Math 290 ELEMENTARY LINEAR ALGEBRA

SOLUTION FOR QUIZ – XII (03/25)

April 3 (Thu), 2008

Instructor: Yasuyuki Kachi

Line #: 74449 / 82588.

[I] (12pts) $\left(2 + \sqrt{-1}\right) + \left(3 + \sqrt{-1} \cdot 2\right)$

$$= 2 + \sqrt{-1} + 3 + \left(\sqrt{-1} \cdot 2\right)$$

$$= \left(2 + 3\right) + \sqrt{-1}\left(1 + 2\right)$$

$$= \left(\boxed{5}\right) + \sqrt{-1}\left(\boxed{3}\right).$$

(1b) $\sqrt{-1} \cdot \sqrt{-1} = -1.$

(1c) $\sqrt{-2} \cdot \sqrt{-5} = \sqrt{-1} \cdot \sqrt{2} \cdot \sqrt{-1} \cdot \sqrt{5}$

$$= \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{2} \cdot \sqrt{5}$$

$$= \left(\sqrt{-1} \cdot \sqrt{-1}\right)\left(\sqrt{2} \cdot \sqrt{5}\right)$$

$$= \left(-1\right)\sqrt{10} = -\sqrt{10}.$$

(1d) $\left(4 + \sqrt{-1} \cdot 3\right)\left(-1 + \sqrt{-1} \cdot 2\right)$

$$= \left[\boxed{4} \cdot \left(\boxed{-1}\right) - \boxed{3} \cdot \boxed{2}\right]$$

$$+ \sqrt{-1}\left[\boxed{4} \cdot \boxed{2} + \boxed{3} \cdot \left(\boxed{-1}\right)\right]$$

$$= -10 + \sqrt{-1} \cdot 5.$$

(2) The complex number $a + \sqrt{-1}\,b$ and the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ correspond to each other (where a and b are real numbers). We write this as

$$a + \sqrt{-1}\,b \quad \longleftrightarrow \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

As for the two complex numbers in (1d), we have

$$4 + \sqrt{-1} \cdot 3 \quad \longleftrightarrow \quad \begin{bmatrix} \boxed{4} & \boxed{-3} \\ \boxed{3} & \boxed{4} \end{bmatrix} = A,$$

and

$$(-1) + \sqrt{-1} \cdot 2 \quad \longleftrightarrow \quad \begin{bmatrix} \boxed{-1} & \boxed{-2} \\ \boxed{2} & \boxed{-1} \end{bmatrix} = B.$$

We calculate AB as

$$AB = \begin{bmatrix} \boxed{4} & \boxed{-3} \\ \boxed{3} & \boxed{4} \end{bmatrix} \begin{bmatrix} \boxed{-1} & \boxed{-2} \\ \boxed{2} & \boxed{-1} \end{bmatrix} = \begin{bmatrix} \boxed{-10} & \boxed{-5} \\ \boxed{5} & \boxed{-10} \end{bmatrix}.$$

Thus

$$\left(\boxed{-10}\right) + \sqrt{-1}\left(\boxed{5}\right) \quad \longleftrightarrow \quad AB.$$

This last complex number is exactly the same as the result of (1d).

[II] (8pts) (1) We may use the formula

$$\left(a + \sqrt{-1}\,b\right)^{-1} = \frac{1}{a^2 + b^2}\left(a - \sqrt{-1}\,b\right),$$

and find

$$\left(5 + \sqrt{-1}\,(-2)\right)^{-1}$$

$$= \frac{1}{\boxed{5^2 + (-2)^2}} \cdot \left[\left(\boxed{5}\right) + \sqrt{-1}\left(\boxed{2}\right) \right]$$

$$= \frac{1}{\boxed{29}} \cdot \left[\left(\boxed{5}\right) + \sqrt{-1}\left(\boxed{2}\right) \right].$$

(2) $\quad 5 + \sqrt{-1}\,(-2) \quad \longleftrightarrow \quad \begin{bmatrix} \boxed{5} & \boxed{2} \\ \boxed{-2} & \boxed{5} \end{bmatrix} = C$

We may calculate C^{-1} as

$$C^{-1} = \frac{1}{\boxed{5 \cdot 5 - 2 \cdot (-2)}} \cdot \begin{bmatrix} \boxed{5} & \boxed{-2} \\ \boxed{2} & \boxed{5} \end{bmatrix}$$

$$= \frac{1}{\boxed{29}} \cdot \begin{bmatrix} \boxed{5} & \boxed{-2} \\ \boxed{2} & \boxed{5} \end{bmatrix}.$$

The complex number $\frac{1}{29}\left(5 + \sqrt{-1} \cdot 2\right)$ and this last matrix clearly correspond to each other.

