Math 290 ELEMENTARY LINEAR ALGEBRA
PROGRESS CHECK – VII

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• Evaluation of determinants – II: Co-factor expansion.

In the previous lecture (Pg. Ch. VI, page 22—) we have examined how we could evaluate the determinant of a square matrix whose entries are concrete numbers, using elementary row and column operations . Such a method works by virtue of Formulas 1, 2 in Pg. Ch. VI, page 13–15. We have found that we could always rewrite the given determinant

$$\det A \;=\; \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

as "a scalar times a triangular determinant " , namely, either

$$\text{(i)}\quad \det \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$= \cdots$$

$$= \cdots$$

$$= \underbrace{\left(\boxed{\;*\;}\right)}_{\substack{\| \\ \text{some scalar}}} \cdot \underbrace{\det \begin{bmatrix} b_{11} & * & * & \cdots & * \\ 0 & b_{22} & * & \cdots & * \\ 0 & 0 & b_{33} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn} \end{bmatrix}}_{\substack{\| \\ b_{11}b_{22}b_{33}\,\cdots\, b_{nn}}} ,$$

or,

$$\text{(ii)}\quad \det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$= \cdots$$

$$= \cdots$$

$$= \underbrace{\left(\boxed{\;*\;}\right)}_{\substack{\| \\ \text{some scalar}}} \cdot \underbrace{\det\begin{bmatrix} c_{11} & 0 & 0 & \cdots & 0 \\ * & c_{22} & 0 & \cdots & 0 \\ * & * & c_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & c_{nn} \end{bmatrix}}_{\substack{\| \\ c_{11}c_{22}c_{33}\cdots c_{nn}}}.$$

Here, in each of (i), (ii) above, the scalar attached in front of the triangular determinant “typcally” comes out as a result of the row and/or the column reduction(s) we have applied to the determinant (see Formulas 1, 2 in Pg. Ch. VI, pages 13–15). Most importantly, this method takes advantage of the formulas on triangular determinants:

$$\det\begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & 0 & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn} \end{bmatrix} = b_{11} \cdot b_{22} \cdot b_{33} \cdot \cdots \cdot b_{nn},$$

and

$$\det \begin{bmatrix} c_{11} & 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & 0 & \cdots & 0 \\ c_{31} & c_{32} & c_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix} = c_{11} \cdot c_{22} \cdot c_{33} \cdot \cdots \cdot c_{nn}$$

(see Formula 4 in Pg. Ch. VI, page 23).

- In the present lecture, we explore another method of evaluating determinants, called $\underline{\text{ co-factor expansions }}$. Prior to the discussion, we introduce one notation:

Notation. From now on, $\underline{\text{ we will often use the notation }}$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$\underline{\text{ as an alternative of }}$ $\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$. Thus,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\,d - b\,c,$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31},$$

and

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\,b_2\,c_3 - a_1\,b_3\,c_2 - a_2\,b_1\,c_3 + a_2\,b_3\,c_1 + a_3\,b_1\,c_2 - a_3\,b_2\,c_1.$$

Formula 1. We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} .$$

- We may avoid double suffixes, and may rewrite Formula 1 as follows:

Formula 1′. We have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} .$$

- **Verification of Formulas 1, 1′.**

Since Formula 1 and Formula 1′ have the same content, it suffices to verify only Formula 1′.

By the definition of determinants, the left-hand side equals

$$a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 .$$

Meanwhile, the right-hand side equals

$$\begin{aligned} & a_1 \Big(b_2 c_3 - b_3 c_2 \Big) - a_2 \Big(b_1 c_3 - b_3 c_1 \Big) + a_3 \Big(b_1 c_2 - b_2 c_1 \Big) \\ = \;& a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 . \end{aligned}$$

These two quantities are identical. Hence the verification of Formula 1′, and hence of Formula 1, is complete.

- The content of Formula 1 (and Formula 1′) is as follows. The left-hand side of the formula is a determinant of a 3×3 matrix. The right-hand side of the formula is a linear combination of determinants of 2×2 matrices. Thus, the formula $\underline{\text{reduces the evaluation of a } 3 \times 3 \text{ determinant to evaluations of } 2 \times 2 \text{ determinants}}$.

Example 1. Let us evaluate the same determinant as Example 31 of Pg. Ch. VI, page 26, but this time using Formula 1 above.

