Math 290 ELEMENTARY LINEAR ALGEBRA

QUIZ – XI

March 11 (Tue), 2008

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Line #: 74449 / 82588 .

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[I] (20pts)

(1) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= \quad 1 \cdot \begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - a \cdot \begin{vmatrix} 1 & 1 \\ b^2 & c^2 \end{vmatrix} + a^2 \cdot \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} \quad \Big(\text{co-factor expansion}\Big)$$

$$= \quad 1 \cdot \Big(\boxed{} - \boxed{} \Big)$$

$$- a \cdot \Big(\boxed{} - \boxed{} \Big)$$

$$+ a^2 \cdot \Big(\boxed{} - \boxed{} \Big)$$

$$= \quad bc \cdot \Big(\boxed{} - \boxed{} \Big) - a\Big(c + b\Big)\Big(\boxed{} - \boxed{} \Big)$$

$$+ a^2 \Big(\boxed{} - \boxed{} \Big)$$

$$= \quad \Big[bc - \Big(c + b\Big)a + a^2 \Big] \Big(\boxed{} - \boxed{} \Big)$$

$$= \quad \Big(\boxed{b} - \boxed{a} \Big) \Big(\boxed{} - \boxed{} \Big) \Big(\boxed{} - \boxed{} \Big) .$$

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([I] continued)

(2) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$= 1 \cdot \begin{vmatrix} b & c \\ b^3 & c^3 \end{vmatrix} - a \cdot \begin{vmatrix} 1 & 1 \\ b^3 & c^3 \end{vmatrix} + a^3 \cdot \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} \quad \Big(\text{co-factor expansion}\Big)$$

$$= 1 \cdot \Big(\boxed{} - \boxed{} \Big)$$
$$- a \cdot \Big(\boxed{} - \boxed{} \Big)$$
$$+ a^3 \cdot \Big(\boxed{} - \boxed{} \Big)$$

$$= bc \cdot \Big(c + b\Big) \Big(\boxed{} - \boxed{} \Big) - a \Big[\Big(c + b\Big)^2 - bc \Big] \Big(\boxed{} - \boxed{} \Big)$$
$$+ a^3 \cdot \Big(\boxed{} - \boxed{} \Big)$$

$$= \Big[\Big(c + b\Big) \cdot bc - \Big(c + b\Big)^2 \cdot a + abc + a^3 \Big] \Big(\boxed{} - \boxed{} \Big)$$

$$= \Big[\Big(c + b\Big) + \boxed{} \Big] \Big[bc - \Big(c + b\Big) a + a^2 \Big] \Big(\boxed{} - \boxed{} \Big)$$

$$= \Big(\boxed{} + \boxed{} + \boxed{} \Big)$$
$$\cdot \Big(\boxed{b} - \boxed{a} \Big) \Big(\boxed{} - \boxed{} \Big) \Big(\boxed{} - \boxed{} \Big) .$$

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([I] continued)

(3) By the result of (2), if $a \neq 0, \quad b \neq 0, \quad c \neq 0,$ then

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = a^3 b^3 c^3 \begin{vmatrix} a^{-3} & b^{-3} & c^{-3} \\ a^{-1} & b^{-1} & c^{-1} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -\, a^3 b^3 c^3 \begin{vmatrix} 1 & 1 & 1 \\ a^{-1} & b^{-1} & c^{-1} \\ a^{-3} & b^{-3} & c^{-3} \end{vmatrix}$$

$$= -\, a\,b\,c \left(\frac{1}{\boxed{}} + \frac{1}{\boxed{}} + \frac{1}{\boxed{}} \right)$$

$$\cdot\ ab \left(\frac{1}{\boxed{b}} - \frac{1}{\boxed{a}} \right)$$

$$\cdot\ ac \left(\frac{1}{\boxed{}} - \frac{1}{\boxed{}} \right)$$

$$\cdot\ bc \left(\frac{1}{\boxed{}} - \frac{1}{\boxed{}} \right)$$

$$= -\left(\boxed{} + \boxed{} + \boxed{} \right)$$

$$\cdot \left(\boxed{a} - \boxed{b} \right) \left(\boxed{} - \boxed{} \right) \left(\boxed{} - \boxed{} \right)$$

$$= \left(\boxed{} + \boxed{} + \boxed{} \right)$$

$$\cdot \left(\boxed{b} - \boxed{a} \right) \left(\boxed{} - \boxed{} \right) \left(\boxed{} - \boxed{} \right),$$

Agree that the same holds even when either $a = 0, \quad b = 0,$ or $c = 0$.

