Math 290 ELEMENTARY LINEAR ALGEBRA

PROGRESS CHECK – III

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- **Matrices and matrix operations**.

We have associated the augmented matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

to a system of linear equations

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n &= b_1, \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n &= b_2, \\ \cdots \qquad \cdots \qquad \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

We have also associated the coefficient matrix

$$C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

to a homogeneous system of linear equations

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n &= 0, \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n &= 0, \\ \cdots \qquad \cdots \qquad \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

- Irrespective of the context of systems of linear equations, it makes sense to treat the "mere rectangular array" of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

as a mathematical object. It makes sense to build some mathematical structure on the set (collection) of those. First, we give an official name for such "arrays".

Definition. A matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where a_{ij} are scalars $(=$ real numbers$)$. Each individual scalar a_{ij} is called an entry of the matrix A. When we need to emphasize the location of a_{ij}, we say that a_{ij} is the $(i,\, j)$-th entry of the matrix A.

- In the above, the number of rows is m and the number of columns is n. We then call the matrix to have size $m \times n$.

- The size description should always be

$$\boxed{\Big(\text{ the number of rows }\Big) \times \Big(\text{ the number of columns }\Big)}\ .$$

- It is common to omit the bracket $[\quad]$ on a 1×1 matrix. We may write

$$a, \qquad \text{instead of} \qquad [a].$$

Example 1. We may write a 2×2 matrix as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

We may write a 3×3 matrix as $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

We may write a 2×4 matrix as $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$.

- **Equality (= Identity) of two matrices.**

  Two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & a_{12} & \cdots & a_{1\ell} \\ b_{21} & a_{22} & \cdots & a_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & a_{k2} & \cdots & a_{k\ell} \end{bmatrix}$$

are <u>equal</u>, when their sizes match, namely, $m = k$, and $n = \ell$, and moreover their corresponding entries match, namely,

$$a_{ij} = b_{ij} \quad \text{holds for each} \quad i,\ j.$$

In this situation, we write

$$A = B.$$

If the two matrices A and B are not equal, then we write

$$A \neq B.$$

Example 2.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

- **Row vectors, Column vectors.**

A matrix of size $1 \times n$ is called a $\underline{\text{ row vector }}$ $\big(\underline{\text{of length } n}\big)$. A matrix of size $m \times 1$ is called a $\underline{\text{ column vector }}$ $\big(\underline{\text{of length } m}\big)$.

Example 3. A row vector of length 2 is $\begin{bmatrix} a & b \end{bmatrix}$, or $\begin{bmatrix} a_1 & a_2 \end{bmatrix}$.

A column vector of length 2 is $\begin{bmatrix} a \\ b \end{bmatrix}$, or $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

A row vector of length 3 is $\begin{bmatrix} a & b & c \end{bmatrix}$, or $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$.

A column vector of length 3 is $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, or $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

- **Matrix addition.**

For two matrices A and B which are in the same size, their sum $A + B$ is defined. $A + B$ is a matrix having the same size as A and B.

Definition. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}.$$

Note that A and B are both in size $m \times n$. We define

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

- We may paraphrase the definition as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

In short, $\underline{\text{ the matrix addition is defined as an ``entrywise'' addition }}$.

Example 4. For $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$, we have

$$A + B = \begin{bmatrix} 1 + (-3) & 2 + (-2) \\ 2 + 4 & 1 + 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 6 & 3 \end{bmatrix}.$$

- **Scalar multiplication.**

For a matrix A and a $\underline{\text{ scalar }}$ s, the multiplication $s\,A$ is defined. $s\,A$ is a matrix having the same size as A.

Definition. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Let s be a scalar. We define

$$s\,A = \begin{bmatrix} s\,a_{11} & s\,a_{12} & \cdots & s\,a_{1n} \\ s\,a_{21} & s\,a_{22} & \cdots & s\,a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s\,a_{m1} & s\,a_{m2} & \cdots & s\,a_{mn} \end{bmatrix}.$$

- We may paraphrase the definition as

$$s \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} s\,a_{11} & s\,a_{12} & \cdots & s\,a_{1n} \\ s\,a_{21} & s\,a_{22} & \cdots & s\,a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s\,a_{m1} & s\,a_{m2} & \cdots & s\,a_{mn} \end{bmatrix}.$$

In short, a scalar multiplication of a matrix is defined as a multiplication of "the same scalar to each entry".

