Math 290 ELEMENTARY LINEAR ALGEBRA

PROGRESS CHECK – VI

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- **Determinants.**

Now we are ready to extensively discuss the topic of determinants.

- In Pg. Ch. V, we have examined the "solvability" of the matrix equation $AX = I$, mainly when A is in 2×2, and also in 3×3:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- In the course, we have found that there naturally emerges a scalar quantity Δ, which depends on

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} :$$

(i) For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\Delta = ad - bc$

(see Pg. Ch. V, Key Example 4 and Key Formula 1).

(ii) For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

$$\Delta = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

(see Pg. Ch. V, Example 9 and Formula 13).

• We have observed that this Δ completely decides the solvability of the equation $AX = I$: Namely, $\Delta \neq 0$ ensures the existence of a unique solution $X = A^{-1}$, whereas $\Delta = 0$ ensures the non-existence of a solution X. We have agreed that, in view of this, it makes sense to isolate the quantity Δ, and give it a name — the determinant of A, and denote it $\Delta = \det A$. Thus

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc,$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \; .$$

• In the present lecture, we would like to find the $n \times n$ analog of the determinant $\det A$ for an arbitrary n greater than 3: $n = 4, 5, 6, \cdots$, as announced in Pg. Ch. V, pages 34–36. Most importantly, we should keep in mind the following:

• Recall that the following is true in general (not just in the 2×2 and 3×3 cases): For a square $n \times n$ matrix A, the equation $AX = I$ either has a unique solution X, or has no solution X. (See Pg. Ch. V, Formula 22.) Recall that, if the equation $AX = I$ has a solution X, then we call it the inverse of A, write it $X = A^{-1}$. (See Pg. Ch. V, Definition, page 42.) Thus, the ultimate goal of the present lecture is to offer the invertibility criterion of A — we define $\det A$ for a square $n \times n$ matrix A "suitably" so that $\det A \neq 0$ is the exact ("if and only if") condition for the invertibility of A.

• Here is a more specific list of properties the operator det is required to possess:

Required Properties.

(i) $\det A \neq 0$ provides a criterion for a square matrix A to be invertible, namely, $\det A \neq 0$ if and only if A^{-1} exists,

(ii) det possesses the "row-column symmetry", namely, it satisfies

$$\det A \;=\; \det\Big(A^T\Big),$$

(iii) $\det A$ possesses the so-called " multi-linearity " :

(iii–a)

$$\det\Big[\boldsymbol{a}_1 \quad \cdots \quad \boldsymbol{a}_{i-1} \quad \Big(\boldsymbol{a}_i + \boldsymbol{b}_i\Big) \quad \boldsymbol{a}_{i+1} \quad \cdots \quad \boldsymbol{a}_n\Big]$$
$$= \quad \det\Big[\boldsymbol{a}_1 \quad \cdots \quad \boldsymbol{a}_{i-1} \quad \boldsymbol{a}_i \quad \boldsymbol{a}_{i+1} \quad \cdots \quad \boldsymbol{a}_n\Big]$$
$$+ \;\det\Big[\boldsymbol{a}_1 \quad \cdots \quad \boldsymbol{a}_{i-1} \quad \boldsymbol{b}_i \quad \boldsymbol{a}_{i+1} \quad \cdots \quad \boldsymbol{a}_n\Big],$$

in other words, if the three matrices A, B and C consist of identical columns except for the column i, and moreover the column i of the matrix A and the column i of the matrix B add up to the column i of the matrix C, then $\det C \;=\; \det A \,+\, \det B$,

(iii–b)

$$\det\Big[\boldsymbol{a}_1 \quad \cdots \quad \boldsymbol{a}_{i-1} \quad \Big(s\boldsymbol{a}_i\Big) \quad \boldsymbol{a}_{i+1} \quad \cdots \quad \boldsymbol{a}_n\Big]$$
$$= \quad s\;\det\Big[\boldsymbol{a}_1 \quad \cdots \quad \boldsymbol{a}_{i-1} \quad \boldsymbol{a}_i \quad \boldsymbol{a}_{i+1} \quad \cdots \quad \boldsymbol{a}_n\Big],$$

in other words, if the two matrices A and B consist of identical columns except for the column i, and moreover s times the column i of the matrix A equals the column i of the matrix B, then $\det B \;=\; s\,\det A$,

(iii–c) the same as (iii–a) holds for rows instead of columns,

(iii–d) the same as (iii–b) holds for rows instead of columns,

(iv) det possesses the compatibility property with the multiplication of two matrices:

$$\det\Big(A\,B\Big) \;=\; \Big(\det A\Big)\Big(\det B\Big)\;,$$

(v) det evaluated at the identity matrix I is equal to 1:

$$\det I \;=\; 1.$$

- To achieve our goal, we need to introduce the following new concept:

- **Permutations.**

  A permutation of the set $\{1,\ 2,\ \cdots,\ n\}$ is an arrangement of these numbers in some order, without omissions or repetitions.

Example 1. Consider the set $\{1, 2\}$. There are exactly 2 permutations:

$$(1,\ 2), \quad (2,\ 1).$$

Example 2. Consider the set $\{1, 2, 3\}$. There are exactly 6 permutations:

$$(1,\ 2,\ 3), \quad (1,\ 3,\ 2), \quad (2,\ 1,\ 3), \quad (2,\ 3,\ 1), \quad (3,\ 1,\ 2), \quad (3,\ 2,\ 1).$$

Example 3. Consider the set $\{1, 2, 3, 4\}$. There are exactly 24 permutations:

$$\begin{array}{llll}
(1,2,3,4), & (2,1,3,4), & (3,1,2,4), & (4,1,2,3), \\
(1,2,4,3), & (2,1,4,3), & (3,1,4,2), & (4,1,3,2), \\
(1,3,2,4), & (2,3,1,4), & (3,2,1,4), & (4,2,1,3), \\
(1,3,4,2), & (2,3,4,1), & (3,2,4,1), & (4,2,3,1), \\
(1,4,2,3), & (2,4,1,3), & (3,4,1,2), & (4,3,1,2), \\
(1,4,3,2), & (2,4,3,1), & (3,4,2,1), & (4,3,2,1).
\end{array}$$

- More generally, for the set $\{1, 2, \cdots, n\}$, there are exactly

$$\boxed{n! \ = \ n\,(n-1) \cdot \cdots \cdot 2 \cdot 1}$$

permutations.

