Math 290 ELEMENTARY LINEAR ALGEBRA
PROGRESS CHECK – II

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- **Solving a System of Linear Equations Using Augmented Matrix.**

We already know how to solve a system of linear equations, by going through "(Gaussian) elimination". The idea is simple — Alter the given system by adding or subtracting two equations in the system, and also multiplying a non-zero constant to an equation in the system. Most importantly, in doing so, we must make sure that the solution set is preserved. We repeat this process and eventually reduce the original system of equations to a certain fairly trivial system of equations, from which we know the solution set. For example, we have considered the system

$$\begin{aligned} x + y + z &= 2, \\ -x + 3y + 2z &= 8, \\ 4x + y &= 4 \end{aligned}$$

(in Example 6, Progress Ch. I). This can also be rewritten as

$$\begin{aligned} 1 \cdot x + 1 \cdot y + 1 \cdot z &= 2, \\ (-1) \cdot x + 3 \cdot y + 2 \cdot z &= 8, \\ 4 \cdot x + 1 \cdot y + 0 \cdot z &= 4. \end{aligned} \tag{1}$$

Through the elimination method, we have reduced (1) to another system

$$\begin{aligned} 1 \cdot x + 0 \cdot y + 0 \cdot z &= 0, \\ 0 \cdot x + 1 \cdot y + 0 \cdot z &= 4, \\ 0 \cdot x + 0 \cdot y + 1 \cdot z &= -2. \end{aligned} \tag{2}$$

The solution for the system (2) is self-evident: $\big(x,\ y,\ z\big) = \big(0,\ 4,\ -2\big)$. We have reduced (1) to (2) <u>while preserving the solution set</u>. Hence we conclude that the system (1) has the only solution $\big(x,\ y,\ z\big) = \big(0,\ 4,\ -2\big)$.

• We notice that, such an elimination process only involves arithmetic manipulation applied to the coefficients of the system of linear equations. Thus, we will be able to carry out the same process more efficiently, if we omit writing the variables $\Big(x,\ y,\ z,\ \text{or}\ x_1,\ \cdots,\ x_n\Big)$, and just write out the coefficients. This works, as long as we layout the coefficients of the system of equations in some order. Namely, we write out the coefficients in a rectangular, row-column like, array, so that each row in the array is occupied by the coefficients of a same equation. In addition, we make a rule that the first column in the array is occupied by the coefficients of the first variable x, (or x_1), the second column in the array is occupied by the coefficients of the second variable y, (or x_2), and so on.

• Let us look at (1) and (2) in the previous page. The system (1) yields

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 3 & 2 & 8 \\ 4 & 1 & 0 & 4 \end{bmatrix}. \tag{3}$$

(3) is <u>the augmented matrix</u> for the system (1). Similarly, the system (2) yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}. \tag{4}$$

(4) is the augmented matrix for the system (2). Below we re-work the reduction process (1) $\longrightarrow$ (2), using the augmented matrices. We start with the matrix (3).

(Step 1)
$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 3 & 2 & 8 \\ 4 & 1 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 4 & 3 & 10 \\ 4 & 1 & 0 & 4 \end{bmatrix}$$

$\Big[$ (row 1) was added to (row 2) $\Big]$

(Step 2)
$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 4 & 3 & 10 \\ 0 & -3 & -4 & -4 \end{bmatrix}$$

$\Big[$ (-4) times (row 1) was added to (row 3) $\Big]$

(Step 3) $$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 4 & 3 & 10 \\ 0 & 1 & -1 & 6 \end{bmatrix}$$

$\Big[\big(\text{row } 2\big)$ was added to $\big(\text{row } 3\big) \Big]$

(Step 4) $$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 6 \\ 0 & 4 & 3 & 10 \end{bmatrix}$$

$\Big[\big(\text{row } 2\big)$ and $\big(\text{row } 3\big)$ were interchanged$\Big]$