- In particular, we may define $\big(-1\big)\,A$ for a matrix A. We often write it as $-A$:

$$\big(-1\big)\,A = -A.$$

That is, for $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, we have

$$-A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix}.$$

- **A linear combination of matrices.**

For two matrices A and B which are in the same size, and for two scalars s and t, the linear combination $s\,A + t\,B$ is defined as the sum of $s\,A$ and $t\,B$. $s\,A + t\,B$ is a matrix having the same size as A, B.

- As a special case, for example, we may consider $1 \cdot A + \big(-1\big) \cdot B$ for matrices A, B. We may write it as $A - B$: $1 \cdot A + \big(-1\big) \cdot B = A - B$.

- It makes sense to more generally consider a linear combination of form

$$s_1\,A_1 + s_2\,A_2 + \cdots + s_r\,A_r$$

for matrices $A_1,\ A_2,\ \cdots,\ A_r$ in the same size, and for scalars $s_1,\ s_2,\ \cdots,\ s_r$.

Example 5. For

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix},$$

we have

$$2\,A = \begin{bmatrix} 2 \cdot 2 & 2 \cdot 1 & 2 \cdot 1 \\ 2 \cdot (-1) & 2 \cdot (-1) & 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ -2 & -2 & 8 \end{bmatrix},$$

$$-\,B = \begin{bmatrix} -2 & 3 & -4 \\ 3 & -1 & 2 \end{bmatrix}, \qquad \text{and}$$

$$2\,A - B = (2\,A) + (-\,B) = \begin{bmatrix} 4 & 2 & 2 \\ -2 & -2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 3 & -4 \\ 3 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & -2 \\ 1 & -3 & 6 \end{bmatrix}.$$

[I] (1) For $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$, find $A + B$, $A - B$.

(2) For $A = \begin{bmatrix} 4 & 11 & -9 \\ 0 & 3 & 2 \\ -3 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 5 \\ -4 & 6 & 11 \\ -6 & 4 & 9 \end{bmatrix}$, find $5\,A + 2\,B$.

- **Matrix multiplication.**

For two matrices A and B, their product AB is defined, whenever the number of columns of A equals the number of rows of B. AB is a matrix whose number of rows equals that of A, and whose number of columns equals that of B.

Definition. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}.$$

Note that A has size $m \times r$, and B has size $r \times n$. We define

$$AB = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

as a matrix having size $m \times n$, and having entries c_{ij} which are decided by the following rule:

Rule. To find the (i, j)-th entry c_{ij} of the product AB, single out the row i from the matrix A, and the column j from the matrix B. That is, from

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ b_{r1} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix},$$

single out

$$\boldsymbol{a}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ir} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{b}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix}.$$

Multiply the corresponding entries for the row $\boldsymbol{a}_i$ and the column $\boldsymbol{b}_j$ together, then add up the resulting products, and call it c_{ij}:

$$a_{i1}\, b_{1j} + a_{i2} b_{2j} + \cdots + a_{ir} b_{rj} = c_{ij}.$$

This c_{ij} will sit in as the (i, j)-th entry of the product AB.

