Math 290 ELEMENTARY LINEAR ALGEBRA
SOLUTION FOR QUIZ – XI (03/11)

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[I] (20pts) $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$.

(1) $\det A$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - a \cdot \begin{vmatrix} 1 & 1 \\ b^2 & c^2 \end{vmatrix} + a^2 \cdot \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} \quad \Big(\text{co-factor expansion}\Big)$$

$$= 1 \cdot \Big(\boxed{b\,c^2} - \boxed{c\,b^2} \Big)$$
$$- a \cdot \Big(\boxed{c^2} - \boxed{b^2} \Big)$$
$$+ a^2 \cdot \Big(\boxed{c} - \boxed{b} \Big)$$

$$= bc \cdot \Big(\boxed{c} - \boxed{b} \Big) - a\Big(c + b\Big)\Big(\boxed{c} - \boxed{b} \Big) + a^2 \Big(\boxed{c} - \boxed{b} \Big)$$

$$= \Big[bc - \Big(c + b\Big)a + a^2 \Big] \Big(\boxed{c} - \boxed{b} \Big)$$

$$= \Big(\boxed{b} - \boxed{a} \Big)\Big(\boxed{c} - \boxed{a} \Big)\Big(\boxed{c} - \boxed{b} \Big).$$

(2) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$= 1 \cdot \begin{vmatrix} b & c \\ b^3 & c^3 \end{vmatrix} - a \cdot \begin{vmatrix} 1 & 1 \\ b^3 & c^3 \end{vmatrix} + a^3 \cdot \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} \quad \Big(\text{co-factor expansion}\Big)$$

$$= 1 \cdot \Big(\boxed{b\,c^3} - \boxed{c\,b^3} \Big)$$

$$- a \cdot \Big(\boxed{c^3} - \boxed{b^3} \Big)$$

$$+ a^3 \cdot \Big(\boxed{c} - \boxed{b} \Big)$$

$$= bc \cdot \Big(c+b\Big) \Big(\boxed{c} - \boxed{b} \Big) - a \Big[\Big(c+b\Big)^2 - bc \Big] \Big(\boxed{c} - \boxed{b} \Big)$$

$$+ a^3 \cdot \Big(\boxed{c} - \boxed{b} \Big)$$

$$= \Big[\Big(c+b\Big) \cdot bc - \Big(c+b\Big)^2 \cdot a + abc + a^3 \Big] \Big(\boxed{c} - \boxed{b} \Big)$$

$$= \Big[\Big(c+b\Big) + \boxed{a} \Big] \Big[bc - \Big(c+b\Big) a + a^2 \Big] \Big(\boxed{c} - \boxed{b} \Big)$$

$$= \Big(\boxed{a} + \boxed{b} + \boxed{c} \Big)$$

$$\cdot \Big(\boxed{b} - \boxed{a} \Big) \Big(\boxed{c} - \boxed{a} \Big) \Big(\boxed{c} - \boxed{b} \Big) .$$

(3) By the result of (2), if $a \neq 0, \quad b \neq 0, \quad c \neq 0,$ then

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = a^3 b^3 c^3 \begin{vmatrix} a^{-3} & b^{-3} & c^{-3} \\ a^{-1} & b^{-1} & c^{-1} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -a^3 b^3 c^3 \begin{vmatrix} 1 & 1 & 1 \\ a^{-1} & b^{-1} & c^{-1} \\ a^{-3} & b^{-3} & c^{-3} \end{vmatrix}$$

$$= -abc \left(\frac{1}{\boxed{a}} + \frac{1}{\boxed{b}} + \frac{1}{\boxed{c}} \right)$$

$$\cdot\ ab \left(\frac{1}{\boxed{b}} - \frac{1}{\boxed{a}} \right)$$

$$\cdot\ ac \left(\frac{1}{\boxed{c}} - \frac{1}{\boxed{a}} \right)$$

$$\cdot\ bc \left(\frac{1}{\boxed{c}} - \frac{1}{\boxed{b}} \right)$$

$$= -\left(\boxed{bc} + \boxed{ac} + \boxed{ab} \right)$$

$$\cdot \left(\boxed{a} - \boxed{b} \right) \left(\boxed{a} - \boxed{c} \right) \left(\boxed{b} - \boxed{c} \right)$$

$$= \left(\boxed{ab} + \boxed{ac} + \boxed{bc} \right)$$

$$\cdot \left(\boxed{b} - \boxed{a} \right) \left(\boxed{c} - \boxed{a} \right) \left(\boxed{c} - \boxed{b} \right),$$

<u>Agree that the same holds even when either $a = 0, \quad b = 0,$ or $c = 0$</u> .