(Step 5) $$\longrightarrow \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 6 \\ 0 & 4 & 3 & 10 \end{bmatrix}$$

$\Big[\big(-1\big)$ times $\big(\text{row } 2\big)$ was added to $\big(\text{row } 1\big) \Big]$

(Step 6) $$\longrightarrow \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 7 & -14 \end{bmatrix}$$

$\Big[\big(-4\big)$ times $\big(\text{row } 2\big)$ was added to $\big(\text{row } 3\big) \Big]$

(Step 7) $$\longrightarrow \begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$\Big[\big(1/7\big)$ was multiplied to $\big(\text{row } 3\big) \Big]$

(Step 8) $$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$\Big[\big(-2\big)$ times $\big(\text{row } 3\big)$ was added to $\big(\text{row } 1\big)$;

$\big(\text{row } 3\big)$ was added to $\big(\text{row } 2\big) \Big]$.

• **Note.** There are many different ways to reduce the same matrix. For example, for the previous matrix (3), we can do

(Step 1–3) $$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 3 & 2 & 8 \\ 4 & 1 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 4 & 3 & 10 \\ 0 & 1 & -1 & 6 \end{bmatrix}$$

$\Big[$ the same as above $\Big]$

(Step 4)′ $$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3/4 & 5/2 \\ 0 & 1 & -1 & 6 \end{bmatrix}$$

$\Big[\ \big(1/4\big)$ was multiplied to $\big(\text{row } 2\big)\ \Big]$

(Step 5)′ $$\longrightarrow \begin{bmatrix} 1 & 0 & 1/4 & -1/2 \\ 0 & 1 & 3/4 & 5/2 \\ 0 & 1 & -1 & 6 \end{bmatrix}$$

$\Big[\ \big(-1\big)$ times $\big(\text{row } 2\big)$ was added to $\big(\text{row } 1\big)\ \Big]$

(Step 6)′ $$\longrightarrow \begin{bmatrix} 1 & 0 & 1/4 & -1/2 \\ 0 & 1 & 3/4 & 5/2 \\ 0 & 0 & -7/4 & 7/2 \end{bmatrix}$$

$\Big[\ \big(-1\big)$ times $\big(\text{row } 2\big)$ was added to $\big(\text{row } 3\big)\ \Big]$

(Step 7)′ $$\longrightarrow \begin{bmatrix} 1 & 0 & 1/4 & -1/2 \\ 0 & 1 & 3/4 & 5/2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$\Big[\ \big(-4/7\big)$ was multiplied to $\big(\text{row } 3\big)\ \Big]$

(Step 8)′ $$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$\Big[\ \big(-1/4\big)$ times $\big(\text{row } 3\big)$ was added to $\big(\text{row } 1\big)$;
$\big(-3/4\big)$ times $\big(\text{row } 3\big)$ was added to $\big(\text{row } 2\big)\ \Big]$.

• We give a formal definition of the augmented matrix for a general system of linear equations.

Definition. Consider

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \cdots \quad \cdots \quad \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

If we keep track of the location of the $+$'s, the x's, and the $=$'s, then this system of m equations in n variables can be abbreviated by the following rectangular array of numbers, in m rows and $(n+1)$ columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

This is called the augmented matrix for the above system of linear equations.

• When constructing an augmented matrix, we must agree that the variables are written in the same order in each equation and the constants on the right.

Example 1. The augmented matrix for the system

$$\begin{aligned} x_1 - x_2 \qquad &= 3, \\ x_2 - 2x_3 &= 1, \\ x_3 &= -1 \end{aligned}$$

is

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

We are able to reduce this augmented matrix A, and thereby solve the given system of linear equations, as follows:

(Step 1) $$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\Big[$ (row 2) was added to (row 1) $\Big]$

(Step 2) $$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\Big[$ 2 times (row 3) was added to (row 1);
2 times (row 3) was added to (row 2) $\Big]$.