- Consider the special case when B is a column vector $\boldsymbol{b}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}.$$

Then

$$A\boldsymbol{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1r}b_r \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2r}b_r \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mr}b_r \end{bmatrix}.$$

The same can also be written as

$$A\boldsymbol{b} = b_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_r \begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{mr} \end{bmatrix}.$$

- Similarly, consider the special case when A is a <u>row vector</u> $\boldsymbol{a}$:

$$\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}.$$

Then

$$\begin{aligned} \boldsymbol{a}B &= \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix} \\ &= \Big[a_1 b_{11} + a_2 b_{21} + \cdots + a_r b_{r1} \quad a_1 b_{12} + a_2 b_{22} + \cdots + a_r b_{r2} \\ &\qquad \cdots \quad a_1 b_{1n} + a_2 b_{2n} + \cdots + a_r b_{rn} \Big]. \end{aligned}$$

The same can also be written as

$$\begin{aligned} \boldsymbol{a}B = & \quad a_1 \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \end{bmatrix} \\ & + a_2 \begin{bmatrix} b_{21} & b_{22} & \cdots & b_{2n} \end{bmatrix} \\ & + \qquad \cdots \qquad \cdots \\ & + a_r \begin{bmatrix} b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}. \end{aligned}$$

Example 6. For $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$, we have

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) & 1 \cdot (-1) + 2 \cdot 8 \\ 4 \cdot 2 + 2 \cdot (-1) & 4 \cdot (-1) + 2 \cdot 8 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
BA &= \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot 1 + (-1) \cdot 4 & 2 \cdot 2 + (-1) \cdot 2 \\ (-1) \cdot 1 + 8 \cdot 4 & (-1) \cdot 2 + 8 \cdot 2 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}.
\end{aligned}
$$

Example 7. For $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$, we have

$$
\begin{aligned}
&AB \\
&= \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 1 + (-1) \cdot 2 + 7 \cdot 1 & 1 \cdot 1 + (-1) \cdot 1 + 7 \cdot (-3) & 1 \cdot 2 + (-1) \cdot 1 + 7 \cdot 2 \\ 2 \cdot 1 + (-1) \cdot 2 + 8 \cdot 1 & 2 \cdot 1 + (-1) \cdot 1 + 8 \cdot (-3) & 2 \cdot 2 + (-1) \cdot 1 + 8 \cdot 2 \\ 3 \cdot 1 + 1 \cdot 2 + (-1) \cdot 1 & 3 \cdot 1 + 1 \cdot 1 + (-1) \cdot (-3) & 3 \cdot 2 + 1 \cdot 1 + (-1) \cdot 2 \end{bmatrix} \\
&= \begin{bmatrix} 6 & -21 & 15 \\ 8 & -23 & 19 \\ 4 & 7 & 5 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&BA \\
&= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot 1 + 1\cdot 2 + 2\cdot 3 & 1\cdot(-1) + 1\cdot(-1) + 2\cdot 1 & 1\cdot 7 + 1\cdot 8 + 2\cdot(-1) \\ 2\cdot 1 + 1\cdot 2 + 1\cdot 3 & 2\cdot(-1) + 1\cdot(-1) + 1\cdot 1 & 2\cdot 7 + 1\cdot 8 + 1\cdot(-1) \\ 1\cdot 1 + (-3)\cdot 2 + 2\cdot 3 & 1\cdot(-1) + (-3)\cdot(-1) + 2\cdot 1 & 1\cdot 7 + (-3)\cdot 8 + 2\cdot(-1) \end{bmatrix} \\
&= \begin{bmatrix} 9 & 0 & 13 \\ 7 & -2 & 21 \\ 1 & 4 & -19 \end{bmatrix}.
\end{aligned}
$$

- As Examples 6, 7 show, $\underline{AB \text{ and } BA \text{ need not be equal}}$, even if both AB and BA are defined and are in the same size.

Example 8. For $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, we have

$$
\begin{aligned}
AB &= \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \\
&= 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 0 \\
&= 12,
\end{aligned}
$$

$$
\begin{aligned}
BA &= \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot 3 & 2 \cdot 2 & 2 \cdot 1 \\ 3 \cdot 3 & 3 \cdot 2 & 3 \cdot 1 \\ 0 \cdot 3 & 0 \cdot 2 & 0 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 4 & 2 \\ 9 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

- As Example 8 shows, AB and BA need not have the same size, even if both AB and BA are defined.