Example 4. The set $\{1, 2, 3, 4, 5\}$ has

$$5! \ = \ 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ = \ 120$$

permutations.

Example 5. The set $\{1, 2, 3, 4, 5, 6\}$ has

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

permutations.

Example 6. The set $\{1, 2, 3, 4, 5, 6, 7\}$ has

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

permutations.

- **The number of inversions.**

In order to define the determinant det, we specifically need the following:

A <u>general</u> permutation of $\{1, 2, \cdots, n\}$ is written as

$$(j_1, j_2, \cdots, j_n).$$

We say that an <u>inversion</u> takes place in $(j_1, j_2, \cdots, j_n)$ at j_i, <u>when there is j_k further right to j_i, and $j_k < j_i$</u>.

Example 7. Consider the permutation $(2, 1, 3, 4)$. An inversion takes place at 2 in this permutation, since 1 is further right to 2. This is the only inversion taking place in this permutation.

Example 8. Consider the permutation $(2, 4, 3, 1)$. An inversion takes place at 2 in this permutation, since 1 is further right to 2. Another inversion takes place at 4 in the same permutation, since 1 and 3 are both further right to 4. Also, an inversion takes place at 3 in the permutation, since 1 is further right to 3. Therefore, there are four inversions taking place in $(2, 4, 3, 1)$ in total.

- For a general permutation $(j_1, j_2, \cdots, j_n)$, we may find the total number of inversions as follows:

Step 1: Count the number of j_k's further right to the $\boxed{j_1}$ with $j_k < j_1$.

Step 2: Count the number of j_k's further right to the $\boxed{j_2}$ with $j_k < j_2$.

$\cdots \quad \cdots$

Step $(\boldsymbol{n-1})$**:** Count the number of j_k's further right to the $\boxed{j_{n-1}}$ $\big($there is only one such$\big)$ with $j_k < j_{n-1}$ $\big($therefore our count is 0 or $1\big)$.

<u>Sum up those "subtotal counts" in Steps 1 – $(n-1)$</u>.

Example 9. Let us find <u>the total number of inversions</u> in the permutation

$$\big(5,\ 2,\ 1,\ 4,\ 3\big).$$

Step 1: Among $2,\ 1,\ 4,\ 3$ which are further right to 5, all are smaller numbers than 5. Therefore $\Big[$ The number of inversions at 5 $\Big] = \boxed{4}$.

Step 2: Among $1,\ 4,\ 3$ which are further right to 2, 1 is the only number smaller than 2. Therefore $\Big[$ The number of inversions at 2 $\Big] = \boxed{1}$.

Step 3: Among $4,\ 3$ which are further right to 1, there is no smaller number than 1. Therefore $\Big[$ The number of inversions at 4 $\Big] = \boxed{0}$.

Step 4: 3 is the only number sitting further right to 4, and it is smaller a number than 4. Therefore $\Big[$ The number of inversions at 4 $\Big] = \boxed{1}$.

Summing up the numbers of invresions in Steps 1–4, we find

$$4 + 1 + 0 + 1 = \boxed{6}$$

as the total numbers of inversions in $\big(5,\ 2,\ 1,\ 4,\ 3\big)$.

- Clearly $\big(1,\ 2,\ \cdots,\ n\big)$ has 0 as the total number of inversions. Any other permutations of $\big\{1,\ 2,\ \cdots,\ n\big\}$ has a positive number of inversions.

[I] Find the number of inversions in each of the following permutations:

(1) $\big(3,\, 1,\, 2\big)$, (2) $\big(1,\, 4,\, 3,\, 2\big)$, (3) $\big(4,\, 6,\, 1,\, 3,\, 2,\, 5\big)$,

(4) $\big(10,\, 9,\, 8,\, 7,\, 6,\, 5,\, 4,\, 3,\, 2,\, 1\big)$.

- **Elementary products.**

Consider a <u>general</u> square matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$. An <u>elementary product</u> of entries of A associated to the permutation $\big(j_1,\, j_2,\, \cdots,\, j_n\big)$ is

$$\boxed{a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}} \, .$$

Example 10. There are <u>two</u> elementary products for the general 2×2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, which are

$$a_{11}a_{22}, \qquad a_{12}a_{21}.$$

- These correspond to the permutations listed below $\big($in the respective order$\big)$:

$$\big(1,\, 2\big), \qquad \big(2,\, 1\big).$$

Example 11. There are <u>six</u> elementary products for the general 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, which are

$$\begin{array}{lll} a_{11}a_{22}a_{33} & a_{12}a_{21}a_{33}, & a_{13}a_{21}a_{32}, \\ a_{11}a_{23}a_{32} & a_{12}a_{23}a_{31}, & a_{13}a_{22}a_{31}. \end{array}$$

- These correspond to the permutations listed below $\big($in the respective order$\big)$:

$$\begin{array}{lll} \big(1,\, 2,\, 3\big), & \big(2,\, 1,\, 3\big), & \big(3,\, 1,\, 2\big), \\ \big(1,\, 3,\, 2\big), & \big(2,\, 3,\, 1\big), & \big(3,\, 2,\, 1\big). \end{array}$$