(4) By the results of (1–3),

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

$$= \; -1 \cdot \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + d \cdot \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} - d^2 \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} + d^3 \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

(co-factor expansion)

$$= \; -abc \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} + d \cdot \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} - d^2 \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} + d^3 \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \Big[- abc + d \cdot \Big(\boxed{ab} + \boxed{ac} + \boxed{bc} \Big) - d^2 \cdot \Big(\boxed{a} + \boxed{b} + \boxed{c} \Big) + d^3 \Big] \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \Big(\boxed{d} - \boxed{a} \Big) \Big(\boxed{d} - \boxed{b} \Big) \Big(\boxed{d} - \boxed{c} \Big) \Big(\boxed{b} - \boxed{a} \Big) \Big(\boxed{c} - \boxed{a} \Big) \Big(\boxed{c} - \boxed{b} \Big) .$$

(5) To evaluate each of the following determinants, we may use the results of (1–4):

$$\begin{vmatrix} 1 & 1 & 1 \\ -\sqrt{2} & \sqrt{2} & 2 \\ 2 & 2 & 4 \end{vmatrix} = \left(\sqrt{2} - \left(-\sqrt{2}\right)\right)\left(2 - \left(-\sqrt{2}\right)\right)\left(2 - \sqrt{2}\right)$$

$$= 2\sqrt{2}\left(2 + \sqrt{2}\right)\left(2 - \sqrt{2}\right)$$

$$= 2\sqrt{2}\left(4 - 2\right)$$

$$= 4\sqrt{2}.$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 8 & 27 \end{vmatrix} = \left(1 + 2 + 3\right)\left(2 - 1\right)\left(3 - 1\right)\left(3 - 2\right)$$

$$= 6 \cdot 1 \cdot 2 \cdot 1$$

$$= 12.$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 4 \\ 1 & 1 & 4 & 16 \\ -1 & 1 & 8 & 64 \end{vmatrix}$$

$$= \left(4 - \left(-1\right)\right)\left(4 - 1\right)\left(4 - 2\right)$$

$$\cdot \left(1 - \left(-1\right)\right)\left(2 - \left(-1\right)\right)\left(2 - 1\right)$$

$$= 5 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \cdot 1$$

$$= 180.$$

[II] (30pts) For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, define

$$A \otimes B = \begin{bmatrix} aB & bB \\ cB & dB \end{bmatrix} = \begin{bmatrix} ap & aq & bp & bq \\ ar & as & br & bs \\ cp & cq & dp & dq \\ cr & cs & dr & ds \end{bmatrix}.$$

(1)

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & 0\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ 0\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & -1\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = X,$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = Y,$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & 1\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ 1\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & 0\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = Z.$$

(2) Let

$$X = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

as in (1). We will evaluate their determinants. We may either resort to a direct calculation, or, alternatively, we may rely on the fact

$$\begin{aligned} X &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ Y &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ Z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \end{aligned}$$

and the formula

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right) = \left(\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^2 \left(\det \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right)^2.$$

$$\begin{aligned} \det X &= \left(\det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)^2 \left(\det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)^2 \\ &= \Big(1 \cdot (-1) - 0 \cdot 0\Big)^2 \Big(0 \cdot 0 - (-1) \cdot 1\Big)^2 = 1 \cdot 1 = 1, \end{aligned}$$

$$\begin{aligned} \det Y &= \left(\det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)^2 \left(\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^2 \\ &= \Big(0 \cdot 0 - (-1) \cdot 1\Big)^2 \Big(1 \cdot 1 - 0 \cdot 0\Big)^2 = 1 \cdot 1 = 1, \end{aligned}$$

$$\begin{aligned} \det Z &= \left(\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^2 \left(\det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)^2 \\ &= \Big(0 \cdot 0 - 1 \cdot 1\Big)^2 \Big(0 \cdot 0 - (-1) \cdot 1\Big)^2 = 1 \cdot 1 = 1. \end{aligned}$$