[III] (12pts) (1) The absolute value of a complex number is defined as

$$\left| a + \sqrt{-1}\, b \right| = \sqrt{a^2 + b^2} \qquad \left(a,\ b \text{ are real numbers}\right).$$

We may find the absolute value of $12 + \sqrt{-1} \cdot 5$ as

$$\left| 12 + \sqrt{-1} \cdot 5 \right| = \sqrt{12^2 + 5^2}$$

$$= \sqrt{169} = 13.$$

(2) $$12 + \sqrt{-1}\, 5 \quad \longleftrightarrow \quad \begin{bmatrix} \boxed{12} & \boxed{-5} \\ \boxed{5} & \boxed{12} \end{bmatrix} = D.$$

We may calculate $\det D$ as

$$\det D = 12 \cdot 12 - \left(-5 \right) \cdot 5$$

$$= 144 + 25 = 169.$$

The result of (1) above, that is, 13, and the quantity $\sqrt{\det D} = \sqrt{169}$, are equal.

(3) The conjugate of a complex number is defined as

$$\overline{a + \sqrt{-1}\, b} = a - \sqrt{-1}\, b \qquad \left(a,\ b \text{ are real numbers} \right).$$

We may find the conjugate of $12 + \sqrt{-1} \cdot 5$ as

$$\overline{12 + \sqrt{-1} \cdot 5} = 12 - \sqrt{-1} \cdot 5.$$

Also, we may find

$$\left(12 + \sqrt{-1} \cdot 5 \right) \left(\overline{12 + \sqrt{-1} \cdot 5} \right)$$

$$= \left(12 + \sqrt{-1} \cdot 5 \right) \left(12 - \sqrt{-1} \cdot 5 \right)$$

$$= \left[12 \cdot 12 - 5 \cdot \left(-5 \right) \right] + \sqrt{-1} \left[12 \cdot \left(-5 \right) + 5 \cdot 12 \right]$$

$$= 169 + \sqrt{-1} \cdot 0 = 169.$$

Clearly

$$\left(12 + \sqrt{-1} \cdot 5 \right) \left(\overline{12 + \sqrt{-1} \cdot 5} \right) = \left| 12 + \sqrt{-1} \cdot 5 \right|^2.$$

(4) For the same matrix D as in (2),

$$D^T = \begin{bmatrix} \boxed{12} & \boxed{5} \\ \boxed{-5} & \boxed{12} \end{bmatrix}, \qquad \text{and}$$

$$D\ D^T = \begin{bmatrix} \boxed{12} & \boxed{-5} \\ \boxed{5} & \boxed{12} \end{bmatrix} \begin{bmatrix} \boxed{12} & \boxed{5} \\ \boxed{-5} & \boxed{12} \end{bmatrix} = \begin{bmatrix} \boxed{169} & \boxed{0} \\ \boxed{0} & \boxed{169} \end{bmatrix}.$$

Clearly

$$\overline{12 + \sqrt{-1} \cdot 5} \quad \longleftrightarrow \quad D^T,$$

and

$$\left| 12 + \sqrt{-1} \cdot 5 \right|^2 \quad \longleftrightarrow \quad DD^T.$$

[IV] (18pts) Let

$$X = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}.$$

Then

$$\det X = 1, \qquad \det Y == 1, \qquad \det Z == 1.$$

(2) For the above X, Y and Z,

$$X^2 = -I, \qquad Y^2 = -I, \qquad Z^2 = -I,$$

$$XY = -Z, \qquad YZ = -X, \qquad ZX = -Y,$$

$$YX = Z, \qquad ZY = X, \qquad XZ = Y.$$

(3) True or false :

" An arbitrary 2×2 matrix of form

$$\begin{bmatrix} a + \sqrt{-1}\, b & -c + \sqrt{-1}\, d \\ c + \sqrt{-1}\, d & a - \sqrt{-1}\, b \end{bmatrix}$$

is a linear combination of I, X, Y, and Z. "