Example 12. There are <u>twentyfour</u> elementary products for the general 4×4 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, which are

$$\begin{array}{llll}
a_{11}a_{22}a_{33}a_{44}, & a_{12}a_{21}a_{33}a_{44}, & a_{13}a_{21}a_{32}a_{44}, & a_{14}a_{21}a_{32}a_{43}, \\
a_{11}a_{22}a_{34}a_{43}, & a_{12}a_{21}a_{34}a_{43}, & a_{13}a_{21}a_{34}a_{42}, & a_{14}a_{21}a_{33}a_{42}, \\
a_{11}a_{23}a_{32}a_{44}, & a_{12}a_{23}a_{31}a_{44}, & a_{13}a_{22}a_{31}a_{44}, & a_{14}a_{22}a_{31}a_{43}, \\
a_{11}a_{23}a_{34}a_{42}, & a_{12}a_{23}a_{34}a_{41}, & a_{13}a_{22}a_{34}a_{41}, & a_{14}a_{22}a_{33}a_{41}, \\
a_{11}a_{24}a_{32}a_{43}, & a_{12}a_{24}a_{31}a_{43}, & a_{13}a_{24}a_{31}a_{42}, & a_{14}a_{23}a_{31}a_{42}, \\
a_{11}a_{24}a_{33}a_{42}, & a_{12}a_{24}a_{33}a_{41}, & a_{13}a_{24}a_{32}a_{41}, & a_{14}a_{23}a_{32}a_{41}.
\end{array}$$

- These correspond to the permutations listed below (in the respective order):

$$\begin{array}{llll}
(1,2,3,4), & (2,1,3,4), & (3,1,2,4), & (4,1,2,3), \\
(1,2,4,3), & (2,1,4,3), & (3,1,4,2), & (4,1,3,2), \\
(1,3,2,4), & (2,3,1,4), & (3,2,1,4), & (4,2,1,3), \\
(1,3,4,2), & (2,3,4,1), & (3,2,4,1), & (4,2,3,1), \\
(1,4,2,3), & (2,4,1,3), & (3,4,1,2), & (4,3,1,2), \\
(1,4,3,2), & (2,4,3,1), & (3,4,2,1), & (4,3,2,1).
\end{array}$$

- **Distinguishing elementary products from arbitrary products of entries.**

  We may use the following criterion to decide whether a given product of entries of the general matrix A is an elementary product or not.

- Let $\boxed{n}$ be the size of the square matrix A. Then the elementary product of A is

  (i) a product of $\boxed{n}$ out of n^2 entries of A, where

  (ii) those n entries as a whole <u>do not miss any row, any column</u>.

Example 13. Observe

$$A = \begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{bmatrix}.$$

The choice of the three highlighted entries a_{11}, a_{23}, a_{32} meets the conditions (i–ii). Hence $a_{11}a_{23}a_{32}$ _is_ an elementary product of A.

Example 14. Observe

$$A = \begin{bmatrix} \boxed{a_{11}} & a_{12} & \boxed{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{bmatrix}.$$

The highlighted entries a_{11}, a_{13}, a_{32} miss the second row. Hence $a_{11}a_{13}a_{32}$ _is not_ an elementary product of A.

Example 15. Observe

$$A = \begin{bmatrix} \boxed{a_{11}} & a_{12} & \boxed{a_{13}} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{bmatrix}.$$

There are _four_ highlighted entries a_{11}, a_{13}, a_{22}, a_{33}, whereas A is in size 3×3. Hence $a_{11}a_{13}a_{22}a_{33}$ _is not_ an elementary product of A.

• We are now close to our first goal — the definition of the determinant $\det A$ for the general $n \times n$ matrix A:

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The following "signed elementary products" are exactly what appear in the definition

$\big(\text{formula}\big)$ of $\quad \det A = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ as terms.

- **Signed elementary products.**

A signed elementary product of a general $n \times n$ matrix A as above is

$$\Big(-1 \Big)^{i} a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n},$$

where i is the number of inversions in the permutation $\big(j_1, j_2, \cdots, j_n\big)$. Therefore, the signed elementary product is either

$$+a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}, \qquad \text{or} \qquad -a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n},$$

depending on the parity of the number of inversions. Namely,

$$\begin{cases} + \; a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n} \\ \qquad \Big(\text{if the number of inversions for } \big(j_1, j_2, \cdots, j_n\big) \text{ is even}\Big), \\ \\ - \; a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n} \\ \qquad \Big(\text{if the number of inversions for } \big(j_1, j_2, \cdots, j_n\big) \text{ is odd}\Big). \end{cases}$$

Example 16. The two signed elementary products of $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ are

$$+a_{11}a_{22}, \qquad -a_{12}a_{21}.$$

Example 17. The six signed elementary products of $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ are

$$\begin{array}{lll} +a_{11}a_{22}a_{33} & -a_{12}a_{21}a_{33}, & +a_{13}a_{21}a_{32}, \\ -a_{11}a_{23}a_{32} & +a_{12}a_{23}a_{31}, & -a_{13}a_{22}a_{31}. \end{array}$$

