Math 290 ELEMENTARY LINEAR ALGEBRA
SOLUTION FOR QUIZ – XV (05/06)

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[I] (15pts) (1) $k = \mathbb{R}$. The subset

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{R}^2$ is linearly independent over $\mathbb{R}$.

Indeed,

$$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0 \ \Big(\text{in } \mathbb{R}\Big).$$

(2) $k = \mathbb{C}$. The subset

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ \sqrt{-1} \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \sqrt{-1} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{-1} \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{C}^3$ is linearly dependent over $\mathbb{C}$.

Indeed, four vectors in $\mathbb{C}^3$ can never be linearly independent over $\mathbb{C}$.

(3) $k = \mathbb{C}$. The subset

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

of $\mathbb{C}^4$ is linearly independent. over $\mathbb{C}$.

Indeed, put the three vectors together to form a matrix:

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

This matrix has a 3×3 minor that yields a non-zero value:

$$\begin{vmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = \left(-1 \right) \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} + \left(-1 \right) \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix}$$

$$= -3 \neq 0.$$

[II] (45pts) (1a) Let $\alpha, \beta \in \mathbb{C}$. Then

$$\operatorname{rank} \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & \alpha^2 & \alpha\beta \\ \beta & \alpha\beta & \beta^2 \end{bmatrix} = 1.$$

Indeed, the matrix has a non-zero entry 1, at the first row, the first column. Thus the rank of the matrix is at least 1. Meanwhile, no 2×2 minor of the matrix yields a non-zero value. All of

$$\begin{vmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{vmatrix}, \quad \begin{vmatrix} 1 & \beta \\ \alpha & \alpha\beta \end{vmatrix}, \quad \begin{vmatrix} \alpha & \beta \\ \alpha^2 & \alpha\beta \end{vmatrix},$$

$$\begin{vmatrix} 1 & \alpha \\ \beta & \alpha\beta \end{vmatrix}, \quad \begin{vmatrix} 1 & \beta \\ \beta & \beta^2 \end{vmatrix}, \quad \begin{vmatrix} \alpha & \beta \\ \alpha\beta & \beta^2 \end{vmatrix},$$

$$\begin{vmatrix} \alpha & \alpha^2 \\ \beta & \alpha\beta \end{vmatrix}, \quad \begin{vmatrix} \alpha & \alpha\beta \\ \beta & \beta^2 \end{vmatrix}, \quad \begin{vmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{vmatrix}$$

equal 0. Hence the rank of the matrix is 1. Alternatively, the reduced row echelon form of the matrix is $\begin{bmatrix} 1 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which clearly has rank 1.

(1b) Let $\omega = \dfrac{-1 + \sqrt{-3}}{2} \in \mathbb{C}$. Note

$$1 + \omega + \omega^2 = 0, \qquad \text{and} \qquad \omega^3 = 1.$$

Note also

$$\omega^4 = \omega.$$

In the matrix in (1a), substitute $\alpha = \omega$, and $\beta = \omega^2$, to obtain

$$\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}.$$

As we have already seen in (1a), the rank of this matrix is 1:

$$\operatorname{rank} \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} = 1.$$

(2) Over $\mathbb{R}$,

$$\operatorname{rank} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = 2.$$

Indeed, the reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which clearly has rank 2.

(3) Let $b, c, d \in \mathbb{R}$ be such that

$$b \neq 0, \qquad c \neq 0, \qquad \text{and} \qquad d \neq 0.$$

Let

$$\mu = \sqrt{-\left(b^2 + c^2 + d^2\right)} \in \mathbb{C}.$$

Then

$$\operatorname{rank} \begin{bmatrix} \mu & -b & -c & -d \\ b & \mu & d & -c \\ c & -d & \mu & b \\ d & c & -b & \mu \end{bmatrix} = 2.$$

Indeed, first, the rank of the matrix is at least 2, since we find the 2×2 minor that yields a non-zero value:

$$\begin{aligned} \begin{vmatrix} \mu & -b \\ b & \mu \end{vmatrix} \;=\; \mu^2 + b^2 \;&=\; -\left(b^2 + c^2 + d^2\right) + b^2 \\ &=\; -c^2 - d^2. \end{aligned}$$

This quantity does not equal 0 because, by assumption, $c \in \mathbb{R}$, $d \in \mathbb{R}$, $c \neq 0$, and $d \neq 0$.