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & -2 \\ -2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= 1 \cdot \Big[(-1) \cdot (-1) - (-2) \cdot (-2) \Big]$$

$$- 1 \cdot \Big[2 \cdot (-1) - (-2) \cdot 1 \Big]$$

$$+ 1 \cdot \Big[2 \cdot (-2) - (-1) \cdot 1 \Big]$$

$$= 1 \cdot (-3) - 1 \cdot 0 + 1 \cdot (-3)$$

$$= -6.$$

Example 2. $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

$$= a^2 \begin{vmatrix} b^2 & bc \\ bc & c^2 \end{vmatrix} - ab \begin{vmatrix} ab & bc \\ ac & c^2 \end{vmatrix} + ac \begin{vmatrix} ab & b^2 \\ ac & bc \end{vmatrix}$$

$$= a^2 \Big(b^2 \cdot c^2 - bc \cdot bc \Big) - ab \Big(ab \cdot c^2 - bc \cdot ac \Big) + ac \Big(ab \cdot bc - b^2 \cdot ac \Big)$$

$$= a^2 \Big(b^2c^2 - b^2c^2 \Big) - ab \Big(abc^2 - abc^2 \Big) + ac \Big(ab^2c - ab^2c \Big)$$

$$= a^2 \cdot 0 - ab \cdot 0 + ac \cdot 0$$

$$= 0.$$

Example 3. $\begin{vmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{vmatrix}$

$$= 0 \begin{vmatrix} 0 & p \\ -p & 0 \end{vmatrix} - r \begin{vmatrix} -r & p \\ q & 0 \end{vmatrix} + (-q) \begin{vmatrix} -r & 0 \\ q & -p \end{vmatrix}$$

$$= 0 \cdot \Big[0 \cdot 0 - p \cdot (-p)\Big] - r \cdot \Big[(-r) \cdot 0 - p \cdot q\Big]$$

$$+ (-q) \cdot \Big[(-r) \cdot (-p) - 0 \cdot q\Big]$$

$$= 0 \cdot p^2 - r \cdot \Big(-pq\Big) + \Big(-q\Big) \cdot \Big(pr\Big)$$

$$= 0 + pqr - pqr$$

$$= 0.$$

- **How to memorize Formulas 1, $1'$?**

Let us look at the identity in Formula 1 more closely:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

$\underline{\text{The first}}$ 2×2 determinant appearing in the right-hand side is

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

This is nothing but $\underline{\text{the “sub-determinant” of the original determinant obtained}}$ $\underline{\text{by eliminating the row and the column containing } a_{11}}$. Let us denote this by M_{11}:

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

Next, the second 2×2 determinant appearing in the right-hand side of the formula is

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} .$$

This is nothing but the “sub-determinant” of the original determinant obtained by eliminating the row and the column containing a_{12} . Let us denote this by M_{12}:

$$M_{12} \;=\; \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} .$$

Last, the third 2×2 determinant appearing in the right-hand side of the formula is

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} .$$

This is nothing but the “sub-determinant” of the original determinant obtained by eliminating the row and the column containing a_{13} . Let us denote this by M_{13}:

$$M_{13} \;=\; \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} .$$

- In short, using M_{11}, M_{12}, M_{13}, we may paraphrase Formula 1 as follows:

Formula 2a. We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \;=\; a_{11}\, M_{11} \;-\; a_{12}\, M_{12} \;+\; a_{13}\, M_{13},$$

where

$$M_{11} \;=\; \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{12} \;=\; \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{13} \;=\; \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} .$$

- In Formula 2a, we have looked at the three entries a_{11}, a_{12}, a_{13}. These are in the first row of the original determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

- We may instead look at the three entries in the second, or the third, row of the original determinant. We accordingly have the following two formulas:

Formula 2b. We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\, a_{21}\, M_{21} \;+\; a_{22}\, M_{22} \;-\; a_{23}\, M_{23},$$

where

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

Formula 2c. We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31}\, M_{31} \;-\; a_{32}\, M_{32} \;+\; a_{33}\, M_{33},$$

where

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \qquad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \qquad M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

- Again, we may avoid double suffixes and may rewrite Formulas 2a, 2b, 2c as follows:

Formula 2a′, 2b′, 2c′. We have

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= \; a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \\
&= \; c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.
\end{aligned}$$

- We have exactly the same type of formulas for columns, instead of rows:

Formula 3a. We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} M_{11} - a_{21} M_{21} + a_{31} M_{31},$$

where

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Formula 3b. We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - a_{12} M_{12} + a_{22} M_{22} - a_{32} M_{32},$$

where

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

Formula 3c. We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13}\,M_{13} - a_{23}\,M_{23} + a_{33}\,M_{33},$$

where

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \qquad M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

