Math 290 ELEMENTARY LINEAR ALGEBRA

QUIZ – XIV

April 15 (Tue), 2008

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Line #: 74449 / 82588 .

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[I] (20pts) For $A = \begin{bmatrix} t + s & t^{-1} - s \\ t - s^{-1} & t^{-1} + s^{-1} \end{bmatrix}$, where

$$t \neq 0, \quad s \neq 0, \quad t \neq s, \quad ts \neq 1.$$

(1) The characteristic polynomial of A is

$\chi_A(\lambda)$

$$= \lambda^2 - \left(\boxed{} \right) \lambda + \left(\boxed{} \right)\left(\boxed{} \right) - \left(\boxed{} \right)\left(\boxed{} \right)$$

$$= \lambda^2 - \left(\boxed{} \right) \lambda + \boxed{} + \boxed{} + \boxed{} + \boxed{}$$

$$= \left(\lambda - \left(\boxed{} + \boxed{} \right) \right) \left(\lambda - \left(\boxed{} + \boxed{} \right) \right).$$

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([I] continued)

(2) The eigenvalues λ of A are

$$\lambda = \underline{\qquad\qquad} , \quad \underline{\qquad\qquad} .$$

- For $\lambda = \underline{\qquad\qquad}$, the eigenvector $\boldsymbol{x} = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix}$,

- For $\lambda = \underline{\qquad\qquad}$, the eigenvector $\boldsymbol{x} = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix}$.

Work.

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[II] (30pts) Let $a, \ b, \ c, \ d$ be real numbers, such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Let $B = \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix}$.

(1) $B^2 = \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix} \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix}$

$$= \begin{bmatrix} \boxed{-c^2 - d^2} & \boxed{bc} & \boxed{bd} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(2) $\operatorname{Tr} B =$ __________ .

(3) $\begin{vmatrix} 0 & d \\ -d & 0 \end{vmatrix} + \begin{vmatrix} 0 & -c \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ -b & 0 \end{vmatrix} =$ ____________________________ .

(4) $\det B =$ __________ .

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([II] continued)

(5) $\chi_B(\lambda) =$ ______________________ .

(6) $I + 2a\,B + 2B^2$

$$= \begin{bmatrix} a^2 + b^2 + c^2 + d^2 & 0 & 0 \\ 0 & a^2 + b^2 + c^2 + d^2 & 0 \\ 0 & 0 & a^2 + b^2 + c^2 + d^2 \end{bmatrix}$$

$$+ \begin{bmatrix} 2a\boxed{\quad} & 2a\boxed{\quad} & 2a\boxed{\quad} \\ 2a\boxed{\quad} & 2a\boxed{\quad} & 2a\boxed{\quad} \\ 2a\boxed{\quad} & 2a\boxed{\quad} & 2a\boxed{\quad} \end{bmatrix}$$

$$+ \begin{bmatrix} 2\left(\boxed{\quad}\right) & 2\left(\boxed{\quad}\right) & 2\left(\boxed{\quad}\right) \\ 2\left(\boxed{\quad}\right) & 2\left(\boxed{\quad}\right) & 2\left(\boxed{\quad}\right) \\ 2\left(\boxed{\quad}\right) & 2\left(\boxed{\quad}\right) & 2\left(\boxed{\quad}\right) \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2\left(\boxed{\quad}\right) & 2\left(\boxed{\quad}\right) \\ 2\left(\boxed{\quad}\right) & a^2 - b^2 + c^2 - d^2 & 2\left(\boxed{\quad}\right) \\ 2\left(\boxed{\quad}\right) & 2\left(\boxed{\quad}\right) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}.$$

$$= A.$$

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([II] continued)

(7) Let A be your answer in (6).

$$A\,A^T \;=\; \underline{\qquad\qquad}\,.$$

Justification. Using $B = \begin{bmatrix} 0 & d & -c \\ -d & 0 & b \\ c & -b & 0 \end{bmatrix}$, we may write $A\,A^T$ as

$$\begin{aligned} A\,A^T &= \left(I + 2a\,B + 2B^2\right)\left(I - 2a\,B + 2B^2\right) \\ &= I + \left(4 - 4\cdot\boxed{}^{\,2}\right)B^2 + 4B^4 \\ &= I + 4\left(\boxed{}^{\,2} + \boxed{}^{\,2} + \boxed{}^{\,2}\right)B^2 + 4B^4 \\ &= I + 4\,B\underbrace{\left[\left(\boxed{}^{\,2} + \boxed{}^{\,2} + \boxed{}^{\,2}\right)B + B^3\right]}_{\chi_B(B)\;=\;\boxed{}} = I, \end{aligned}$$

where the last step is by ____________________ theorem.

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([II] continued)

(8) $A \begin{bmatrix} b \\ c \\ d \end{bmatrix}$

$$= \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2\big(bc + ad\big) & 2\big(bd - ac\big) \\ 2\big(bc - ad\big) & a^2 - b^2 + c^2 - d^2 & 2\big(cd + ab\big) \\ 2\big(bd + ac\big) & 2\big(cd - ab\big) & a^2 - b^2 - c^2 + d^2 \end{bmatrix} \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a^2 b + b^3 - bc^2 - bd^2 + 2\,bc^2 + 2\,acd + 2\,bd^2 - 2\,acd} \\ \boxed{2\,b^2 c - 2\,abd + a^2 c - b^2 c + c^3 - cd^2 + 2\,cd^2 + 2\,abd} \\ \boxed{2\,b^2 d + 2\,abc + 2\,c^2 d - 2\,abc + a^2 d - b^2 d - c^2 d + d^3} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a^2 b + b^3 + bc^2 + bd^2} \\ \boxed{} \\ \boxed{} \end{bmatrix}$$

$$= \left(\boxed{} \right) \begin{bmatrix} b \\ c \\ d \end{bmatrix} = 1 \cdot \begin{bmatrix} b \\ c \\ d \end{bmatrix} .$$

Hence A has eigenvalue $\lambda = 1$.

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([II] continued)

(9) From (8), and the fact $\det A = 1$, the characteristic polynomial $\chi_A(\lambda) = \det\left(\lambda I - A\right)$ is

$$\chi_A(\lambda) = \lambda^3 - \Big(\underbrace{\boxed{}}_{\mathrm{Tr}\,A}\Big)\lambda^2$$

$$+ \Big(\boxed{\quad * \quad}\Big)\lambda - \underbrace{\boxed{}}_{\det A}.$$

Denote the other two eigenvalues of A by $\lambda = p$, $\lambda = q$. Thus

$$\chi_A(\lambda) = \lambda^3 - \Big(\boxed{1} + p + q\Big)\lambda^2$$

$$+ \Big(\boxed{\quad * \quad}\Big)\lambda - \Big(\boxed{1} \cdot p\,q\Big).$$

By comparing the above two expressions of $\chi_A(\lambda)$,

$p + q =$ ______________ , $p\,q =$ __________ .

Solve $\lambda^2 - \left(p + q\right)\lambda + pq = 0$, and

$$\lambda = \boxed{} \pm 2a\sqrt{-\Big(\boxed{}\Big)}.$$