Now, taking into account $\mu \neq 0$, we may apply the elementary row operation to the matrix:

$$\begin{bmatrix} \mu & -b & -c & -d \\ b & \mu & d & -c \\ c & -d & \mu & b \\ d & c & -b & \mu \end{bmatrix} \longrightarrow \begin{bmatrix} \mu & -b & -c & -d \\ 0 & -c^2 - d^2 & bc + \mu d & bd - \mu c \\ 0 & bc - \mu d & -b^2 - d^2 & cd + \mu b \\ 0 & bd + \mu c & cd - \mu b & -b^2 - c^2 \end{bmatrix}.$$

Then we may apply the elementary column operation to the matrix:

$$\longrightarrow \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & -c^2 - d^2 & bc + \mu d & bd - \mu c \\ 0 & bc - \mu d & -b^2 - d^2 & cd + \mu b \\ 0 & bd + \mu c & cd - \mu b & -b^2 - c^2 \end{bmatrix}.$$

Here, no 2×2 minor of the matrix

$$\begin{bmatrix} -c^2 - d^2 & bc + \mu d & bd - \mu c \\ bc - \mu d & -b^2 - d^2 & cd + \mu b \\ bd + \mu c & cd - \mu b & -b^2 - c^2 \end{bmatrix}$$

yields a non-zero value.

Indeed, it is straightforward to check that all of

$$\begin{vmatrix} -c^2-d^2 & bc+\mu d \\ bc-\mu d & -b^2-d^2 \end{vmatrix}, \quad \begin{vmatrix} -c^2-d^2 & bd-\mu c \\ bc-\mu d & cd+\mu b \end{vmatrix}, \quad \begin{vmatrix} bc+\mu d & bd-\mu c \\ -b^2-d^2 & cd+\mu b \end{vmatrix},$$

$$\begin{vmatrix} -c^2-d^2 & bc+\mu d \\ bd+\mu c & cd-\mu b \end{vmatrix}, \quad \begin{vmatrix} -c^2-d^2 & bd-\mu c \\ bd+\mu c & -b^2-c^2 \end{vmatrix}, \quad \begin{vmatrix} bc+\mu d & bd-\mu c \\ cd-\mu b & -b^2-c^2 \end{vmatrix},$$

$$\begin{vmatrix} bc-\mu d & -b^2-d^2 \\ bd+\mu c & cd-\mu b \end{vmatrix}, \quad \begin{vmatrix} bc-\mu d & cd+\mu b \\ bd+\mu c & -b^2-c^2 \end{vmatrix}, \quad \begin{vmatrix} -b^2-d^2 & cd+\mu b \\ cd-\mu b & -b^2-c^2 \end{vmatrix}$$

equal 0. Accordingly, no 3×3 minor of the matrix

$$\begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & -c^2-d^2 & bc+\mu d & bd-\mu c \\ 0 & bc-\mu d & -b^2-d^2 & cd+\mu b \\ 0 & bd+\mu c & cd-\mu b & -b^2-c^2 \end{bmatrix}$$

yields a non-zero value. Hence this last matrix has rank 2. Consequently, the original matrix $\begin{bmatrix} \mu & -b & -c & -d \\ b & \mu & d & -c \\ c & -d & \mu & b \\ d & c & -b & \mu \end{bmatrix}$ has rank 2.