Example 18. The twentyfour <u>signed</u> elementary products of $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ are

$$\begin{array}{llll}
+a_{11}a_{22}a_{33}a_{44}, & -a_{12}a_{21}a_{33}a_{44}, & +a_{13}a_{21}a_{32}a_{44}, & -a_{14}a_{21}a_{32}a_{43}, \\
-a_{11}a_{22}a_{34}a_{43}, & +a_{12}a_{21}a_{34}a_{43}, & -a_{13}a_{21}a_{34}a_{42}, & +a_{14}a_{21}a_{33}a_{42}, \\
-a_{11}a_{23}a_{32}a_{44}, & +a_{12}a_{23}a_{31}a_{44}, & -a_{13}a_{22}a_{31}a_{44}, & +a_{14}a_{22}a_{31}a_{43}, \\
+a_{11}a_{23}a_{34}a_{42}, & -a_{12}a_{23}a_{34}a_{41}, & +a_{13}a_{22}a_{34}a_{41}, & -a_{14}a_{22}a_{33}a_{41}, \\
+a_{11}a_{24}a_{32}a_{43}, & -a_{12}a_{24}a_{31}a_{43}, & +a_{13}a_{24}a_{31}a_{42}, & -a_{14}a_{23}a_{31}a_{42}, \\
-a_{11}a_{24}a_{33}a_{42}, & +a_{12}a_{24}a_{33}a_{41}, & -a_{13}a_{24}a_{32}a_{41}, & +a_{14}a_{23}a_{32}a_{41}.
\end{array}$$

- **The definition of determinants.**

  Now we come to the definition of determinants.

Definition. Define <u>the determinant</u> of the general $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

as <u>the sum of all the signed elementary products</u>. We write it

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} .$$

- For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we have

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

• Note that we often write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ instead of $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ for the general 2×2 matrix. Accordingly,

$$\det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

• For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we have

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32}$$
$$-a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}.$$

• For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, we have

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\begin{aligned} = \quad & a_{11}a_{22}a_{33}a_{44} - a_{12}a_{21}a_{33}a_{44} + a_{13}a_{21}a_{32}a_{44} - a_{14}a_{21}a_{32}a_{43} \\ & -a_{11}a_{22}a_{34}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{13}a_{21}a_{34}a_{42} + a_{14}a_{21}a_{33}a_{42} \\ & -a_{11}a_{23}a_{32}a_{44} + a_{12}a_{23}a_{31}a_{44} - a_{13}a_{22}a_{31}a_{44} + a_{14}a_{22}a_{31}a_{43} \\ & +a_{11}a_{23}a_{34}a_{42} - a_{12}a_{23}a_{34}a_{41} + a_{13}a_{22}a_{34}a_{41} - a_{14}a_{22}a_{33}a_{41} \\ & +a_{11}a_{24}a_{32}a_{43} - a_{12}a_{24}a_{31}a_{43} + a_{13}a_{24}a_{31}a_{42} - a_{14}a_{23}a_{31}a_{42} \\ & -a_{11}a_{24}a_{33}a_{42} + a_{12}a_{24}a_{33}a_{41} - a_{13}a_{24}a_{32}a_{41} + a_{14}a_{23}a_{32}a_{41}. \end{aligned}$$

• These are exactly what we used to write as Δ. Agree that the above $\det A$ matches Δ in Pg. Ch. V, pages 34–35.

Example 19. Let us compute $\det A$ for

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

From the definition,

$$\begin{aligned} \det A &= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} \\ &= 1 \cdot 5 \cdot 9 - 2 \cdot 4 \cdot 9 + 3 \cdot 4 \cdot 8 - 1 \cdot 6 \cdot 8 + 2 \cdot 6 \cdot 7 - 3 \cdot 5 \cdot 7 \\ &= 45 - 72 + 96 - 48 + 84 - 105 = 0. \end{aligned}$$

- **Elementary row reductions and determinants.**

In finding the determinant of a given square matrix, generally it is not wise to just apply the above definition. Indeed, there is a theory that allows us to reduce the matrix to another, simpler, matrix while keeping track of its determinant value.

Formula 1. Let A and B be $n \times n$ matrices.

(1) Suppose A and B are identical except the row j, and the row j of B is equal to the row j of A plus s times the row i of A:

$$A = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boxed{\boldsymbol{a}_j} \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}, \qquad B = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boxed{\boldsymbol{a}_j + s\,\boldsymbol{a}_i} \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}.$$

Then

$$\boxed{\det B = \det A}.$$

(2) Suppose A and B are identical except the row j, and the row j of B is equal to s times the row j of A:

$$A = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boxed{\boldsymbol{a}_j} \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}, \qquad B = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boxed{s\,\boldsymbol{a}_j} \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}.$$

Then

$$\boxed{\det B = s \det A}\ .$$

(3) Suppose A and B are identical except the rows i and j, and the row j of B is the row i of A, and the row i of B is the row j of A:

$$A = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boxed{\boldsymbol{a}_i} \\ \vdots \\ \boxed{\boldsymbol{a}_j} \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}, \qquad B = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boxed{\boldsymbol{a}_j} \\ \vdots \\ \boxed{\boldsymbol{a}_i} \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}.$$

Then

$$\boxed{\det B = -\det A}\ .$$

Example 20. We have

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + s\,a_{11} & a_{22} + s\,a_{12} & a_{23} + s\,a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ s\,a_{31} & s\,a_{32} & s\,a_{33} \end{bmatrix} = s \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\det \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} = -\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

- The same formula as in Formula 1 hold for <u>columns</u>, instead of rows:

Formula 2. Let A and B be $n \times n$ matrices.

