Math 290 ELEMENTARY LINEAR ALGEBRA

EXTRA CREDIT HOMEWORK – II

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Due on April 24 (Thu), 2008

Instructor: Yasuyuki Kachi

Line #: 74449 / 82588.

[I] (30pts) Let

$$A = \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2\big(bc + ad\big) & 2\big(bd - ac\big) \\ 2\big(bc - ad\big) & a^2 - b^2 + c^2 - d^2 & 2\big(cd + ab\big) \\ 2\big(bd + ac\big) & 2\big(cd - ab\big) & a^2 - b^2 - c^2 + d^2 \end{bmatrix},$$

where a, b, c and d are real numbers. Here, we do not assume anything about the value $a^2 + b^2 + c^2 + d^2$ (unlike Quiz – XIV; problem [II]).

(1)
$$\begin{aligned} \operatorname{Tr} A = \ & a^2 + b^2 - c^2 - d^2 \\ & + a^2 - b^2 + c^2 - d^2 \\ & + a^2 - b^2 - c^2 + d^2 \\ = \ & \\ & \underline{\qquad\qquad\qquad\qquad\qquad\qquad} . \end{aligned}$$

(2) Let $X = \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix}$. Note that X is skew-symmetric.

Calculate

$$\Big(a^2 + b^2 + c^2 + d^2\Big) I + 2\,a\,X + 2\,X^2.$$

Show work in the next page.

$$\left(a^2 + b^2 + c^2 + d^2\right) I + 2\,a\,X + 2\,X^2$$

$$= \begin{bmatrix} a^2 + b^2 + c^2 + d^2 & 0 & 0 \\ 0 & a^2 + b^2 + c^2 + d^2 & 0 \\ 0 & 0 & a^2 + b^2 + c^2 + d^2 \end{bmatrix}$$

$$+ \begin{bmatrix} 2a \boxed{\qquad} & 2a \boxed{\qquad} & 2a \boxed{\qquad} \\ 2a \boxed{\qquad} & 2a \boxed{\qquad} & 2a \boxed{\qquad} \\ 2a \boxed{\qquad} & 2a \boxed{\qquad} & 2a \boxed{\qquad} \end{bmatrix}$$

$$+ \begin{bmatrix} 2\left(\boxed{\qquad\qquad}\right) & 2\left(\boxed{\qquad\qquad}\right) & 2\left(\boxed{\qquad\qquad}\right) \\ 2\left(\boxed{\qquad\qquad}\right) & 2\left(\boxed{\qquad\qquad}\right) & 2\left(\boxed{\qquad\qquad}\right) \\ 2\left(\boxed{\qquad\qquad}\right) & 2\left(\boxed{\qquad\qquad}\right) & 2\left(\boxed{\qquad\qquad}\right) \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2\left(\boxed{\qquad\qquad}\right) & 2\left(\boxed{\qquad\qquad}\right) \\ 2\left(\boxed{\qquad\qquad}\right) & a^2 - b^2 + c^2 - d^2 & 2\left(\boxed{\qquad\qquad}\right) \\ 2\left(\boxed{\qquad\qquad}\right) & 2\left(\boxed{\qquad\qquad}\right) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}.$$

Verify that this indeed equals A in page 1 .

(3) In the identity

$$A = \left(a^2 + b^2 + c^2 + d^2\right) I + 2\,a\,X + 2\,X^2,$$

apply the transpose operator to the both sides:

$$A^T = \left(a^2 + b^2 + c^2 + d^2\right) \boxed{\qquad}^{T} + 2\,a \boxed{\qquad}^{T} + 2\left(\boxed{\qquad}\right)^{T}.$$

Here, note $I^T = I$ and $X^T = -X$. (Remember that X is skew-symmetric.) Moreover,

$$\left(X^2\right)^T = \left(X^T\right)^2 = \left(-X\right)^2 = X^2.$$

Use these information and write A^T as a linear combination of I, X and X^2:

$$A^T = \left(a^2 + b^2 + c^2 + d^2\right)\boxed{} - 2\,a\boxed{} + 2\boxed{}\ .$$