The system of linear equations that yields the last augmented matrix is

$$\begin{aligned} x_1 \qquad\qquad &= 2, \\ x_2 \qquad &= -1, \\ x_3 &= -1. \end{aligned}$$

Accordingly, the solution to the system in Example 1 is

$$\big(x_1,\ x_2,\ x_3\big) = \big(2,\ -1,\ -1\big).$$

Example 2. The augmented matrix for the system

$$\begin{aligned} x_1 + 2x_2 \qquad\quad + \quad x_4 &= 4, \\ x_2 + 2\,x_3 + \quad x_4 &= 3, \\ x_3 + 2\,x_4 &= 1, \\ x_4 &= 4 \end{aligned}$$

is

$$B = \begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

We are able to reduce this augmented matrix B, and thereby solve the given system of linear equations, as follows:

(Step 1)
$$\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & -1 & -2 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$\Big[\ (-2)$ times (row 2) was added to (row 1) $\Big]$

(Step 2)
$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 7 & 2 \\ 0 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$\Big[\ \ 4$ times (row 3) was added to (row 1);
(-2) times (row 3) was added to (row 2) $\Big]$

(Step 3)
$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -26 \\ 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$\Big[\ (-7)$ times (row 4) was added to (row 1);
3 times (row 4) was added to (row 2);
(-2) times (row 4) was added to (row 3) $\Big]$.

The system of linear equations that yields the last augmented matrix is

$$\begin{aligned} x_1 \qquad\qquad\qquad &= -26, \\ x_2 \qquad\qquad &= 13, \\ x_3 \qquad &= -7, \\ x_4 &= 4. \end{aligned}$$

Accordingly, the solution to the system in Example 2 is

$$\big(x_1,\ x_2,\ x_3,\ x_4\big) = \big(-26,\ 13,\ -7,\ 4\big).$$

Example 3. The augmented matrix for the system

$$\begin{aligned} x_1 + \quad x_2 - 5\,x_3 &= 3, \\ x_1 \qquad\quad - 2\,x_3 &= 1, \\ 2\,x_1 - \quad x_2 - \quad x_3 &= 0 \end{aligned}$$

is

$$C = \begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix}.$$

We are able to reduce this augmented matrix C, and thereby solve the given system of linear equations, as follows:

(Step 1)
$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & -3 & 9 & -6 \end{bmatrix}$$

$\Big[\ \big(-1\big)$ times $\big($row 1$\big)$ was added to $\big($row 2$\big)$;
$\big(-2\big)$ times $\big($row 1$\big)$ was added to $\big($row 3$\big)\ \Big]$

(Step 2)
$$\longrightarrow \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{bmatrix}$$

$\Big[\ \big(-1\big)$ was multiplied to $\big($row 2$\big)\ \Big]$

(Step 3)
$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Big[\ \big(-1\big)$ times $\big($row 2$\big)$ was added to $\big($row 1$\big)$;
3 times $\big($row 2$\big)$ was added to $\big($row 3$\big)\ \Big]$.

The system of linear equations that yields the last augmented matrix is

$$\begin{aligned} x_1 \qquad - 2\,x_3 &= 1, \\ x_2 - 3\,x_3 &= 2, \\ 0 &= 0. \end{aligned}$$

Thus, we set $x_3 = t$. The last system of equations reads

$$\begin{aligned} x_1 \qquad - 2t &= 1, \\ x_2 - 3t &= 2. \end{aligned}$$

Hence $x_1 = 1 + 2t$, $x_2 = 2 + 3t$. Accordingly, the solution set to the system in Example 3 is

$$\big(x_1,\ x_2,\ x_3\big) = \big(1 + 2t,\ \ 2 + 3t,\ \ t\big).$$

- **Elementary Row Operations, (Reduced) Row Echelon Forms.**

Our goal is to establish a systematic method to find the solution set to any system of linear equations. We dealt with a few concrete examples of systems of linear equations. We have performed calculation and successfully found the solution set each time. In the course, we have used the augmented matrix of the given system. We have repeatedly applied <u>row operations</u> to the augmented matrix. We intuitively feel that basically the same method should work for an arbitrary system of linear equations, at least theoretically. To justify, we must answer the following two basic questions A and B:

A. What exactly kind of row operations are allowed to augmented matrices?

Answer for A: The following three operations are allowed, and nothing else is allowed:

Operation 1. <u>Multiply</u> a <u>row</u> through by a <u>non-zero</u> constant.