Example 9. For $A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$, we have

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)\cdot 1 + 3\cdot 0 & (-1)\cdot 2 + 3\cdot 7 \\ 4\cdot 1 + (-5)\cdot 0 & 4\cdot 2 + (-5)\cdot 7 \\ 0\cdot 1 + 2\cdot 0 & 0\cdot 2 + 2\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 19 \\ 4 & -27 \\ 0 & 14 \end{bmatrix},$$

BA is undefined.

- As Example 9 shows, BA needs not be defined, even if AB is defined.

[II] (1) For $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & -1/8 & 0 \\ 0 & 0 & 1/7 \end{bmatrix}$, find AB and BA. If undefined, write undefined.

(2) For $A = \begin{bmatrix} 6 \\ -2 \\ 1 \\ 6 \end{bmatrix}$, $B = \begin{bmatrix} 10 & 12 \end{bmatrix}$, find AB and BA. If undefined, write undefined.

- **A matrix partition.**

  A matrix can be "partitioned" into its columns. If B is a matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix},$$

then we may let

$$\boldsymbol{b}_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{r1} \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{r2} \end{bmatrix}, \qquad \cdots \qquad \boldsymbol{b}_n = \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{rn} \end{bmatrix},$$

and think of B as $\begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{bmatrix}$. We call $\boldsymbol{b}_j$ <u>the j-th column vector</u> of B.

Formula 1. Let $B = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{bmatrix}$ be as above. Let A be any matrix whose number of columns is r. Then

$$AB = \begin{bmatrix} A\boldsymbol{b}_1 & A\boldsymbol{b}_2 & \cdots & A\boldsymbol{b}_n \end{bmatrix}.$$

Example 10. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}.$$

The first column of the resulting matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$.

The second column of the resulting matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}$.

- Similarly, a matrix can be "partitioned" into its rows. If A is a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix},$$

then we may let

$$\begin{aligned} \boldsymbol{a}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \end{bmatrix}, \\ \boldsymbol{a}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2r} \end{bmatrix}, \\ \vdots & \qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ \boldsymbol{a}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \end{aligned}$$

and think of A as $\begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix}$. We call $\boldsymbol{a}_i$ $\underline{\text{ the } i\text{-th row vector }}$ of A.

Formula 2. Let $A = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix}$ be as above. Let B be any matrix whose number of rows is r. Then $AB = \begin{bmatrix} \boldsymbol{a}_1 B \\ \boldsymbol{a}_2 B \\ \vdots \\ \boldsymbol{a}_m B \end{bmatrix}$.

Example 11. Look at the same matrix multiplication as in Example 10.

The first row of the resulting matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$.

The second row of the resulting matrix is $\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$.

- **Systems of linear equations in matrix form.**

Let us return to the consideration of systems of linear equations. Keeping the definition of the matrix multiplication in mind, we realize that the system

$$\begin{aligned} a_{11}\,x_1 \;+\; a_{12}\,x_2 \;+\; \cdots \;+\; a_{1n}\,x_n \;&=\; b_1, \\ a_{21}\,x_1 \;+\; a_{22}\,x_2 \;+\; \cdots \;+\; a_{2n}\,x_n \;&=\; b_2, \\ \cdots \qquad \cdots \qquad \cdots& \\ a_{m1}x_1 \;+ a_{m2}x_2 \;+ \cdots \;+ a_{mn}x_n \;&=\; b_m \end{aligned}$$

can be rewritten as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

In other words, letting

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \boldsymbol{x} \;=\; \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{b} \;=\; \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

the above system of linear equations can be rewritten as

$$A\boldsymbol{x} \;=\; \boldsymbol{b}.$$

- The <u>homogeneous</u> system of linear equations

$$\begin{aligned} a_{11}\,x_1 \;+\; a_{12}\,x_2 \;+\; \cdots \;+\; a_{1n}\,x_n \;&=\; 0, \\ a_{21}\,x_1 \;+\; a_{22}\,x_2 \;+\; \cdots \;+\; a_{2n}\,x_n \;&=\; 0, \\ \cdots \qquad \cdots \qquad \cdots& \\ a_{m1}x_1 \;+ a_{m2}x_2 \;+ \cdots \;+ a_{mn}x_n \;&=\; 0 \end{aligned}$$

can be rewritten as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In other words, letting A and $\boldsymbol{x}$ be as above, and