The answer is "true". Indeed,

$$\begin{bmatrix} a + \sqrt{-1}\, b & -c + \sqrt{-1}\, d \\ c + \sqrt{-1}\, d & a - \sqrt{-1}\, b \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} \sqrt{-1}\, b & 0 \\ 0 & -\sqrt{-1}\, b \end{bmatrix} + \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{-1}\, d \\ \sqrt{-1}\, d & 0 \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix} + c \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}$$

$$= a\, I + b\, X + c\, Y + d\, Z.$$

This shows that an arbitrary 2×2 matrix of form $\begin{bmatrix} a + \sqrt{-1}\, b & -c + \sqrt{-1}\, d \\ c + \sqrt{-1}\, d & a - \sqrt{-1}\, b \end{bmatrix}$ is a linear combination of I, X, Y and Z.

(4) $\begin{vmatrix} a + \sqrt{-1}\, b & -c + \sqrt{-1}\, d \\ c + \sqrt{-1}\, d & a - \sqrt{-1}\, b \end{vmatrix}$

$$= \left(a + \sqrt{-1}\, b\right)\left(a - \sqrt{-1}\, b\right) - \left(-c + \sqrt{-1}\, d\right)\left(c + \sqrt{-1}\, d\right)$$

$$= \left(a + \sqrt{-1}\, b\right)\left(a - \sqrt{-1}\, b\right) + \left(c - \sqrt{-1}\, d\right)\left(c + \sqrt{-1}\, d\right)$$

$$= \left(a^2 + b^2\right) + \sqrt{-1}\left(-\, ab + ba\right) + \left(c^2 + d^2\right) + \sqrt{-1}\left(-\, cd + dc\right)$$

$$= \left(a^2 + b^2 + c^2 + d^2\right) + \sqrt{-1}\left(-\, ab + ba - cd + dc\right)$$

$$= a^2 + b^2 + c^2 + d^2.$$

(5) True or false :

" For

$$A = \begin{bmatrix} a + \sqrt{-1}\, b & -\, c + \sqrt{-1}\, d \\ c + \sqrt{-1}\, d & a - \sqrt{-1}\, b \end{bmatrix},$$

$$B = \begin{bmatrix} a' + \sqrt{-1}\, b' & -\, c' + \sqrt{-1}\, d' \\ c' + \sqrt{-1}\, d' & a' - \sqrt{-1}\, b' \end{bmatrix},$$

their product AB is of form $\begin{bmatrix} a'' + \sqrt{-1}\, b'' & -\, c'' + \sqrt{-1}\, d'' \\ c'' + \sqrt{-1}\, d'' & a'' - \sqrt{-1}\, b'' \end{bmatrix}$. "

The answer is "true". Indeed, AB equals

$$\begin{bmatrix} \begin{matrix}\left(aa' - bb' - cc' - dd'\right) \\ + \sqrt{-1}\left(ab' + ba' - cd' + dc'\right)\end{matrix} & \begin{matrix} -\left(ac' + bd' + ca' - db'\right) \\ + \sqrt{-1}\left(ad' - bc' + cb' + da'\right)\end{matrix} \\ \\ \begin{matrix}\left(ac' + bd' + ca' - db'\right) \\ + \sqrt{-1}\left(ad' - bc' + cb' + da'\right)\end{matrix} & \begin{matrix}\left(aa' - bb' - cc' - dd'\right) \\ - \sqrt{-1}\left(ab' + ba' - cd' + dc'\right)\end{matrix} \end{bmatrix}.$$

This exactly falls into the pattern $\begin{bmatrix} a'' + \sqrt{-1}\, b'' & -c'' + \sqrt{-1}\, d'' \\ c'' + \sqrt{-1}\, d'' & a'' - \sqrt{-1}\, b'' \end{bmatrix}$, where

$$a'' = a\,a' - b\,b' - c\,c' - d\,d',$$

$$b'' = a\,b' + b\,a' - c\,d' + d\,c',$$

$$c'' = a\,c' + b\,d' + c\,a' - d\,b',$$

$$d'' = a\,d' - b\,c' + c\,b' + d\,a'.$$