(1) Suppose A and B are identical except the column j, and the column j of B is equal to the column j of A plus s times the column i of A. Then

$$\boxed{\det B = \det A}.$$

(2) Suppose A and B are identical except the column j, and the column j of B is equal to s times the column j of A. Then

$$\boxed{\det B = s \det A}.$$

(3) Suppose A and B are identical except the columns i and j, and the column j of B is the column i of A, and the column i of B is the column j of A. Then

$$\boxed{\det B = -\det A}.$$

Example 21. We have

$$\det \begin{bmatrix} a_{11} + s\,a_{12} & a_{12} & a_{13} \\ a_{21} + s\,a_{22} & a_{22} & a_{23} \\ a_{31} + s\,a_{32} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\det \begin{bmatrix} s\,a_{11} & a_{12} & a_{13} \\ s\,a_{21} & a_{22} & a_{23} \\ s\,a_{31} & a_{32} & a_{33} \end{bmatrix} = s \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\det \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix} = -\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Formula 3. (1) If the matrix A contains a row (or a column) which consists entirely of zeroes, then

$$\boxed{\det A = 0} \, .$$

(2) If the matrix A contains two rows (or two columns) which are <u>identical</u>, then

$$\boxed{\det A = 0} \, .$$

(3) More generally, if the matrix A contains two rows (or two columns) which are <u>proportionate</u>, then

$$\boxed{\det A = 0} \, .$$

Example 22. $$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} = 0,$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{bmatrix} = 0,$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ s\,a_{11} & s\,a_{12} & s\,a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & s\,a_{12} \\ a_{21} & a_{22} & s\,a_{22} \\ a_{31} & a_{32} & s\,a_{32} \end{bmatrix} = 0.$$

- The following explains the logical dependence of Formulas 1–3. (This is given for the completeness sake.)

- **Logical dependence of Formulas 1–3.**

We may first prove Formula 3 (2) from the definition of the determinant. Indeed, in the formula of the determinant of the general matrix A, by substituting one entire row i with another entire row j of the original matrix A, we find a complete paring of identical terms with mutually opposing signs. For example, in the general 3×3 matrix, we may replace the entire second row with the original entire first row and

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\,a_{12}\,a_{33} - a_{12}\,a_{11}\,a_{33} + a_{13}\,a_{11}\,a_{32}$$
$$- a_{11}\,a_{13}\,a_{32} + a_{12}\,a_{13}\,a_{31} - a_{13}\,a_{12}\,a_{31}$$
$$= 0.$$

We may then prove Formula 1 (2) also from the definition of the determinant. Indeed, in the formula of the determinant of the general matrix A, by multiplying one entire row i by a scalar s, we find that each term (elementary product) is multiplied by s. Consequently, the determinant is multiplied by s. For example, in the general 3×3 matrix, we may multiply the entire second row with s, and

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ s\,a_{21} & s\,a_{22} & s\,a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
$$= a_{11}\left(s\,a_{22}\right)a_{33} - a_{12}\left(s\,a_{21}\right)a_{33} + a_{13}\left(s\,a_{21}\right)a_{32}$$
$$- a_{11}\left(s\,a_{23}\right)a_{32} + a_{12}\left(s\,a_{23}\right)a_{31} - a_{13}\left(s\,a_{22}\right)a_{31}$$
$$= s\left(a_{11}\,a_{22}\,a_{33}\right) - s\left(a_{12}\,a_{21}\,a_{33}\right) + s\left(a_{13}\,a_{21}\,a_{32}\right)$$
$$- s\left(a_{11}\,a_{23}\,a_{32}\right) + s\left(a_{12}\,a_{23}\,a_{31}\right) - s\left(a_{13}\,a_{22}\,a_{31}\right)$$
$$= s\,\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Now, notice that Formula 3 (1) is a special case of Formula 1 (2). Indeed, in Formula 1 (2), set $s = 0$. For example, in the general 3×3 matrix, we may replace the entire second row with 0, and

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 \cdot a_{21} & 0 \cdot a_{22} & 0 \cdot a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
$$= 0 \cdot \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0.$$

Also, we may deduce Formula 3 (3) easily from Formula 1 (2) and Formula 3 (2). For example, in the general 3×3 matrix, we may replace the entire second row with s times the original entire first row, and

$$\begin{aligned}
\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ s\,a_{11} & s\,a_{12} & s\,a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= s \cdot \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&= s \cdot 0 \\
&= 0.
\end{aligned}$$

Now, we may prove the property (iii–c), listed as a part of Required Properties, in page 3, based on the same idea as Formula 1 (2). For example, in the general 3×3 matrix, we may add the entire second row with an extra row $\begin{bmatrix} b_{21} & b_{22} & b_{23} \end{bmatrix}$, and

$$\begin{aligned}
&\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
= \;& a_{11} \left(a_{22} + b_{22} \right) a_{33} - a_{12} \left(a_{21} + b_{21} \right) a_{33} + a_{13} \left(a_{21} + b_{21} \right) a_{32} \\
& - a_{11} \left(a_{23} + b_{23} \right) a_{32} + a_{12} \left(a_{23} + b_{23} \right) a_{31} - a_{13} \left(a_{22} + b_{22} \right) a_{31} \\
= \;& \left(a_{11}\, a_{22}\, a_{33} \right) - \left(a_{12}\, a_{21}\, a_{33} \right) + \left(a_{13}\, a_{21}\, a_{32} \right) \\
& - \left(a_{11}\, a_{23}\, a_{32} \right) + \left(a_{12}\, a_{23}\, a_{31} \right) - \left(a_{13}\, a_{22}\, a_{31} \right) \\
& + \left(a_{11}\, b_{22}\, a_{33} \right) - \left(a_{12}\, b_{21}\, a_{33} \right) + \left(a_{13}\, b_{21}\, a_{32} \right) \\
& - \left(a_{11}\, b_{23}\, a_{32} \right) + \left(a_{12}\, b_{23}\, a_{31} \right) - \left(a_{13}\, b_{22}\, a_{31} \right) \\
= \;& \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} .
\end{aligned}$$

Now, we may deduce Formula 1 (1) easily from the property (iii–c), Formula 1 (2), and Formula 3 (2). For example, in the general 3×3 matrix, we may add s times the first row to the original second row, and