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

the same system can be rewritten as

$$A\boldsymbol{x} = \mathbf{0}.$$

Example 12. The system of linear equations

$$\begin{aligned} 6x_2 + 4x_3 &= -12, \\ 3x_1 + 3x_2 \phantom{{}+4x_3} &= 9, \\ 2x_1 \phantom{{}+6x_2} - 3x_3 &= 10 \end{aligned}$$

is rewritten as

$$\begin{bmatrix} 0 & 6 & 4 \\ 3 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 9 \\ 10 \end{bmatrix}.$$

Using the elimination method, we find the only solution to this system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}.$$

Example 13. The homogeneous system of linear equations

$$\begin{aligned} 8x_1 - 8x_2 &= 0, \\ -3x_1 + 2x_2 &= 0 \end{aligned}$$

is rewritten as

$$\begin{bmatrix} 8 & -8 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using the elimination method $\big($or by speculation$\big)$, we find the only solution to this system as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[III] Rewrite the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & & & + & 2\,x_3 & = & 4, \\ 2\,x_1 & - & 4\,x_2 & & & = & -6, \\ 3\,x_1 & + & 2\,x_2 & + & 5\,x_3 & = & 10 \end{array}$$

in the form $A\boldsymbol{x} = \boldsymbol{b}$. Solve the equation.

- **System of linear equations in linear combination form.**

We look at an alternative way to rewrite systems of linear equations. We realize that the system

$$\begin{array}{rcl} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n & = & b_1, \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n & = & b_2, \\ \cdots \quad \cdots \quad \cdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

can be rewritten as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

In other words, letting

$$\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

the above system of linear equations can be rewritten as

$$x_1\,\boldsymbol{a}_1 + x_2\,\boldsymbol{a}_2 + \cdots + x_n\,\boldsymbol{a}_n = \boldsymbol{b}.$$

- The homogeneous system of linear equations

$$\begin{aligned} a_{11}\,x_1 \;+\; a_{12}\,x_2 \;+\; \cdots \;+\; a_{1n}\,x_n \;&=\; 0, \\ a_{21}\,x_1 \;+\; a_{22}\,x_2 \;+\; \cdots \;+\; a_{2n}\,x_n \;&=\; 0, \\ \cdots \qquad \cdots \qquad \cdots \\ a_{m1}x_1 \;+ a_{m2}x_2 \;+ \cdots + a_{mn}x_n \;&=\; 0 \end{aligned}$$

can be rewritten as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \;+\; x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \;+\; \cdots \;+\; x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In other words, letting $\boldsymbol{a}_1, \; \boldsymbol{a}_2, \; \cdots \;, \; \boldsymbol{a}_n$ be as above, and

$$\mathbf{0} \;=\; \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

the same system can be rewritten as

$$x_1\,\boldsymbol{a}_1 \;+\; x_2\,\boldsymbol{a}_2 \;+\; \cdots \;+\; x_n\,\boldsymbol{a}_n \;=\; \mathbf{0}.$$

- The above observation is useful, since it leads us to the following two theorems:

Theorem 1. A system of linear equations $A\boldsymbol{x} \;=\; \boldsymbol{b}$ is consistent, if and only if $\boldsymbol{b}$ is a linear combination of the column vectors of A:

$$\boldsymbol{b} \;=\; s_1\,\boldsymbol{a}_1 \;+\; s_2\,\boldsymbol{a}_2 \;+\; \cdots \;+\; s_n\,\boldsymbol{a}_n.$$

Theorem 2. A system of homogeneous linear equations $A\boldsymbol{x} \;=\; \mathbf{0}$ has a non-trivial solution, if and only if $\mathbf{0}$ is a non-trivial linear combination of column vectors $\boldsymbol{a}_1, \; \boldsymbol{a}_2, \; \cdots \; \boldsymbol{a}_n,$ of A, namely, $\mathbf{0}$ is expressed as

$$\mathbf{0} \;=\; s_1\,\boldsymbol{a}_1 \;+\; s_2\,\boldsymbol{a}_2 \;+\; \cdots \;+\; s_n\,\boldsymbol{a}_n,$$

where $s_1, \; s_2, \; , \; \cdots \;, \; s_n$ are scalars not all of which equal 0.