(4) Now, simplify $A\,A^T$:

$$A\,A^T = \left(\left(a^2 + b^2 + c^2 + d^2\right) I + 2a\,B + 2B^2\right)$$

$$\cdot \left(\left(a^2 + b^2 + c^2 + d^2\right) I - 2a\,B + 2B^2\right)$$

$$= \left(\boxed{}^2 + \boxed{}^2 + \boxed{}^2 + \boxed{}^2\right)^2 I$$

$$+ 4\left(\boxed{b}^2 + \boxed{c}^2 + \boxed{d}^2\right) B^2 + 4B^4$$

$$= \left(\boxed{}^2 + \boxed{}^2 + \boxed{}^2 + \boxed{}^2\right)^2 I$$

$$+ 4\,B\underbrace{\left[\left(\boxed{b}^2 + \boxed{c}^2 + \boxed{d}^2\right) B + B^3\right]}_{\chi_B(B) \,=\, \boxed{}}$$

[by Cayley–Hamilton's theorem]

$$= \left(\boxed{}^2 + \boxed{}^2 + \boxed{}^2 + \boxed{}^2\right)^2 I\ .$$

In short, we have obtained

$$A\,A^T$$

$$= \begin{bmatrix} \left(a^2 + b^2 + c^2 + d^2\right)^2 & 0 & 0 \\ 0 & \left(a^2 + b^2 + c^2 + d^2\right)^2 & 0 \\ 0 & 0 & \left(a^2 + b^2 + c^2 + d^2\right)^2 \end{bmatrix}.$$

(5) Keep in mind the formula $\det\left(A\,A^T\right) = \left(\det A\right)^2$, to find

$$\left(\det A\right)^2 = \left(a^2 + b^2 + c^2 + d^2\right)^{\boxed{}}.$$

(6) From (5),

$$\square \quad \det A = \left(a^2 + b^2 + c^2 + d^2\right)^3$$

$$\square \quad \det A = -\left(a^2 + b^2 + c^2 + d^2\right)^3$$

(Check one).

Reason for (6): Assume $a \neq 0$, and $b = c = d = 0$. Then

$$A = \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc + ad) & 2(bd - ac) \\ 2(bc - ad) & a^2 - b^2 + c^2 - d^2 & 2(cd + ab) \\ 2(bd + ac) & 2(cd - ab) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}$$

becomes

$$A = \begin{bmatrix} \boxed{a^2} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{a^2} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{a^2} \end{bmatrix},$$

which clearly has a positive determinant $\boxed{}^{\,6}$.

[II] (70pts) Let A be as in problem [I] $\big($page 1$\big)$. Let

$$B = \begin{bmatrix} p^2 + q^2 - r^2 - s^2 & 2\big(qr + ps\big) & 2\big(qs - pr\big) \\ 2\big(qr - ps\big) & p^2 - q^2 + r^2 - s^2 & 2\big(rs + pq\big) \\ 2\big(qs + pr\big) & 2\big(rs - pq\big) & p^2 - q^2 - r^2 + s^2 \end{bmatrix},$$

where p, q, r and s are real numbers. In particular, by the result of [I],

$$\det B = \Big(p^2 + q^2 + r^2 + s^2\Big)^3.$$

Is it possible to express AB in a form

$$C = \begin{bmatrix} x^2 + y^2 - z^2 - w^2 & 2\big(yz + xw\big) & 2\big(yw - xz\big) \\ 2\big(yz - xw\big) & x^2 - y^2 + z^2 - w^2 & 2\big(zw + xy\big) \\ 2\big(yw + xz\big) & 2\big(zw - xy\big) & x^2 - y^2 - z^2 + w^2 \end{bmatrix},$$

for suitable real numbers x, y, z and w? If the answer is yes, then write out each of x, y, z and w concretely in terms of a, b, c, d, p, q, r and s.