Operation 2. <u>Interchange</u> two <u>rows</u>.

Operation 3. <u>Add</u> a multiple of one <u>row</u> to another <u>row</u>.

- These operations are called <u>elementary row operations</u>.

We must also answer:

B. From a given augmented matrix we must ultimately create, through the three elementary row operations, a matrix of a certain form. What exactly form?

Answer for B: A matrix in <u>reduced row echelon form</u>.

Here, an augmented matrix is said to be in <u>reduced row echelon form</u>, when it satisfies all the Properties 1–4 below:

Property 1. If a row does <u>not</u> consist entirely of zeroes, then the first non-zero number in the row is 1. We call this <u>the leading 1</u>.

Property 2. If there are any rows that consist entirely of zeroes, then they are grouped together at the bottom of the matrix.

Property 3. In any two successive rows that do not consist entirely of zeroes, the leading 1 in the <u>lower</u> row occurs farther to the <u>right</u> than the leading 1 in the higher row.

Property 4. Each <u>column</u> $\big($not row (!)$\big)$ that contains the leading 1 has zero everywhere else.

- A matrix having Properties 1–3, and not necessarily of 4, is said to be in <u>row echelon form</u>.

Example 4. Each of the following matrices is in <u>reduced row echelon form</u>:

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}, \quad \begin{bmatrix} 0 & \boxed{1} & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 5. Each of the following matrices is in <u>row echelon form</u>, but is <u>not</u> in <u>reduced</u> row echelon form:

$$\begin{bmatrix} \boxed{1} & 4 & 5 & 0 \\ 0 & \boxed{1} & -2 & -1 \\ 0 & 0 & \boxed{1} & -3/2 \end{bmatrix}, \quad \begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 7 \\ 0 & 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & -1 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- **How to distinguish <u>reduced</u> row echelon forms from mere row echelon forms?**

A matrix in <u>reduced</u> row echelon form has zeroes <u>both below and above</u> each leading 1. A matrix in row echelon form has zeroes <u>below</u> each leading 1.

[I] Which are in row echelon form? Which are in reduced row echelon form?

(1) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, (2) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (3) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

(4) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, (5) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$, (6) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$,

(7) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, (8) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (9) $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,

(10) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (11) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, (12) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

(13) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$.

[II] Describe all possible reduced row echelon form having 3 rows and 3 columns. Use $*$ to denote an arbitrary entry.

- **Homogeneous system of linear equations.**

A system of linear equations is said to be homogeneous, if each of the equations in the system has 0 as the constant term (the right hand side). Thus, the general form of a homogeneous system of a linear equations is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ \cdots \quad \cdots \quad \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Example 6. Each of the following systems of linear equations is homogeneous:

$$x_1 - 3x_2 + 6x_3 = 0,$$

$$\begin{aligned} x_1 + 4x_2 &= 0, \\ -2x_1 + 3x_2 &= 0, \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0, \\ 2x_1 + 5x_2 + 4x_3 &= 0, \end{aligned}$$

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0, \\ x_1 - x_2 + x_3 &= 0, \\ x_1 + x_2 - x_3 &= 0, \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 \qquad &= 0, \\ x_1 + x_2 \qquad + x_4 &= 0, \\ x_1 \qquad + x_3 + x_4 &= 0, \\ x_2 + x_3 + x_4 &= 0, \end{aligned}$$

$$\begin{aligned} 2x_1 + x_2 - 3x_3 \qquad &= 0, \\ x_1 - 2x_2 + 4x_3 \qquad &= 0, \\ -5x_2 + x_3 + x_4 + x_5 &= 0. \end{aligned}$$