[IV] Spell out the following equation as a system of linear equations with a, b, c, d as variables:

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then solve the system. In other words, find $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

[V] Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Find the exact condition on a, b, c and d such that $AB = BA$ holds.

[VI] Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $B = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$, where θ and ϕ are scalars. Prove that $AB = BA$ holds.

[VII] Verify

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}.$$

Use this formula to calculate

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}.$$

- **The trace of a matrix.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be a matrix <u>in size $n \times n$</u>. We define <u>the trace</u> of A as

$$\operatorname{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

[VIII] (1) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 3 & 1 & 3 \end{bmatrix}$. Find $\operatorname{Tr}\big(A\big)$.

(2) Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find $\operatorname{Tr}\big(B\big)$.

[IX] Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Let s be a scalar.

(1) Prove $\operatorname{Tr}\big(A + B\big) = \operatorname{Tr}\big(A\big) + \operatorname{Tr}\big(B\big)$.

(2) Prove $\operatorname{Tr}\big(s\,A\big) = s\operatorname{Tr}\big(A\big)$.

(3) Prove $\operatorname{Tr}\big(AB\big) = \operatorname{Tr}\big(BA\big)$.

[X] Prove that $\underline{\text{no}}$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ satisfy

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solutions to problems [I–X].

[I] (1) $A + B = \begin{bmatrix} 7 & 3 \\ 1 & 9 \\ -2 & 15 \end{bmatrix}$, $A - B = \begin{bmatrix} 5 & -5 \\ 3 & -1 \\ -4 & -5 \end{bmatrix}$.

(2) $5\,A = \begin{bmatrix} 20 & 55 & -45 \\ 0 & 15 & 10 \\ -15 & 5 & 5 \end{bmatrix}$, $2\,B = \begin{bmatrix} 2 & 0 & 10 \\ -8 & 12 & 22 \\ -12 & 8 & 18 \end{bmatrix}$. Hence

$$5A + 2B = \begin{bmatrix} 20 & 55 & -45 \\ 0 & 15 & 10 \\ -15 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 10 \\ -8 & 12 & 22 \\ -12 & 8 & 18 \end{bmatrix} = \begin{bmatrix} 22 & 55 & -35 \\ -8 & 27 & 32 \\ -27 & 13 & 23 \end{bmatrix}.$$

[II] (1) $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(2) $AB = \begin{bmatrix} 60 & 72 \\ -20 & -24 \\ 10 & 12 \\ 60 & 72 \end{bmatrix}$, BA is undefined.

[III] $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -4 & 0 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 10 \end{bmatrix}.$

The only solution to this system is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1/2 \\ 3 \end{bmatrix}.$

[IV]
$$\begin{aligned} 2a \quad\quad - c \quad\quad &= 1, \\ 2b \quad\quad - d &= 0, \\ 3a \quad\quad - 2c \quad\quad &= 0, \\ 3b \quad\quad - 2d &= 1. \end{aligned}$$

The only solution to this system is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -2 \end{bmatrix}$. Hence the matrix to find is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$

[V] $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a - b & a + b \\ c - d & c + d \end{bmatrix},$

$BA = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ -a + c & -b + d \end{bmatrix}.$

Hence the condition $AB = BA$ reads

$$\begin{bmatrix} a - b & a + b \\ c - d & c + d \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ -a + c & -b + d \end{bmatrix}.$$