⋆ [Instruction]: Assume that AB is indeed written as C as above. So $C = AB$. Then, by the product formula, $\det C = \Big(\det A\Big)\Big(\det B\Big)$. By the result of [I], this reads

$$\Big(x^2 + y^2 + z^2 + w^2\Big)^3 = \Big(a^2 + b^2 + c^2 + d^2\Big)^3 \Big(p^2 + q^2 + r^2 + s^2\Big)^3.$$

Since a, b, c, d, p, q, r, s, x, y, z and w are all real numbers, the above implies

$$x^2 + y^2 + z^2 + w^2 = \Big(a^2 + b^2 + c^2 + d^2\Big)\Big(p^2 + q^2 + r^2 + s^2\Big).$$

For notational convenience, write

$$\begin{aligned}\operatorname{Nm} A &= a^2 + b^2 + c^2 + d^2,\\ \operatorname{Nm} B &= p^2 + q^2 + r^2 + s^2,\\ \operatorname{Nm} C &= x^2 + y^2 + z^2 + w^2.\end{aligned}$$

$\big(\operatorname{Nm}$ stands for 'norm'.$\big)$ Thus

$$(*) \qquad \operatorname{Nm} C = \Big(\operatorname{Nm} A\Big)\Big(\operatorname{Nm} B\Big).$$

Then x^2 is written using $\operatorname{Tr} C$ and $\operatorname{Nm} C$. Indeed, from

$$\operatorname{Tr} C = 3x^2 - y^2 - z^2 - w^2, \qquad \text{and} \qquad \operatorname{Nm} C = x^2 + y^2 + z^2 + w^2,$$

agree that x^2 is found as

$$x^2 = \frac{\Big(\operatorname{Tr} C\Big) + \Big(\operatorname{Nm} C\Big)}{4}.$$

Taking into account $C = AB$, and $(*)$, this reads

$$x^2 = \frac{\Big(\operatorname{Tr}(AB)\Big) + \Big(\operatorname{Nm} A\Big)\Big(\operatorname{Nm} B\Big)}{4}.$$

To find $T = \operatorname{Tr}(AB)$, you have to first calculate the three main diagonal entries of AB, call them K, L and M. Then add them up: $T = K + L + M$. Use the chart in page 7. Fill in the answer for T in the chart in page 8. Meanwhile, $N = \Big(\operatorname{Nm} A\Big)\Big(\operatorname{Nm} B\Big)$ is easily found. Fill in the answer in the chart in page 8. So $x^2 = \Big(T + N\Big)/4$ is found (use the chart in page 8). Finally, look at the matrix C, and agree

$$x^2 + y^2 = \frac{L + M}{2}, \quad x^2 + z^2 = \frac{K + M}{2}, \quad x^2 + w^2 = \frac{K + L}{2}.$$

Fill in the answers for these three quantities in the chart in page 9. Then use your answer for x^2 (in the chart in page 8) to give answers for y^2, z^2 and w^2 in the chart in page 10.

$K = (1,1)$-entry of AB, $L = (2,2)$-entry of AB, $M = (3,3)$-entry of AB.

$$
\begin{aligned}
\frac{\frac{K}{L}}{M} = {} & \frac{\frac{\square}{\square}}{\square}\, a^2p^2 + \frac{\frac{\square}{\square}}{\square}\, a^2q^2 + \frac{\frac{\square}{\square}}{\square}\, a^2r^2 + \frac{\frac{\square}{\square}}{\square}\, a^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, b^2p^2 + \frac{\frac{\square}{\square}}{\square}\, b^2q^2 + \frac{\frac{\square}{\square}}{\square}\, b^2r^2 + \frac{\frac{\square}{\square}}{\square}\, b^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, c^2p^2 + \frac{\frac{\square}{\square}}{\square}\, c^2q^2 + \frac{\frac{\square}{\square}}{\square}\, c^2r^2 + \frac{\frac{\square}{\square}}{\square}\, c^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, d^2p^2 + \frac{\frac{\square}{\square}}{\square}\, d^2q^2 + \frac{\frac{\square}{\square}}{\square}\, d^2r^2 + \frac{\frac{\square}{\square}}{\square}\, d^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, bcqr + \frac{\frac{\square}{\square}}{\square}\, bcps + \frac{\frac{\square}{\square}}{\square}\, adqr + \frac{\frac{\square}{\square}}{\square}\, adps \\
& + \frac{\frac{\square}{\square}}{\square}\, bdqs + \frac{\frac{\square}{\square}}{\square}\, bdpr + \frac{\frac{\square}{\square}}{\square}\, acqs + \frac{\frac{\square}{\square}}{\square}\, acpr \\
& + \frac{\frac{\square}{\square}}{\square}\, cdrs + \frac{\frac{\square}{\square}}{\square}\, cdpq + \frac{\frac{\square}{\square}}{\square}\, abrs + \frac{\frac{\square}{\square}}{\square}\, abpq.
\end{aligned}
$$

Fill in either $\boxed{+1}$, $\boxed{-1}$, $\boxed{+4}$, $\boxed{-4}$ or $\boxed{0}$.