⋆ [Alternative Solution] : For $A = \begin{bmatrix} \mu & -b & -c & -d \\ b & \mu & d & -c \\ c & -d & \mu & b \\ d & c & -b & \mu \end{bmatrix}$, we have

$$A^T = \begin{bmatrix} \mu & b & c & d \\ -b & \mu & -d & c \\ -c & d & \mu & -b \\ -d & -c & b & \mu \end{bmatrix}.$$

It follows

$$A\,A^T \;=\; \begin{bmatrix} \mu & -b & -c & -d \\ b & \mu & d & -c \\ c & -d & \mu & b \\ d & c & -b & \mu \end{bmatrix} \begin{bmatrix} \mu & b & c & d \\ -b & \mu & -d & c \\ -c & d & \mu & -b \\ -d & -c & b & \mu \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the column vectors of A^T, namely,

$$\boldsymbol{x}_1 = \begin{bmatrix} \mu \\ -b \\ -c \\ -d \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} b \\ \mu \\ d \\ -c \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} c \\ -d \\ \mu \\ b \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} d \\ c \\ -b \\ \mu \end{bmatrix},$$

are all in $\operatorname{Ker} A$. Let U be the $\mathbb{C}$-subspace of $\mathbb{C}^4$ spanned by these four vectors over $\mathbb{C}$:

$$U \;=\; \mathbb{C}\left\langle \boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{x}_3,\ \boldsymbol{x}_4 \right\rangle .$$

$\underline{\text{To establish } \operatorname{rank} A = 2, \text{ it suffices to prove } \operatorname{rank} A^T = 2, \text{ that is, } \dim_{\mathbb{C}} U = 2}$.

For this matter, first note $U \subseteq \operatorname{Ker} A$. We claim $\dim_{\mathbb{C}} U \geq 2$. Indeed, the matrix

$$\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 \end{bmatrix}$$

has a 2×2 minor that yields a non-zero value, as we have seen. In particular, $\dim_{\mathbb{C}} \operatorname{Ker} A \geq 2$. We claim $\dim_{\mathbb{C}} \operatorname{Ker} A = 2$. Indeed, set

$$B \;=\; \begin{bmatrix} 0 & b & c & d \\ -b & 0 & -d & c \\ -c & d & 0 & -b \\ -d & -c & b & 0 \end{bmatrix}.$$

Note that $\underline{B \text{ has entries in } \mathbb{R}}$. It easily follows that μ is one of the eigenvalues of B, and moreover

$$A \;=\; \mu I \;-\; B.$$

In particular, $\operatorname{Ker} A$ exactly equals the eigenspace V_μ of B associated to the eigenvalue μ, as a $\mathbb{C}$-subspace of $\mathbb{C}^4$. Since the matrix B has entries in $\mathbb{R}$, the characteristic polynomial of B has coefficients in $\mathbb{R}$. Hence the complex conjugate $\overline{\mu}$ of μ is also an eigenvalue of B. Let $V_{\overline{\mu}}$ be the eigenspace of B associated to the eigenvalue $\overline{\mu}$, as a $\mathbb{C}$-subspace of $\mathbb{C}^4$. Then the sum

$$V_\mu \,+\, V_{\overline{\mu}}$$

inside $\mathbb{C}^4$ is a direct sum. We have alraedy proved $\dim_{\mathbb{C}} \operatorname{Ker} A \;\geq\; 2$, that is,

$$\dim_{\mathbb{C}} V_\mu \;\geq\; 2.$$

The entry-wise complex conjugate operation takes the $\mathbb{C}$-subspace V_μ of $\mathbb{C}^4$ into the $\mathbb{C}$-subspace $V_{\overline{\mu}}$ of $\mathbb{C}^4$ and vice versa. In particular,

$$\dim_{\mathbb{C}} V_{\overline{\mu}} \;\geq\; 2.$$

From these,

$$\mathbb{C}^4 \;=\; V_\mu \oplus V_{\overline{\mu}},$$

and moreover $\dim_{\mathbb{C}} V_\mu \;=\; \dim_{\mathbb{C}} V_{\overline{\mu}} \;=\; 2$. Recall $U \;\subseteq\; \operatorname{Ker} A \;=\; V_\mu$, and also $\dim_{\mathbb{C}} U \;\geq\; 2$. Thus we have proved

$$\dim_{\mathbb{C}} U \;=\; 2.$$

By the definition of U, it follows $\operatorname{rank} A \;=\; 2$.