- **The trivial solution and a non-trivial solution to a homogeneous system.**

  Clearly, a homogeneous system of linear equations

$$\begin{aligned} a_{11}\,x_1 \,+\, a_{12}\,x_2 \,+\, \cdots \,+\, a_{1n}\,x_n \;&=\; 0, \\ a_{21}\,x_1 \,+\, a_{22}\,x_2 \,+\, \cdots \,+\, a_{2n}\,x_n \;&=\; 0, \\ \cdots \qquad \cdots \qquad \cdots& \\ a_{m1}x_1 \,+ a_{m2}x_2 \,+ \cdots + a_{mn}x_n \;&=\; 0 \end{aligned}$$

always has a solution

$$\Big(x_1,\ \cdots\ ,\ x_n\Big) \;=\; \Big(0,\ \cdots\ ,\ 0\Big).$$

This is called the trivial solution to the system. On the other hand, sometimes a homogeneous system of equations has a solution other than

$$\Big(x_1,\ \cdots\ ,\ x_n\Big) \;=\; \Big(0,\ \cdots\ ,\ 0\Big).$$

Such a solution is called a non-trivial solution to the system.

- The following formula is useful:

Formula 1. Consider a homogeneous system

$$\begin{aligned} a_{11}\,x_1 \,+\, a_{12}\,x_2 \,+\, \cdots \,+\, a_{1n}\,x_n \;&=\; 0, \\ a_{21}\,x_1 \,+\, a_{22}\,x_2 \,+\, \cdots \,+\, a_{2n}\,x_n \;&=\; 0, \\ \cdots \qquad \cdots \qquad \cdots& \\ a_{m1}x_1 \,+ a_{m2}x_2 \,+ \cdots + a_{mn}x_n \;&=\; 0, \end{aligned}$$

where m denotes the number of equations, and n denotes the number of variables. If $\boxed{m < n}$, then the system has a non-trivial solution.

- To paraphrase Formula 1 in one sentence:

Formula 1. A homogeneous system of linear equations whose number of variables exceeds the number of equations has a non-trivial solution.

• Note that the above Formula <u>does not say anything</u> about the homogeneous system whose number of variables is less than or equal to the number of equations.

• **Homogeneous System with Two Equations, Two Variables.**

It is of interest to study the existence/non-existence of a non-trivial solution to a homogeneous system of linear equations whose number of equations <u>equals</u> the number of variables. The easiest case is in two equations, two variables:

$$\begin{aligned} a\,x \;+\; b\,y \;&=\; 0, \\ c\,x \;+\; d\,y \;&=\; 0. \end{aligned}$$

Formula 2. The above system has a <u>non-trivial</u> solution, precisely when

$$\boxed{\,a\,d \;-\; b\,c \;=\; 0\,}\;.$$

• **Application.** We can use Formula 2 to solve the following problem.

Problem. Find all values of λ such that the homogeneous system of linear equations

$$\begin{aligned} \big(\lambda - 1\big)\,x \;+\; 2\,y \;&=\; 0, \\ x \;+\; \lambda\,y \;&=\; 0 \end{aligned}$$

has a <u>non-trivial</u> solution.

Solution. By Formula 2, the system has a non-trivial solution precisely when

$$\big(\lambda - 1\big)\,\lambda \;-\; 2\cdot 1 \;=\; 0.$$

We may simplify this last equation as

$$\lambda^2 \;-\; \lambda \;-\; 2 \;=\; 0.$$

We may factor the left-hand side as

$$\Big(\lambda + 1\Big)\Big(\lambda - 2\Big) \;=\; 0.$$

Hence we have $\qquad \lambda \;=\; 2,\;\; -1.$

- **The augmented matrix versus the coefficient matrix.**

  The augmented matrix for a homogeneous system of linear equations is of form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{bmatrix}.$$