- To avoid double suffixes and rewrite Formulas 3a, 3b, 3c:

Formula 3a′, 3b′, 3c′. We have

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ &= -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ &= a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \end{aligned}$$

- **Minors and Co-factors.**

We call the 2×2 determinants

$$M_{11},\ M_{12},\ M_{13},\ M_{21},\ M_{22},\ M_{23},\ M_{31},\ M_{32},\ M_{33}$$

that appeared in Formulas 2a, 2b, 2c, 3a, 3b and 3c <u>the minors</u> of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

We notice that, the signs of the terms appearing in each of these six formulas are "alternating". Indeed, we may rewrite the six formulas in a more uniform fashion, if we introduce "the signed minors", or, the co-factors :

$$\begin{array}{lll} C_{11} = +M_{11}, & C_{12} = -M_{12}, & C_{13} = +M_{13}, \\ C_{21} = -M_{21}, & C_{22} = +M_{22}, & C_{23} = -M_{23}, \\ C_{31} = +M_{31}, & C_{32} = -M_{32}, & C_{33} = +M_{33}. \end{array}$$

Observe the pattern of signs for C_{ij} s against M_{ij} s:

+	−	+
−	+	−
+	−	+

Formula 4 (Co-factor expansion with respect to rows). We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \ ,$$

where

$$C_{11} = + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{13} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

$$C_{21} = - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{22} = + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{23} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

$$C_{31} = + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad C_{32} = - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad C_{33} = + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Formula 5 (Co-factor expansion with respect to columns). We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{aligned} &= a_{11}\,C_{11} + a_{21}\,C_{21} + a_{31}\,C_{31} \\ &= a_{12}\,C_{12} + a_{22}\,C_{22} + a_{32}\,C_{32} \\ &= a_{13}\,C_{13} + a_{23}\,C_{23} + a_{33}\,C_{33}\ , \end{aligned}$$

where

$$C_{11} = +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{13} = +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

$$C_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{22} = +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

$$C_{31} = +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad C_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad C_{33} = +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

- Again, there is a way to rewrite Formulas 4, 5 without using the double suffixes:

Formula 4′ (Co-factor expansion with respect to rows). We have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{aligned} &= a_1\,A_1 + a_2\,A_2 + a_3\,A_3 \\ &= b_1\,B_1 + b_2\,B_2 + b_3\,B_3 \\ &= c_1\,C_1 + c_2\,C_2 + c_3\,C_3\ , \end{aligned}$$

where

$$A_1 = +\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, \quad A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \quad A_3 = +\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

$$B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}, \quad B_2 = +\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}, \quad B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix},$$

$$C_1 = +\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad C_3 = +\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Formula 5′ (Co-factor expansion with respect to columns). We have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\,A_1 + b_1\,B_1 + c_1\,C_1$$

$$= a_2\,A_2 + b_2\,B_2 + c_2\,C_2$$

$$= a_3\,A_3 + b_3\,B_3 + c_3\,C_3\ ,$$

where

$$A_1 = +\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, \quad A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \quad A_3 = +\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

$$B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}, \quad B_2 = +\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}, \quad B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix},$$

$$C_1 = +\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad C_3 = +\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

[I] Re-do Pg. Ch. VI, problem [IV], (1), (2), using co-factor expansions:

(1) $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix}$. (2) $\begin{vmatrix} 4 & 3 & -2 \\ 5 & 4 & 1 \\ -2 & 3 & 4 \end{vmatrix}$.

[II] Re-do Pg. Ch. VI, problem [V], (1–3), using co-factor expansions:

(1) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{vmatrix}$. (2) $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$. (3) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{vmatrix}$.

[III] Re-do Pg. Ch. VI, problem [VI], (1–4), using co-factor expansions:

(1) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$. (2) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

(3) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$. (4) $\begin{vmatrix} x & 0 & c \\ -1 & x & b \\ 0 & -1 & a \end{vmatrix}$.

- **Co-factor expansion – The case of larger size determinants.**

We have exactly the same formula as Formulas 4, 5 for larger size determinants. First, we define the minors and co-factors.

Definition. Let us consider the general $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}$$

and its determinant

$$\det A = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}.$$

Note that in the above,

(i) the rows $1,\ i-1,\ i,\ i+1,\ n,$ and (ii) the columns $1,\ j-1,\ j,\ j+1,\ n,$

are highlighted. $\underline{\text{The } (i, j)\text{-th minor}}$ $M_{ij}(A)$ of the matrix A is the following $(n-1) \times (n-1)$ determinant:

$$M_{ij}(A) = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}.$$

In other words, the $(i,\, j)$-th minor $M_{ij}(A)$ of A is the determinant of the $(n-1) \times (n-1)$ matrix obtained by taking out the row i and the column j of the original matrix A.