[III] (30pts) Let k be a field. Let $a_1,\ a_2,\ \cdots,\ a_n \in k$. Then

$$\operatorname{rank} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ {a_1}^2 & {a_2}^2 & {a_3}^2 & \cdots & {a_n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$$

equals the number of <u>distinct</u> values of $a_1,\ a_2,\ \cdots,\ a_n$.

Indeed, let us assume that

$$\Big\{ a_{j_1},\ a_{j_2},\ \cdots,\ a_{j_r} \Big\}$$

are distinct elements in k, and that any other a_ℓ is identical to one of a_{j_1}, a_{j_2}, $\cdots$, a_{j_r}. Let us take the following $r \times r$ minor of the original matrix:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_{j_1} & a_{j_2} & a_{j_3} & \cdots & a_{j_r} \\ {a_{j_1}}^2 & {a_{j_2}}^2 & {a_{j_3}}^2 & \cdots & {a_{j_r}}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{j_1}^{r-1} & a_{j_2}^{r-1} & a_{j_3}^{r-1} & \cdots & a_{j_r}^{r-1} \end{vmatrix}.$$

Such a minor is formed by choosing the first r consecutive rows, and columns j_1, j_2, $\cdots$, j_r from the original matrix. We know that this minor equals

$$\begin{array}{l} \Big(a_{j_2} - a_{j_1} \Big) \\ \Big(a_{j_3} - a_{j_1} \Big)\Big(a_{j_3} - a_{j_2} \Big) \\ \Big(a_{j_4} - a_{j_1} \Big)\Big(a_{j_4} - a_{j_2} \Big)\Big(a_{j_4} - a_{j_3} \Big) \\ \qquad\vdots \qquad\qquad \vdots \qquad\qquad \ddots \\ \Big(a_{n-1} - a_{j_1} \Big)\Big(a_{n-1} - a_{j_2} \Big) \cdots \Big(a_{n-1} - a_{n-2} \Big) \\ \Big(a_{j_r} - a_{j_1} \Big)\Big(a_{j_r} - a_{j_2} \Big) \cdots \Big(a_{j_r} - a_{j_{r-2}} \Big)\Big(a_{j_r} - a_{j_{r-1}} \Big). \end{array}$$

Since a_{j_1}, a_{j_2}, $\cdots$, a_{j_r} are all distinct in k, each of the factors in this last expression is non-zero in k. By the integral domain property of a field, this last expression is non-zero in k. Now, clearly there is no $\Big((r + 1) \times (r + 1) \Big)$ minor of the original matrix that yields a non-zero value. Indeed, if you choose $(r + 1)$ columns of the original matrix, then you always find two columns that are identical.

[IV] (50pts) Let Let A be a square, $n \times n$, matrix, whose entries are in $\mathbb{C}$.

(1) Let $\lambda = \lambda_0$ be an arbitrary eigenvalue of A. The definition of the eigenspace V_{λ_0} of A associated with the eigenvalue $\lambda = \lambda_0$, as a subspace of the $\mathbb{C}$-vector space $\mathbb{C}^n$, is as follows:

$$V_{\lambda_0} = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{C}^n \;\middle|\; A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \lambda_0 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \right\}.$$

$\star$ $\left[\,\underline{\text{Alternative Answer}}\,\right]$: $V_{\lambda_0} = \operatorname{Ker}\left(\lambda_0 I - A\right)$.