- To find $\boxed{x^2}$, set

$$T \;=\; K + L + M, \qquad N \;=\; \left(a^2 + b^2 + c^2 + d^2\right)\left(p^2 + q^2 + r^2 + s^2\right).$$

$$\begin{aligned}
\frac{\frac{T}{N}}{x^2} \;=\; & \frac{\frac{\square}{\square}}{\square}\, a^2p^2 + \frac{\frac{\square}{\square}}{\square}\, a^2q^2 + \frac{\frac{\square}{\square}}{\square}\, a^2r^2 + \frac{\frac{\square}{\square}}{\square}\, a^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, b^2p^2 + \frac{\frac{\square}{\square}}{\square}\, b^2q^2 + \frac{\frac{\square}{\square}}{\square}\, b^2r^2 + \frac{\frac{\square}{\square}}{\square}\, b^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, c^2p^2 + \frac{\frac{\square}{\square}}{\square}\, c^2q^2 + \frac{\frac{\square}{\square}}{\square}\, c^2r^2 + \frac{\frac{\square}{\square}}{\square}\, c^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, d^2p^2 + \frac{\frac{\square}{\square}}{\square}\, d^2q^2 + \frac{\frac{\square}{\square}}{\square}\, d^2r^2 + \frac{\frac{\square}{\square}}{\square}\, d^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, bcqr + \frac{\frac{\square}{\square}}{\square}\, bcps + \frac{\frac{\square}{\square}}{\square}\, adqr + \frac{\frac{\square}{\square}}{\square}\, adps \\
& + \frac{\frac{\square}{\square}}{\square}\, bdqs + \frac{\frac{\square}{\square}}{\square}\, bdpr + \frac{\frac{\square}{\square}}{\square}\, acqs + \frac{\frac{\square}{\square}}{\square}\, acpr \\
& + \frac{\frac{\square}{\square}}{\square}\, cdrs + \frac{\frac{\square}{\square}}{\square}\, cdpq + \frac{\frac{\square}{\square}}{\square}\, abrs + \frac{\frac{\square}{\square}}{\square}\, abpq.
\end{aligned}$$

First calculate T from the previous chart. Meanwhile, calculate N independently.

x^2 is found as $\boxed{x^2 \;=\; \frac{T + N}{4}}$.

- To find each of $\boxed{y^2}$, $\boxed{z^2}$, $\boxed{w^2}$, first do

$$\frac{\frac{\left(\frac{L+M}{2}\right)}{\left(\frac{K+M}{2}\right)}}{\left(\frac{K+L}{2}\right)} = \frac{\frac{\Box}{\Box}}{\Box} a^2p^2 + \frac{\frac{\Box}{\Box}}{\Box} a^2q^2 + \frac{\frac{\Box}{\Box}}{\Box} a^2r^2 + \frac{\frac{\Box}{\Box}}{\Box} a^2s^2$$

$$+ \frac{\frac{\Box}{\Box}}{\Box} b^2p^2 + \frac{\frac{\Box}{\Box}}{\Box} b^2q^2 + \frac{\frac{\Box}{\Box}}{\Box} b^2r^2 + \frac{\frac{\Box}{\Box}}{\Box} b^2s^2$$

$$+ \frac{\frac{\Box}{\Box}}{\Box} c^2p^2 + \frac{\frac{\Box}{\Box}}{\Box} c^2q^2 + \frac{\frac{\Box}{\Box}}{\Box} c^2r^2 + \frac{\frac{\Box}{\Box}}{\Box} c^2s^2$$

$$+ \frac{\frac{\Box}{\Box}}{\Box} d^2p^2 + \frac{\frac{\Box}{\Box}}{\Box} d^2q^2 + \frac{\frac{\Box}{\Box}}{\Box} d^2r^2 + \frac{\frac{\Box}{\Box}}{\Box} d^2s^2$$