Definition. Let A and $M_{ij}(A)$ be as above. The $(i,\, j)$-th co-factor $C_{ij}(A)$ of the matrix A is defined as

$$C_{ij}(A) \;=\; \Big(-1 \Big)^{i+j} M_{ij}(A).$$

- Agree that the above definitions of $M_{ij} = M_{ij}(A)$ and $C_{ij} = C_{ij}(A)$ coincide with the previous one we gave, when the original matrix A is 3×3. Below is the pattern of the signs for $C_{ij}(A)$ s against $M_{ij}(A)$ s:

$$\begin{array}{c|c|c|c|c}
+ & - & + & \cdots & + \\ \hline
- & + & - & \cdots & - \\ \hline
+ & - & + & \cdots & + \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
+ & - & + & \cdots & +
\end{array}
\qquad\qquad
\begin{array}{c|c|c|c|c}
+ & - & + & \cdots & - \\ \hline
- & + & - & \cdots & + \\ \hline
+ & - & + & \cdots & - \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
- & + & - & \cdots & +
\end{array}$$

$n \times n$, where n is odd $\qquad\qquad$ $n \times n$, where n is even

- Now we may state the larger size counterpart of Formulas 4, 5:

Formula 6 (Co-factor expansion with respect to rows). Let

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be the general $n \times n$ matrix. Let $C_{ij} = C_{ij}(A)$ be the co-factors of A, as defined above. Then

$$\begin{aligned}
\det A \;&=\; \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&=\; a_{11}\,C_{11} \;+\; a_{12}\,C_{12} \;+\; \cdots \;+\; a_{1n}C_{1n} \\
&=\; a_{21}\,C_{21} \;+\; a_{22}\,C_{22} \;+\; \cdots \;+\; a_{2n}C_{2n} \\
&=\; \cdots \quad \cdots \quad \cdots \\
&=\; a_{n1}\,C_{n1} \;+\; a_{n2}\,C_{n2} \;+\; \cdots \;+\; a_{nn}C_{nn} \;.
\end{aligned}$$

Formula 7 (Co-factor expansion with respect to columns). Let

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be the general $n \times n$ matrix. Let $C_{ij} = C_{ij}\big(A\big)$ be the co-factors of A, as defined above. Then

$$\begin{aligned}
\det A \;&=\; \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&=\; a_{11}\,C_{11} \;+\; a_{21}\,C_{21} \;+\; \cdots \;+\; a_{n1}C_{n1} \\
&=\; a_{12}\,C_{12} \;+\; a_{22}\,C_{22} \;+\; \cdots \;+\; a_{n2}C_{n2} \\
&=\; \cdots \quad \cdots \quad \cdots \\
&=\; a_{1n}\,C_{1n} \;+\; a_{2n}\,C_{2n} \;+\; \cdots \;+\; a_{nn}C_{nn} \;.
\end{aligned}$$

- To highlight the 4×4 case of Formulas 6 and 7:

Formula 8 (Co-factor expansion with respect to rows, case 4×4). Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

be the general 4×4 matrix. The sixteen <u>minors</u> of A are

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}, \; M_{12} = \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}, \; M_{13} = \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}, \; M_{14} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix},$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}, \; M_{22} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}, \; M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}, \; M_{24} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix},$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}, \; M_{32} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}, \; M_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}, \; M_{34} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix},$$

$$M_{41} = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix}, \; M_{42} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix}, \; M_{43} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}, \; M_{44} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Moreover, the sixteen <u>co-factors</u> of A are

$$\begin{array}{llll} C_{11} = +M_{11}, & C_{12} = -M_{12}, & C_{13} = +M_{13}, & C_{14} = -M_{14}, \\ C_{21} = -M_{21}, & C_{22} = +M_{22}, & C_{23} = -M_{23}, & C_{24} = +M_{24}, \\ C_{31} = +M_{31}, & C_{32} = -M_{32}, & C_{33} = +M_{33}, & C_{34} = -M_{34}, \\ C_{41} = -M_{41}, & C_{42} = +M_{42}, & C_{43} = -M_{43}, & C_{44} = +M_{44}. \end{array}$$

Observe the pattern of the signs for C_{ij} s against M_{ij} s :