This is equivalent to $a = d, \;\; b + c = 0.$

[VI] $$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta)(\cos\phi) - (\sin\theta)(\sin\phi) & -(\cos\theta)(\sin\phi) - (\sin\theta)(\cos\phi) \\ (\sin\theta)(\cos\phi) + (\cos\theta)(\sin\phi) & -(\sin\theta)(\sin\phi) + (\cos\theta)(\cos\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix},$$

and similarly

$$BA = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\phi + \theta) & -\sin(\phi + \theta) \\ \sin(\phi + \theta) & \cos(\phi + \theta) \end{bmatrix}.$$

[VII] Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$, where we assume

$$(*) \qquad a_{ij} = b_{ij} = 0, \qquad \text{for} \qquad i \neq j.$$

Under the assumption $(*)$, the form

$$a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + a_{i3}\, b_{3j}$$

equals $a_{ii} b_{ii}$ whenever $i = j$, because we have the following (i–iii):

(i) if $i = j = 1$, then

$$\begin{aligned} a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31} &= a_{11} \cdot b_{11} + 0 \cdot 0 + 0 \cdot 0 \\ &= a_{11} \cdot b_{11}, \end{aligned}$$

(ii) if $i = j = 2$, then

$$\begin{aligned} a_{21}\, b_{12} + a_{22}\, b_{22} + a_{23}\, b_{32} &= 0 \cdot 0 + a_{22} \cdot b_{22} + 0 \cdot 0 \\ &= a_{22} \cdot b_{22}, \end{aligned}$$

(iii) if $i = j = 3$, then

$$\begin{aligned} a_{31}\, b_{13} + a_{32}\, b_{23} + a_{33}\, b_{33} &= 0 \cdot 0 + 0 \cdot 0 + a_{33}\, b_{33} \\ &= a_{33} \cdot b_{33}. \end{aligned}$$

Under the same assumption $(*)$, the form

$$a_{i1}\,b_{1j} \;+\; a_{i2}\,b_{2j} \;+\; a_{i3}\,b_{3j}$$

equals 0 whenever $i \neq j$, because we have the following (i–iii):

(i) if $i = 1$, then $j \neq 1$, and $b_{1j} = a_{12} = a_{13} = 0$, hence

$$a_{11}\,b_{1j} \;=\; a_{12}\,b_{2j} \;=\; a_{13}\,b_{3j} \;=\; 0,$$

(ii) if $i = 2$, then $j \neq 2$, and $a_{21} = b_{2j} = a_{23} = 0$, hence

$$a_{21}\,b_{1j} \;=\; a_{22}\,b_{2j} \;=\; a_{23}\,b_{3j} \;=\; 0,$$

(iii) if $i = 3$, then $j \neq 3$, and $a_{31} = a_{32} = b_{3j} = 0$, hence

$$a_{31}\,b_{1j} \;=\; a_{32}\,b_{2j} \;=\; a_{33}\,b_{3j} \;=\; 0.$$

In sum, under the assumption $(*)$, we have

$$AB \;=\; \begin{bmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}.$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \;=\; \begin{bmatrix} \big(-1\big)\cdot\big(-1\big) & 0 & 0 \\ 0 & 2\cdot 2 & 0 \\ 0 & 0 & 3\cdot 3 \end{bmatrix}$$

$$\;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix} \;=\; \begin{bmatrix} 3\cdot\big(-7\big) & 0 & 0 \\ 0 & \big(-5\big)\cdot 4 & 0 \\ 0 & 0 & 0\cdot 12 \end{bmatrix}$$

$$\;=\; \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

[VIII] (1) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 3 & 1 & 3 \end{bmatrix}$, $\operatorname{Tr}\big(A\big) = 1 + \big(-2\big) + 3 = 2.$

(2) For $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\operatorname{Tr}\big(B\big) = 1 + 1 + 1 = 3.$

[IX] (1)

$$\begin{aligned} \operatorname{Tr}\big(A + B\big) &= \operatorname{Tr}\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \\ &= \Big(a_{11}+b_{11}\Big) + \Big(a_{22}+b_{22}\Big) \\ &= \Big(a_{11}+a_{22}\Big) + \Big(b_{11}+b_{22}\Big) \\ &= \operatorname{Tr}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \operatorname{Tr}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \operatorname{Tr}\big(A\big) + \operatorname{Tr}\big(B\big). \end{aligned}$$