(3) Let

$$X = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

as in (1). We will find

$$X^2, \; Y^2, \; Z^2, \; XY, \; YZ, \; ZX, \; YX, \; ZY, \text{ and } XZ.$$

We may either resort to a direct calculation, or, alternatively, we may once again rely on the fact

$$\begin{aligned} X &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ Y &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ Z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \end{aligned}$$

and the formula

$$\Big(A \otimes B\Big)\Big(C \otimes D\Big) = \Big(AC\Big) \otimes \Big(BD\Big).$$

$$X^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$Y^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$Z^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$XY = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \otimes \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= -\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right),$$

$$YZ = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \otimes \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= -\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right),$$

$$ZX = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \otimes \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right),$$

$$YX = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \otimes \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$ZY = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \otimes \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$XZ = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \otimes \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In sum,

$$\begin{aligned} X^2 &= -I, & Y^2 &= -I, & Z^2 &= -I, \\ XY &= -Z, & YZ &= -X, & ZX &= -Y, \\ YX &= Z, & ZY &= X, & XZ &= Y. \end{aligned}$$

(4) True or false :

" An arbitrary 4×4 matrix of form

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

is a linear combination of I, X, Y, and Z. "

The answer is "true". Indeed,

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & -b & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -c & 0 \\ 0 & 0 & 0 & -c \\ c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -d \\ 0 & 0 & d & 0 \\ 0 & -d & 0 & 0 \\ d & 0 & 0 & 0 \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$+ d \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= aI + bX + cY + dZ.$$

This shows that an arbitrary 4×4 matrix of form $\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$ is a linear combination of I, X, Y and Z.

(5) We evaluate $\det \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$. Because of the special pattern of entries, Formula 11 in Pg. Ch. VII is applicable:

$$\det \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

$$= \det \left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} - \begin{bmatrix} -c & -d \\ d & -c \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \right)$$

$$= \det \left(\begin{bmatrix} \boxed{a^2 + b^2} & \boxed{0} \\ \boxed{0} & \boxed{a^2 + b^2} \end{bmatrix} - \begin{bmatrix} \boxed{-c^2 - d^2} & \boxed{0} \\ \boxed{0} & \boxed{-c^2 - d^2} \end{bmatrix} \right)$$

$$= \det \left(\begin{bmatrix} \boxed{a^2 + b^2 + c^2 + d^2} & \boxed{0} \\ \boxed{0} & \boxed{a^2 + b^2 + c^2 + d^2} \end{bmatrix} \right)$$

$$= \left(a^2 + b^2 + c^2 + d^2 \right)^2.$$

(6) <u>True or false</u> : " For two matrices

$$A = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a' & -b' & -c' & -d' \\ b' & a' & d' & -c' \\ c' & -d' & a' & b' \\ d' & c' & -b' & a' \end{bmatrix},$$

their product AB is of form $\begin{bmatrix} a'' & -b'' & -c'' & -d'' \\ b'' & a'' & d'' & -c'' \\ c'' & -d'' & a'' & b'' \\ d'' & c'' & -b'' & a'' \end{bmatrix}$. "

The answer is "true". Indeed, AB equals

$$\begin{bmatrix} aa'-bb'-cc'-dd' & -ab'-ba'+cd'-dc' & -ac'-bd'-ca'+db' & -ad'+bc'-cb'-da' \\ ab'+ba'-cd'+dc' & aa'-bb'-cc'-dd' & ad'-bc'+cb'+da' & -ac'-bd'-ca'+db' \\ ac'+bd'+ca'-db' & -ad'+bc'-cb'-da' & aa'-bb'-cc'-dd' & ab'+ba'-cd'+dc' \\ ad'-bc'+cb'+da' & ac'+bd'+ca'-db' & -ab'-ba'+cd'-dc' & aa'-bb'-cc'-dd' \end{bmatrix}.$$

This exactly falls into the pattern $\begin{bmatrix} a'' & -b'' & -c'' & -d'' \\ b'' & a'' & d'' & -c'' \\ c'' & -d'' & a'' & b'' \\ d'' & c'' & -b'' & a'' \end{bmatrix}$, where

$$a'' = a\,a' - b\,b' - c\,c' - d\,d',$$

$$b'' = a\,b' + b\,a' - c\,d' + d\,c',$$

$$c'' = a\,c' + b\,d' + c\,a' - d\,b',$$

$$d'' = a\,d' - b\,c' + c\,b' + d\,c'.$$