$+$	$-$	$+$	$-$
$-$	$+$	$-$	$+$
$+$	$-$	$+$	$-$
$-$	$+$	$-$	$+$

Then we have

$$\begin{aligned}
\det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \\
&= a_{11}\,C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13} + a_{14}\,C_{14} \\
&= a_{21}\,C_{21} + a_{22}\,C_{22} + a_{23}\,C_{23} + a_{24}\,C_{24} \\
&= a_{31}\,C_{31} + a_{32}\,C_{32} + a_{33}\,C_{33} + a_{34}\,C_{34} \\
&= a_{41}\,C_{41} + a_{42}\,C_{42} + a_{43}\,C_{43} + a_{44}\,C_{44} \ .
\end{aligned}$$

Formula 9 (Co-factor expansion with respect to columns, case 4×4). Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

be the general 4×4 matrix, as in Formula 8. Then, under the same notation as in Formula 8, we have

$$\begin{aligned}\det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} + a_{42}C_{42} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \\ &= a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}\ .\end{aligned}$$

[IV] Evaluate each of the following determinants using co-factor expansions:

(1) $$\begin{vmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc + ad) & 2(bd - ac) \\ 2(bc - ad) & a^2 - b^2 + c^2 - d^2 & 2(cd + ab) \\ 2(bd + ac) & 2(cd - ab) & a^2 - b^2 - c^2 + d^2 \end{vmatrix}.$$

(2) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}.$$ (3) $$\begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ p & -q & r & -s \\ q & p & s & r \end{vmatrix}.$$ (4) $$\begin{vmatrix} ap & aq & bp & bq \\ ar & as & br & bs \\ cp & cq & dp & dq \\ cr & cs & dr & ds \end{vmatrix}.$$

[V] Evaluate each of the following determinants using co-factor expansions:

(1) $$\begin{vmatrix} t & -1 & 0 \\ 0 & t & -1 \\ c & b & t+a \end{vmatrix}.$$ (2) $$\begin{vmatrix} t & -1 & 0 & 0 \\ 0 & t & -1 & 0 \\ 0 & 0 & t & -1 \\ d & c & b & t+a \end{vmatrix}.$$

(3) $$\begin{vmatrix} t & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & t & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & t & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & t & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & t & -1 \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_3 & a_2 & t+a_1 \end{vmatrix}.$$

Solution to problems [I–V].

[I] (1) $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix} = 223.$ (2) $\begin{vmatrix} 4 & 3 & -2 \\ 5 & 4 & 1 \\ -2 & 3 & 4 \end{vmatrix} = -60.$

[II] (1) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{vmatrix} = t.$ (2) $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1.$ (3) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{vmatrix} = 1.$

[III] (1) $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = ab + bc + ca + abc.$$

(2) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \Big(b-a\Big)\Big(c-a\Big)\Big(c-b\Big).$$

(3) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \Big(b-a\Big)\Big(c-a\Big)\Big(c-b\Big)\Big(a+b+c\Big).$$

(4) $$\begin{vmatrix} x & 0 & c \\ -1 & x & b \\ 0 & -1 & a \end{vmatrix} = ax^2 + bx + c.$$

[IV] (1) $$\begin{vmatrix} a^2+b^2-c^2-d^2 & 2\big(bc+ad\big) & 2\big(bd-ac\big) \\ 2\big(bc-ad\big) & a^2-b^2+c^2-d^2 & 2\big(cd+ab\big) \\ 2\big(bd+ac\big) & 2\big(cd-ab\big) & a^2-b^2-c^2+d^2 \end{vmatrix}$$

$$= \Big(a^2+b^2+c^2+d^2\Big)^3.$$

(2) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

$$= \Big(b-a\Big)\Big(c-a\Big)\Big(d-a\Big)\Big(c-b\Big)\Big(d-b\Big)\Big(d-c\Big).$$

[V] (1) $$\begin{vmatrix} t & -1 & 0 \\ 0 & t & -1 \\ c & b & t+a \end{vmatrix} = t^3 + at^2 + bt + c.$$

(2) $$\begin{vmatrix} t & -1 & 0 & 0 \\ 0 & t & -1 & 0 \\ 0 & 0 & t & -1 \\ d & c & b & t+a \end{vmatrix} = t^4 + at^3 + bt^2 + ct + d.$$

(3) $$\begin{vmatrix} t & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & t & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & t & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & t & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & t & -1 \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_3 & a_2 & t+a_1 \end{vmatrix}$$

$$= t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_{n-2} t^2 + a_{n-1} t + a_n.$$