(2) We will prove the following statement:

" Assume that the characteristic equation $\chi_A(\lambda) = 0$ has no multiple roots in $\mathbb{C}$, namely, $\chi_A(\lambda)$ is written as

$$\chi_A(\lambda) = \left(\lambda - \lambda_1\right)\left(\lambda - \lambda_2\right)\cdots\left(\lambda - \lambda_n\right),$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{C}$ are all distinct. Then

$$\mathbb{C}^n = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n}.$$

Also,

$$\dim_{\mathbb{C}} V_{\lambda_1} = \dim_{\mathbb{C}} V_{\lambda_2} = \cdots = \dim_{\mathbb{C}} V_{\lambda_n} = 1. \text{ "}$$

Proof. First we prove that the sum $V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_n}$ is a direct sum. Thus, for each $j = 2, 3, \cdots, n$, we prove

$$\left(V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_{j-1}}\right) \cap V_{\lambda_j} = \left\{\mathbf{0}\right\}.$$

We will prove this by induction. Thus, we assume that

$$\begin{aligned} V_{\lambda_1} \cap V_{\lambda_2} &= \Big\{ \mathbf{0} \Big\}, \\ \Big(V_{\lambda_1} + V_{\lambda_2} \Big) \cap V_{\lambda_3} &= \Big\{ \mathbf{0} \Big\}, \\ \Big(V_{\lambda_1} + V_{\lambda_2} + V_{\lambda_3} \Big) \cap V_{\lambda_4} &= \Big\{ \mathbf{0} \Big\}, \\ \vdots \qquad\qquad & \vdots \\ \Big(V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_{j-2}} \Big) \cap V_{\lambda_{j-1}} &= \Big\{ \mathbf{0} \Big\}, \end{aligned}$$

are already proved. $\Big($Strictly speaking, in addition to this, we need to prove that $V_{\lambda_1} \cap V_{\lambda_2} = \Big\{ \mathbf{0} \Big\}$. However, the argument below covers the proof for this fact.$\Big)$

Let $\boldsymbol{x}_j$ be an arbitrary element in the intersection

$$\Big(V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_{j-1}} \Big) \cap V_{\lambda_j}.$$

It suffices to prove $\boldsymbol{x}_j = \mathbf{0}$. Since $\boldsymbol{x}_j \in V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_{j-1}}$, there exist

$$\boldsymbol{x}_1 \in V_{\lambda_1}, \qquad \boldsymbol{x}_2 \in V_{\lambda_2}, \qquad \cdots, \qquad \boldsymbol{x}_{j-1} \in V_{\lambda_{j-1}},$$

such that

$$(*) \qquad \boldsymbol{x}_j = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots + \boldsymbol{x}_{j-1}.$$

By the definition of the eigenspaces, we have

$$A\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1, \qquad A\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2, \qquad \cdots, \qquad A\boldsymbol{x}_{j-1} = \lambda_{j-1} \boldsymbol{x}_{j-1}.$$

Also, since $\boldsymbol{x}_j \in V_{\lambda_j}$, we have

$$A\boldsymbol{x}_j = \lambda_j \boldsymbol{x}_j.$$

Now, multiply A to the both sides of the identity $(*)$ from the left:

$$A\boldsymbol{x}_j \;=\; A\boldsymbol{x}_1 \;+\; A\boldsymbol{x}_2 \;+\; \cdots \;+\; A\boldsymbol{x}_{j-1}.$$

By the above data, we may rewrite this identity as

$$\lambda_j\,\boldsymbol{x}_j \;=\; \lambda_1\,\boldsymbol{x}_1 \;+\; \lambda_2\,\boldsymbol{x}_2 \;+\; \cdots \;+\; \lambda_{j-1}\,\boldsymbol{x}_{j-1}.$$

Compare this with the identity

$$\lambda_j\,\boldsymbol{x}_j \;=\; \lambda_j\,\boldsymbol{x}_1 \;+\; \lambda_j\,\boldsymbol{x}_2 \;+\; \cdots \;+\; \lambda_j\,\boldsymbol{x}_{j-1},$$

which is obtained by multiplying λ_j to the both sides of $(*)$. The left-hand sides of the last two identities are identical. Hence we may equate the right-hand sides of the last two identities:

$$\lambda_1\,\boldsymbol{x}_1 \;+\; \lambda_2\,\boldsymbol{x}_2 \;+\; \cdots \;+\; \lambda_{j-1}\,\boldsymbol{x}_{j-1} \;=\; \lambda_j\,\boldsymbol{x}_1 \;+\; \lambda_j\,\boldsymbol{x}_2 \;+\; \cdots \;+\; \lambda_j\,\boldsymbol{x}_{j-1}.$$