(2)

$$\begin{aligned} \operatorname{Tr}\big(s\,A\big) &= \operatorname{Tr}\begin{bmatrix} s\,a_{11} & s\,a_{12} \\ s\,a_{21} & s\,a_{22} \end{bmatrix} \\ &= \Big(s\,a_{11}\Big) + \Big(s\,a_{22}\Big) \\ &= s\Big(a_{11}+a_{22}\Big) \\ &= s\ \operatorname{Tr}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= s\ \operatorname{Tr}\big(A\big). \end{aligned}$$

(3)

$$\begin{aligned} \operatorname{Tr}\big(AB\big) &= \operatorname{Tr}\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix} \\ &= \Big(a_{11}b_{11}+a_{12}b_{21}\Big) + \Big(a_{21}b_{12}+a_{22}b_{22}\Big) \end{aligned}$$

$$
\begin{aligned}
&= a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22} \\
&= b_{11}a_{11} + b_{21}a_{12} + b_{12}a_{21} + b_{22}a_{22} \\
&= b_{11}a_{11} + b_{12}a_{21} + b_{21}a_{12} + b_{22}a_{22} \\
&= \Big(b_{11}a_{11} + b_{12}a_{21} \Big) + \Big(b_{21}a_{12} + b_{22}a_{22} \Big) \\
&= \operatorname{Tr} \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix} \\
&= \operatorname{Tr}\big(BA\big).
\end{aligned}
$$

[X] For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, we have

$$
\begin{aligned}
AB - BA = &\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \\
&- \begin{bmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \end{bmatrix} .
\end{aligned}
$$

Thus, the $\big(1,\,1\big)$ entry of $AB - BA$ equals

$$
\Big(a_{11}b_{11} + a_{12}b_{21} \Big) - \Big(a_{11}b_{11} + a_{21}b_{12} \Big) = a_{12}b_{21} - a_{21}b_{12},
$$

whereas the $\big(2,\,2\big)$ entry of $AB - BA$ equals

$$
\begin{aligned}
\Big(a_{21}b_{12} + a_{22}b_{22} \Big) - \Big(a_{12}b_{21} + a_{22}b_{22} \Big) &= a_{21}b_{12} - a_{12}b_{21} \\
&= -\Big(a_{12}b_{21} - a_{21}b_{12} \Big) .
\end{aligned}
$$

By the requirement $AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$
a_{12}b_{21} - a_{21}b_{12} = 1 = -\Big(a_{12}b_{21} - a_{21}b_{12} \Big) .
$$

Hence

$$a_{12}b_{21} - a_{21}b_{12} \;=\; -\Big(a_{12}b_{21} - a_{21}b_{12} \Big) ,$$

and

$$a_{12}b_{21} - a_{21}b_{12} \;=\; 0.$$

This is absurd, because at the same time we have

$$a_{12}b_{21} - a_{21}b_{12} \;=\; 1.$$

⋆ **An alternative solution for [X].**

Assume $AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, to derive a contradiction. Take the trace of the both sides, and

$$(*) \qquad \operatorname{Tr}\big(AB - BA\big) \;=\; \operatorname{Tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

The right-hand side of $(*)$ equals 2, whereas the left-hand side of $(*)$ equals

$$\begin{aligned} \operatorname{Tr}\big(AB - BA\big) &= \operatorname{Tr}\Big(AB + \big(- BA\big)\Big) \\ &= \operatorname{Tr}\big(AB\big) + \operatorname{Tr}\big(- BA\big) \\ &= \operatorname{Tr}\big(AB\big) - \operatorname{Tr}\big(BA\big) \\ &= \operatorname{Tr}\big(AB\big) - \operatorname{Tr}\big(AB\big) \;=\; 0, \end{aligned}$$

where we have used the result of [IX]. Thus we arrive at a contradiction.