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + s\,a_{11} & a_{22} + s\,a_{12} & a_{23} + s\,a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ s\,a_{11} & s\,a_{12} & s\,a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + s \cdot \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + s \cdot 0$$

$$= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Finally, we may deduce Formula 1 (3) as follows. In the general matrix A, add the entire row i to the original entire row j. Call it B. Then add $\left(-1\right)$ times the entire row j of B to the row i of B. Call it C. Then add the entire row i of C to the row j of C. Call it D. Then multiply $\left(-1\right)$ to the entire row i of D. Call it F. It is easy to check that F is obtained by interchanging the row i and the row j of the original matrix A. By Formula 1 (1–2), we have

$$\det A = \det B = \det C = \det D = -\det F.$$

For example, in the general 3×3 matrix, we may modify the matrix while preserving the determinants, as follows:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} + a_{21} & a_{12} + a_{22} & a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} - \big(a_{11} + a_{21}\big) & a_{12} - \big(a_{12} + a_{22}\big) & a_{13} - \big(a_{13} + a_{23}\big) \\ a_{11} + a_{21} & a_{12} + a_{22} & a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det \begin{bmatrix} -a_{21} & -a_{22} & -a_{23} \\ a_{11} + a_{21} & a_{12} + a_{22} & a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det \begin{bmatrix} -a_{21} & -a_{22} & -a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

This last determinant equals

$$-\det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Proof of Formula 2 is completely parallel to the proof of Formula 1. We may simply replace "rows" with "columns". In the course we prove the property (iii-a) listed in page 3.

Example 23. $$\det \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 5 & 6 & -7 \end{bmatrix} = 0.$$

Notice that the second row of the determinant consists entirely of 0. Hence Formula 3 (1) applies.

Example 24. $$\det \begin{bmatrix} -4 & 3 & 2 \\ 8 & 0 & 0 \\ -4 & 3 & 2 \end{bmatrix} = 0.$$

Notice that the first and the third rows of the determinant are identical. Hence Formula 3 (2) applies.

Example 25. $$\det \begin{bmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{bmatrix} = - \det \begin{bmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{bmatrix}.$$

Notice that the difference between the two determinants is that the second and the third columns are interchanged. Hence Formula 2 (3) applies.

Example 26. $$\det \begin{bmatrix} 5 & 0 & 10 \\ 25 & -30 & 40 \\ -15 & 5 & 20 \end{bmatrix} = 5^3 \det \begin{bmatrix} 1 & 0 & 2 \\ 5 & -6 & 8 \\ -3 & 1 & 4 \end{bmatrix}.$$

Notice that the difference between the two determinants is that each of the first, the second, and the third rows in the first determinant is obtained by multiplying 5 to the first, the second, and the third rows in the second determinant. Hence Formula 1 (2) applies.

Example 27. $$\det \begin{bmatrix} 1 & -3 & 2 \\ 5 & 2 & -1 \\ -1 & 0 & 6 \end{bmatrix} = \det \begin{bmatrix} 1 & -3 & 2 \\ 0 & 17 & -11 \\ -1 & 0 & 6 \end{bmatrix}.$$

Notice that the difference between the two determinants is that the second row in the second determinant is obtained by adding (-5) times the first row to the second row in the first determinant. Hence Formula 1 (1) applies.

[II] Explain which (combination of) formula(s) among Formula 1 (1), (2), (3), Formula 2 (1), (2), (3), Formula 3 (1), (2), (3) will be needed to establish each of the following identities:

(1) $$\det \begin{bmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{bmatrix} = - \det \begin{bmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{bmatrix}.$$

(2) $$\det \begin{bmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 9 \end{bmatrix} = 12 \det \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{bmatrix}.$$

(3) $$\det \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 6^4 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- **Evaluation of determinants.**

The general strategy to evaluate the determinant of a concrete matrix is to apply elementary row operations (and often elementary column operations) to reduce the given matrix to another matrix having many 0s, while keeping track of the change of the determinant in each of the operations as prescribed in Formulas 1–3. As for this, it is extremely useful to rely on the following notion.

Definition. A square matrix of form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

is called an upper triangular matrix. Thus, an upper triangular matrix is a square matrix whose entries below the main diagonal are all 0. Similarly, a square matrix of form

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

is called a lower triangular matrix. Thus, a lower triangular matrix is a square matrix whose entries above the main diagonal are all 0. We call a matrix triangular, if it is either upper or lower triangular.

Example 28.

(1a) A general upper triangular 2×2 matrix is $\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$, or $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

(1b) A general lower triangular 2×2 matrix is $\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$, or $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$.

(2a) A general upper triangular 3×3 matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$.

(2b) A general lower triangular 3×3 matrix is $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

(3a) A general upper triangular 4×4 matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{14} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$.

(3b) A general lower triangular 4×4 matrix is $\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$.

Formula 4. The determinant of the general triangular matrix is the product of its diagonal entries :

(1)
$$\boxed{\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}} ,$$

(2)
$$\boxed{\det \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}} .$$

- For example,

$$\det\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = \det\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} \ ,$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = \det\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} \ ,$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} = \det\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}a_{22}a_{33}a_{44} \ .$$

- It is worth highlighting a special case of Formula 4. If a square matrix A is upper triangular and at the same time lower triangular, then it must look like

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} .$$

This is nothing else but a diagonal matrix . Formula 4 for the diagonal matrices is highlighted as follows:

Formula 5. The determinant of the general diagonal matrix is the product of its diagonal entries :

$$\boxed{\det\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}} \ .$$

- For example,

$$\det \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = a_{11}a_{22} ,$$

$$\det \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} ,$$

$$\det \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} = a_{11}a_{22}a_{33}a_{44} .$$

- **Verification of Formulas 4, 5.**

Let us verify Formula 4 (1). In the formula of the determinant of the general matrix A, if we substitute all the entries a_{ij} for $i > j$ with 0, then all the elementary products except $a_{11}\,a_{22}\cdots a_{nn}$ contain 0 as a factor. In other words, the only possibly non-zero term in the formula for $\det A$ is $a_{11}\,a_{22}\cdots a_{nn}$. Consequently, $\det A$ equals $a_{11}\,a_{22}\cdots a_{nn}$. Verification of Formula 4 (2) is entirely parallel. Finally, Formula 5 is a special case of Formula 4.