$$+ \frac{\frac{\Box}{\Box}}{\Box} bcqr + \frac{\frac{\Box}{\Box}}{\Box} bcps + \frac{\frac{\Box}{\Box}}{\Box} adqr + \frac{\frac{\Box}{\Box}}{\Box} adps$$

$$+ \frac{\frac{\Box}{\Box}}{\Box} bdqs + \frac{\frac{\Box}{\Box}}{\Box} bdpr + \frac{\frac{\Box}{\Box}}{\Box} acqs + \frac{\frac{\Box}{\Box}}{\Box} acpr$$

$$+ \frac{\frac{\Box}{\Box}}{\Box} cdrs + \frac{\frac{\Box}{\Box}}{\Box} cdpq + \frac{\frac{\Box}{\Box}}{\Box} abrs + \frac{\frac{\Box}{\Box}}{\Box} abpq.$$

For this, simply use chart in page 7.

- <u>Now we may find</u>

$$
\begin{aligned}
\frac{\frac{y^2}{z^2}}{w^2} = {} & \frac{\frac{\square}{\square}}{\square}\, a^2p^2 + \frac{\frac{\square}{\square}}{\square}\, a^2q^2 + \frac{\frac{\square}{\square}}{\square}\, a^2r^2 + \frac{\frac{\square}{\square}}{\square}\, a^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, b^2p^2 + \frac{\frac{\square}{\square}}{\square}\, b^2q^2 + \frac{\frac{\square}{\square}}{\square}\, b^2r^2 + \frac{\frac{\square}{\square}}{\square}\, b^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, c^2p^2 + \frac{\frac{\square}{\square}}{\square}\, c^2q^2 + \frac{\frac{\square}{\square}}{\square}\, c^2r^2 + \frac{\frac{\square}{\square}}{\square}\, c^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, d^2p^2 + \frac{\frac{\square}{\square}}{\square}\, d^2q^2 + \frac{\frac{\square}{\square}}{\square}\, d^2r^2 + \frac{\frac{\square}{\square}}{\square}\, d^2s^2 \\
& + \frac{\frac{\square}{\square}}{\square}\, bcqr + \frac{\frac{\square}{\square}}{\square}\, bcps + \frac{\frac{\square}{\square}}{\square}\, adqr + \frac{\frac{\square}{\square}}{\square}\, adps \\
& + \frac{\frac{\square}{\square}}{\square}\, bdqs + \frac{\frac{\square}{\square}}{\square}\, bdpr + \frac{\frac{\square}{\square}}{\square}\, acqs + \frac{\frac{\square}{\square}}{\square}\, acpr \\
& + \frac{\frac{\square}{\square}}{\square}\, cdrs + \frac{\frac{\square}{\square}}{\square}\, cdpq + \frac{\frac{\square}{\square}}{\square}\, abrs + \frac{\frac{\square}{\square}}{\square}\, abpq.
\end{aligned}
$$

<u>For this simply use</u>

$$\boxed{y^2 = x^2 - \frac{L+M}{2}} \;,\quad \boxed{z^2 = x^2 - \frac{K+M}{2}} \;,\quad \boxed{w^2 = x^2 - \frac{K+L}{2}} \;.$$

<u>Use x^2 from chart in page 8</u>.

• Each of x^2 (from page 8), y^2 (from page 10), z^2 (from page 10), and w^2 (from page 10), is a complete square.

$$x^2 = \Big(\underline{\quad a\,p \quad - \quad b\,q \quad - \quad c\,r \quad - \quad d\,s \quad} \Big)^2 ,$$

$$y^2 = \Big(\underline{\qquad\qquad\qquad\qquad\qquad\qquad} \Big)^2 ,$$

$$z^2 = \Big(\underline{\qquad\qquad\qquad\qquad\qquad\qquad} \Big)^2 ,$$

$$w^2 = \Big(\underline{\qquad\qquad\qquad\qquad\qquad\qquad} \Big)^2 .$$

From these, conclude

$$x = \underline{\qquad\qquad\qquad\qquad\qquad\qquad} ,$$

$$y = \underline{\qquad\qquad\qquad\qquad\qquad\qquad} ,$$

$$z = \underline{\qquad\qquad\qquad\qquad\qquad\qquad} ,$$

$$w = \underline{\qquad\qquad\qquad\qquad\qquad\qquad} .$$