The last column of the augmented matrix consists entirely of 0. When we deal with nothing else but homogeneous systems, the last column is redundant. It makes sense to introduce the <u>coefficient matrix</u> of the homogneous system

$$\begin{aligned} a_{11}\, x_1 \;+\; a_{12}\, x_2 \;+\; \cdots + \; a_{1n}\, x_n \;&=\; 0, \\ a_{21}\, x_1 \;+\; a_{22}\, x_2 \;+\; \cdots + \; a_{2n}\, x_n \;&=\; 0, \\ \cdots \qquad \cdots \qquad & \cdots \\ a_{m1}x_1 \;+ a_{m2}x_2 \;+ \cdots + a_{mn}x_n \;&=\; 0 \end{aligned}$$

as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

[III] Consider the system of linear equations

$$\begin{aligned} x \;+\; 2\,y \;+\; 3z \;&=\; 0, \\ 4\,x \;+\; 5\,y \;+\; 6z \;&=\; 0, \\ 7\,x \;+\; 8\,y \;+\; 9z \;&=\; 0. \end{aligned}$$

(1) Is this a homogeneous equation?

(2) Find the <u>coefficient matrix</u>.

(3) Find the <u>reduced row echelon form</u> of the coefficient matrix.

(4) Does the system have a <u>non-trivial</u> solution? If it does, find the solution set.

[IV] Consider the system of linear equations

$$\begin{aligned} x + k\,y + 2\,z &= 0, \\ -3\,x + 4\,y + z &= 0. \end{aligned}$$

(1) Is this a homogeneous equation?

(2) Find the coefficient matrix.

(3) Find all the values k such that the system has a non-trivial solution.

[V] Consider the system of linear equations

$$\begin{aligned} x + y &= 0, \\ y + z &= 0, \\ x + z &= 0. \end{aligned}$$

(1) Is this a homogeneous equation?

(2) Find the coefficient matrix.

(3) Does the system have a non-trivial solution? If it does, find the solution set.

[VI] Consider the system of linear equations

$$\begin{aligned} x + y &= 0, \\ y + z &= 0, \\ x + z &= 0, \\ a\,x + b\,y + c\,z &= 0. \end{aligned}$$

(1) Is this a homogeneous equation?

(2) Find the coefficient matrix.

(3) Is it possible that for some $\left(a,\ b,\ c\right)$, the system has a non-trivial solution? Explain.

Solution to problems [I–VI].

[I] (1), (2), (4), (6), (10), (12) are in reduced row echelon form.

(1), (2), (4), (6), (8), (9), (10), (12) are in row echelon form.

[II] $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

[III] (1) The system is homogeneous.

(2) The coefficient matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

(3) The reduced row echelon form of the coefficient matrix is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

(4) The system has a non-trivial solution. The solution set:

$$\Big(x,\ \ y,\ \ z\Big) \ = \ \Big(t,\ \ -2t,\ \ t\Big).$$

[IV] (1) The system is homogeneous.

(2) The coefficient matrix is $\begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix}$.

(3) The system always has a non-trivial solution, for an arbitrary value k.

Reason: the number of equation is 2, which is less than the number of variables, which is 3.

[V] (1) The system is homogeneous.

(2) The coefficient matrix is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(3) The system does not have any non-trivial solution.

Reason: The reduced row echelon form of the coefficient matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which corresponds to the homogeneous system

$$\begin{aligned} x \qquad\qquad &= 0, \\ y \qquad &= 0, \\ z &= 0. \end{aligned}$$

[VI] (1) The system is homogeneous.

(2) The coefficient matrix is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & b & c \end{bmatrix}$.

(3) The system does not have any non-trivial solution, regardless of the values of $\big(a, \ b, \ c\big)$.

Reason: The system in [V] does not have a non-trivial solution. The system in [VI] consists of all the equations in the system in [V] and another equation.