- With Formulas 1–5 in our hand, we have a good command to evaluate determinants.

Example 29. $$\det \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix} = 2 \cdot 3 \cdot \big(-5\big) = -30.$$

Notice that the determinant is upper triangular. Hence we have applied Formula 4 (1).

Example 30. $$\det \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix} = 5 \cdot 6 \cdot \big(-3\big) = -90.$$

Notice that the determinant is diagonal. Hence we have applied Formula 5.

[III] Use Formulas 4, 5 to evaluate the following determinants:

(1) $\det \begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$.   (2) $\det \begin{bmatrix} 5 & 8 & -4 & 2 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

(3) $\det \begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \\ 3 & 5 & 3 & 0 \\ -8 & 7 & 0 & -2 \end{bmatrix}$.

• **The general case.**

Example 31. Let us evaluate $\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{bmatrix}$.

We may proceed as follows:

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & -3 & -2 \end{bmatrix}$$

$\Big[\ \big(-2\big)$ times $\big($row 1$\big)$ was added to $\big($row 2$\big)$;
$\big(-1\big)$ times $\big($row 1$\big)$ was added to $\big($row 3$\big)\ \Big]$

$$= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$\big(-1\big)$ times $\big($row 2$\big)$ was added to $\big($row 3$\big)\ \Big]$

$$= 1 \cdot \big(-3\big) \cdot 2$$

$$= -6.$$

Example 32. Let us evaluate $\det \begin{bmatrix} 3 & -1 & -3 \\ -1 & -4 & -2 \\ 3 & -1 & -1 \end{bmatrix}$.

We may proceed as follows:

$$\det \begin{bmatrix} 3 & -1 & -3 \\ -1 & -4 & -2 \\ 3 & -1 & -1 \end{bmatrix} = -\det \begin{bmatrix} 3 & -1 & -3 \\ 1 & 4 & 2 \\ 3 & -1 & -1 \end{bmatrix}$$

$\Big[\ \big(-1\big)$ was multiplied to $\big($row 2$\big) \ \Big]$

$$= -\det \begin{bmatrix} 3 & -1 & -3 \\ 1 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Big[\ \big(-1\big)$ times $\big($row 1$\big)$ was added to $\big($row 3$\big) \ \Big]$

$$= -\det \begin{bmatrix} 0 & -1 & -3 \\ 13 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Big[$ 3 times $\big($column 2$\big)$ was added to $\big($column 1$\big) \ \Big]$

$$= \det \begin{bmatrix} 13 & 4 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Big[\ \big($row 1$\big)$ and $\big($row 2$\big)$ were interchanged $\Big]$

$$= 13 \cdot \big(-1\big) \cdot 2$$

$$= -26.$$

Example 33. Let us evaluate $\det \begin{bmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{bmatrix}$.

We may proceed as follows:

$$\det \begin{bmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{bmatrix} = \big(-7\big) \det \begin{bmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

$\Big[\ \big(-1/7\big)$ was multiplied to $\big($row 4$\big)\ \Big]$

$$= \big(-7\big) \det \begin{bmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 22 \\ -4 & 6 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\Big[$ 2 times $\big($column 1$\big)$ was added to $\big($column 4$\big)\ \Big]$

$$= \big(-7\big) \det \begin{bmatrix} 0 & -15 & 8 & 2 \\ 8 & -131 & -1 & 22 \\ -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\Big[\ \big(-6\big)$ times $\big($column 4$\big)$ was added to $\big($column 2$\big)\ \Big]$

$$= \big(-7\big) \det \begin{bmatrix} 0 & -1063 & 8 & 2 \\ 8 & 0 & -1 & 22 \\ -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\Big[\ \big(-131\big)$ times $\big($column 3$\big)$ was added to $\big($column 2$\big)\ \Big]$

$$= \ 7 \det \begin{bmatrix} -1063 & 0 & 8 & 2 \\ 0 & 8 & -1 & 22 \\ 0 & -4 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\Big[\ \big(\text{column } 1\big) \ \text{ and } \ \big(\text{column } 2\big) \ \text{ were interchanged } \ \Big]$

$$= \ \big(-7\big) \det \begin{bmatrix} -1063 & 8 & 0 & 2 \\ 0 & -1 & 8 & 22 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Big[\ \big(\text{column } 2\big) \ \text{ and } \ \big(\text{column } 3\big) \ \text{ were interchanged } \ \Big]$

$$= \ 7 \det \begin{bmatrix} -1063 & 8 & 2 & 0 \\ 0 & -1 & 22 & 8 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Big[\ \big(\text{column } 3\big) \ \text{ and } \ \big(\text{column } 4\big) \ \text{ were interchanged } \ \Big]$

$$= \ 7 \cdot \big(-1063\big) \cdot \big(-1\big) \cdot 1 \cdot 1$$

$$= \ 7441.$$

[IV] Evaluate the following determinants:

(1) $\det \begin{bmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{bmatrix}$. (2) $\det \begin{bmatrix} 4 & 3 & -2 \\ 5 & 4 & 1 \\ -2 & 3 & 4 \end{bmatrix}$.

(3) $\det \begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix}$.

[V] Evaluate the following determinants:

(1) $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (2) $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (3) $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}$.

[VI] Evaluate the following determinants:

(1) $\det \begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{bmatrix}$. (2) $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$.

(3) $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$. (4) $\det \begin{bmatrix} x & 0 & c \\ -1 & x & b \\ 0 & -1 & a \end{bmatrix}$.

Solution to problem [I–VI].

[I] The number of inversions of

(1) $(3, 1, 2)$ is $2 + 0 = 2$.

