)$ is the dimension of the ____________ of A,

associated with the eigenvalue $\lambda = \lambda_1$.

Justification for (4c):

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[IV] (80pts) (1) Let $a, \ b, \ c, \ d \in \mathbb{C}$. Find

$$\begin{vmatrix} a & b+c+d & b^2+c^2+d^2 & b^3+c^3+d^3 \\ b+c+d & b^2+c^2+d^2 & b^3+c^3+d^3 & b^4+c^4+d^4 \\ b^2+c^2+d^2 & b^3+c^3+d^3 & b^4+c^4+d^4 & b^5+c^5+d^5 \\ b^3+c^3+d^3 & b^4+c^4+d^4 & b^5+c^5+d^5 & b^6+c^6+d^6 \end{vmatrix}$$

$=$

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Work:

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([IV] continued)

(2) Let $\theta \in \mathbb{R}$. Use the result of (1) to find

$$\begin{vmatrix} 0 & 1 + 2\cos\theta & 1 + 2\cos\left(2\theta\right) & 1 + 2\cos\left(3\theta\right) \\ 1 + 2\cos\theta & 1 + 2\cos\left(2\theta\right) & 1 + 2\cos\left(3\theta\right) & 1 + 2\cos\left(4\theta\right) \\ 1 + 2\cos\left(2\theta\right) & 1 + 2\cos\left(3\theta\right) & 1 + 2\cos\left(4\theta\right) & 1 + 2\cos\left(5\theta\right) \\ 1 + 2\cos\left(3\theta\right) & 1 + 2\cos\left(4\theta\right) & 1 + 2\cos\left(5\theta\right) & 1 + 2\cos\left(6\theta\right) \end{vmatrix}$$

=

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Write your answer in terms of $\sin\theta$ and $\sin\left(2\theta\right)$, or alternatively, in terms of $\cos\theta$ $\left(\text{and } \sin\theta\right)$.

Work:

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[V] (Extra 40pts) Let $a_i, \; b_i \in \mathbb{C}$. Let

$$\Delta_{12} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \qquad \Delta_{13} = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \qquad \Delta_{23} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

Let $\lambda_1, \; \lambda_2, \; \mu_1,$ and $\mu_2 \in \mathbb{C}$ be such that the following polynomial identities hold:

$$a_1 x^2 + a_2 x + a_3 = \left(x - \lambda_1 \right) \left(x - \lambda_2 \right),$$

and

$$b_1 x^2 + b_2 x + b_3 = \left(x - \mu_1 \right) \left(x - \mu_2 \right).$$

Prove that

$$\det \begin{bmatrix} \Delta_{12} & \Delta_{13} \\ \Delta_{13} & \Delta_{23} \end{bmatrix} = 0$$

if and only if either one of λ_1 or λ_2 coincides with either one of μ_1 or μ_2.

Proof:

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([V] continued)