This can be rewritten as

$$\begin{aligned} \Big(\lambda_1 - \lambda_j\Big)\,\boldsymbol{x}_1 \;+\; \Big(\lambda_2 - \lambda_j\Big)\,\boldsymbol{x}_2 \;+\; \cdots \;+\; \Big(\lambda_{j-2} - \lambda_j\Big)\,\boldsymbol{x}_{j-2} & \\ = \;\Big(\lambda_j \;-\; \lambda_{j-1}\Big)\,\boldsymbol{x}_{j-1}. & \end{aligned}$$

By assumption, $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_j \in \mathbb{C}$ are all <u>distinct</u>. In particular,

$$\lambda_j \;-\; \lambda_{j-1} \;\neq\; 0.$$

Hence we may multiply the reciprocal of $\lambda_j - \lambda_{j-1}$ to the both sides of the above last identity, and obtain an expression of $\boldsymbol{x}_{j-1}$ as a $\mathbb{C}$-linear combination of $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\cdots$, $\boldsymbol{x}_{j-2}$. In particular,

$$\boldsymbol{x}_{j-1} \;\in\; V_{\lambda_1} \;+\; V_{\lambda_2} \;+\; \cdots \;+\; V_{\lambda_{j-2}}.$$

It is also true that $\boldsymbol{x}_{j-1} \in V_{\lambda_{j-1}}$. Hence

$$\boldsymbol{x}_{j-1} \;\in\; \Big(\, V_{\lambda_1} \;+\; V_{\lambda_2} \;+\; \cdots \;+\; V_{\lambda_{j-2}} \,\Big) \;\cap\; V_{\lambda_{j-1}}.$$

By the induction hypothesis, it follows that $\boldsymbol{x}_{j-1} \;=\; 0.$

The same argument works to prove

$$\boldsymbol{x}_{j-2} \;=\; \mathbf{0}, \qquad \boldsymbol{x}_{j-3} \;=\; \mathbf{0}. \qquad \cdots\,, \qquad \boldsymbol{x}_1 \;=\; \mathbf{0}.$$

By $(*)$, we conclude $\boldsymbol{x}_j \;=\; \mathbf{0}$. Hence our proof for the fact that the sum of $V_{\lambda_1},\ V_{\lambda_2},\ \cdots\ ,\ V_{\lambda_r}$ is a direct sum:

$$V_{\lambda_1} \;+\; V_{\lambda_2} \;+\; \cdots \;+\; V_{\lambda_n} \;\;=\;\; V_{\lambda_1} \;\oplus\; V_{\lambda_2} \;\oplus\; \cdots \;\oplus\; V_{\lambda_n},$$

is complete.

Now, we have the direct sum $V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n}$ as a $\mathbb{C}$-subspace of $\mathbb{C}^n$. Since for each j

$$\dim_{\mathbb{C}} V_{\lambda_j} \;\geq\; 1,$$

we have

$$\begin{aligned}
&\dim_{\mathbb{C}} \Big(V_{\lambda_1} \;\oplus\; V_{\lambda_2} \;\oplus\; \cdots \;\oplus\; V_{\lambda_n} \Big) \\
&\qquad \geq\; \dim_{\mathbb{C}} V_{\lambda_1} \;+ \dim_{\mathbb{C}} V_{\lambda_2} \;+\; \cdots \;+\; \dim_{\mathbb{C}} V_{\lambda_n} \\
&\qquad \geq\; n \\
&\qquad =\; \dim_{\mathbb{C}} \mathbb{C}^n.
\end{aligned}$$

Hence $V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n}$ must equal $\mathbb{C}^n$, and

$$\dim_{\mathbb{C}} V_{\lambda_1} \;=\; \dim_{\mathbb{C}} V_{\lambda_2} \;=\; \cdots \;=\; \dim_{\mathbb{C}} V_{\lambda_n} \;=\; 1.$$

Now our proof is complete.

[V] (60pts) Coming up.