(2) $(1, 4, 3, 2)$ is $0 + 2 + 1 = 3$.

(3) $(4, 6, 1, 3, 2, 5)$ is $3 + 4 + 0 + 1 + 0 = 8$.

(4) $(10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$ is $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$.

[II] (1) The identity $\det \begin{bmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{bmatrix} = -\det \begin{bmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ holds true.
Indeed, the difference between the two determinants is that the first and the third rows are interchanged. Hence Formula 1 (3) applies.

(2) The identity $\det \begin{bmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 9 \end{bmatrix} = 12 \det \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{bmatrix}$ holds true.

Indeed, the difference between the two determinants is that the second, and the third, columns in the first determinant is obtained by multiplying 4 to the second column, and multiplying 3 to the third column, in the second determinant. Hence Formula 2 (2) applies.

(3) The identity $\det \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 6^4 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ holds true.

Indeed, the difference between the two determinants is that all four rows in the first determinant is obtained by multiplying 6 to all four rows in the second determinant. Hence Formula 1 (2) applies.

[III] (1) $\det \begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix} = \big(-3\big) \cdot 11 \cdot 2 = -66.$

(2) $\det \begin{bmatrix} 5 & 8 & -4 & 2 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = 5 \cdot 0 \cdot 2 \cdot \big(-1\big) = 0.$

(3) $\det \begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \\ 3 & 5 & 3 & 0 \\ -8 & 7 & 0 & -2 \end{bmatrix} = 4 \cdot \big(1/2\big) \cdot 3 \cdot \big(-2\big) = -12.$

[IV] (1) $\det \begin{bmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{bmatrix} = 223.$ (2) $\det \begin{bmatrix} 4 & 3 & -2 \\ 5 & 4 & 1 \\ -2 & 3 & 4 \end{bmatrix} = -60.$

(3) $\det \begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix} = 0.$

Notice that in (3), the third row in the determinant consists entirely of 0s.

[V] (1) $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix} = t.$ (2) $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1.$

(3) $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix} = 1.$

[VI] (1) Let us first assume $b = 0$, then

$$\det \begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{bmatrix} = \det \begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1+c \end{bmatrix}$$

$$= \det \begin{bmatrix} a & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & c \end{bmatrix}$$

$$= \det \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} = ac.$$

Next, assume $b \neq 0$. Then

$$\det \begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{bmatrix}$$

$$= \det \begin{bmatrix} 0 & -a & 1 - (1+a)(1+c) \\ 0 & b & -c \\ 1 & 1 & 1+c \end{bmatrix}$$

$$= \det \begin{bmatrix} 0 & -a & -a-c-ac \\ 0 & b & -c \\ 1 & 1 & 1+c \end{bmatrix}$$

$$= \det \begin{bmatrix} 0 & -a & (-ac/b) - a - c - ac \\ 0 & b & 0 \\ 1 & 1 & (c/b) + 1 + c \end{bmatrix}$$

$$= -\det\begin{bmatrix} -a & 0 & \left(-\,ac/b\right) - a - c - ac \\ b & 0 & 0 \\ 1 & 1 & \left(c/b\right) + 1 + c \end{bmatrix}$$

$$= \quad \det\begin{bmatrix} -a & \left(-\,ac/b\right) - a - c - ac & 0 \\ b & 0 & 0 \\ 1 & \left(c/b\right) + 1 + c & 1 \end{bmatrix}$$

$$= -\det\begin{bmatrix} b & 0 & 0 \\ -a & \left(-\,ac/b\right) - a - c - ac & 0 \\ 1 & \left(c/b\right) + 1 + c & 1 \end{bmatrix}$$

$$= -b \cdot \left(-\frac{ac}{b} - a - c - ac \right) \cdot 1$$

$$= ac + ab + bc + abc.$$

Hence we conclude that, regardless of whether $b = 0$ or not, we have

$$\det\begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{bmatrix} = ac + ab + bc + abc.$$

(2) $$\det\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & \left(c-a\right)\left(c-b\right) \end{bmatrix}$$

$$= \left(b-a\right)\left(c-a\right)\left(c-b\right) .$$

(3)
$$\det\begin{bmatrix}1&1&1\\a&b&c\\a^3&b^3&c^3\end{bmatrix} = \det\begin{bmatrix}1&1&1\\0&b-a&c-a\\0&b^3-a^3&c^3-a^3\end{bmatrix}$$

$$= \det\begin{bmatrix}1&1&1\\0&b-a&c-a\\0&0&c^3-a^3-(c-a)(b^2+ab+a^2)\end{bmatrix}$$

$$= \det\begin{bmatrix}1&1&1\\0&b-a&c-a\\0&0&(c-a)(c-b)(a+b+c)\end{bmatrix}$$

$$= \Big(b-a\Big)\Big(c-a\Big)\Big(c-b\Big)\Big(a+b+c\Big)\ .$$

(4)
$$\det\begin{bmatrix}x&0&c\\-1&x&b\\0&-1&a\end{bmatrix} = \det\begin{bmatrix}0&x^2&bx+c\\-1&x&b\\0&-1&a\end{bmatrix}$$

$$= \det\begin{bmatrix}0&x^2&ax^2+bx+c\\-1&x&ax+b\\0&-1&0\end{bmatrix}$$

$$= -\det\begin{bmatrix}0&ax^2+bx+c&x^2\\-1&ax+b&x\\0&0&-1\end{bmatrix}$$

$$= \det\begin{bmatrix}-1&ax+b&x\\0&ax^2+bx+c&x^2\\0&0&-1\end{bmatrix}$$

$$= (-1)\cdot\Big(ax^2+bx+c\Big)\cdot(-1)$$

$$= ax^2+bx+c.$$