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([I] continued)

(4) By the results of (1–3),

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

$$= -1 \cdot \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + d \cdot \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} - d^2 \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} + d^3 \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$\Big(\text{co-factor expansion}\Big)$$

$$= -abc \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} + d \cdot \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} - d^2 \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} + d^3 \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \Big[-abc + d \cdot \Big(\boxed{} + \boxed{} + \boxed{} \Big) - d^2 \cdot \Big(\boxed{} + \boxed{} + \boxed{} \Big) + d^3 \Big] \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \Big(\boxed{d} - \boxed{} \Big) \Big(\boxed{d} - \boxed{} \Big) \Big(\boxed{d} - \boxed{} \Big) \Big(\boxed{} - \boxed{} \Big) \Big(\boxed{} - \boxed{} \Big) \Big(\boxed{} - \boxed{} \Big).$$

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([I] continued)

(5) Use the results of (1–4), to evaluate:

$$\begin{vmatrix} 1 & 1 & 1 \\ -\sqrt{2} & \sqrt{2} & 2 \\ 2 & 2 & 4 \end{vmatrix} = \underline{\qquad\qquad},$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 8 & 27 \end{vmatrix} = \underline{\qquad\qquad},$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 4 \\ 1 & 1 & 4 & 16 \\ -1 & 1 & 8 & 64 \end{vmatrix} = \underline{\qquad\qquad}.$$

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[II] (30pts) For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, define

$$A \otimes B = \begin{bmatrix} aB & bB \\ cB & dB \end{bmatrix}.$$

In other words, define

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap & aq & bp & bq \\ ar & as & br & bs \\ cp & cq & dp & dq \\ cr & cs & dr & ds \end{bmatrix}.$$

(1) $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} = X,$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} = Y,$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} = Z.$$

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([II] continued)

(2) For the above X, Y and Z,

$\det X =$ ____________ , $\det Y =$ ____________ , $\det Z =$ ____________ .

You may use the formula

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = \left(\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^2 \left(\det\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right)^2.$$

(3) For the above X, Y and Z,

$X^2 =$ ____________ , $Y^2 =$ ____________ , $Z^2 =$ ____________ ,

$XY =$ ____________ , $YZ =$ ____________ , $ZX =$ ____________ .

$YX =$ ____________ , $ZY =$ ____________ , $XZ =$ ____________ .

As for (3): Choose your answers from the following :

I, X, Y, Z, $-I$, $-X$, $-Y$, or $-Z$.

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([II] continued)

(4) Let X, Y and Z be as above.

True or false :

" An arbitrary 4×4 matrix of form

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

is a linear combination of I, X, Y, and Z. "

☐ True ☐ False (Check one).

Justification for (4).

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([II] continued)

(5) Evaluate the determinant:

$$\det \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} = \underline{\hspace{6cm}} .$$

Justification for (5). Because of the special pattern of entries, Formula 11 in Pg. Ch. VII is applicable:

$$\det \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

$$= \det \left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} - \begin{bmatrix} -c & -d \\ d & -c \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \right)$$

$$= \det \left(\begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix} - \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix} \right)$$

$$= \det \left(\begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix} \right)$$

$$= \underline{\hspace{6cm}} .$$

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([II] continued)

(6) True or false :

" For two matrices

$$A = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a' & -b' & -c' & -d' \\ b' & a' & d' & -c' \\ c' & -d' & a' & b' \\ d' & c' & -b' & a' \end{bmatrix},$$

their product AB is of form $\begin{bmatrix} a'' & -b'' & -c'' & -d'' \\ b'' & a'' & d'' & -c'' \\ c'' & -d'' & a'' & b'' \\ d'' & c'' & -b'' & a'' \end{bmatrix}$. "

☐ True ☐ False (Check one).

If your answer is "true", then express a'', b'', c'' and d'' in terms of a, b, c, d, a', b', c' and d'.

a'' =

______________________________ ,

b'' =

______________________________ ,

c'' =

______________________________ ,

d'' =

______________________________ .