Math 290 ELEMENTARY LINEAR ALGEBRA

SOLUTION FOR QUIZ – IV (02/07)

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[I] (12pts) (1) $\operatorname{Tr}\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 3 + 2 + 1 = 6.$

(2) True or false :

" For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we always have

$$\operatorname{Tr}\Big(AB \Big) = \operatorname{Tr}\Big(BA \Big). \text{ ''}$$

The answer is "true". Indeed, from

$$AB = \begin{bmatrix} ap + br & * \\ * & cq + ds \end{bmatrix}, \qquad BA = \begin{bmatrix} ap + cq & * \\ * & br + ds \end{bmatrix},$$

we have

$$\operatorname{Tr}\Big(AB \Big) = ap + br + cq + ds,$$

and

$$\operatorname{Tr}\Big(BA \Big) = ap + cq + br + ds.$$

These two are equal.

[II] (8pts) Let

$$\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

(1) “ The system

$$\begin{aligned} a_{11}\,x_1 \;+\; a_{12}\,x_2 \;+\; \cdots \;+\; a_{1n}\,x_n \;&=\; b_1, \\ a_{21}\,x_1 \;+\; a_{22}\,x_2 \;+\; \cdots \;+\; a_{2n}\,x_n \;&=\; b_2, \\ \cdots \qquad \cdots \qquad \cdots & \\ a_{m1}x_1 \;+ a_{m2}x_2 \;+ \cdots + a_{mn}x_n \;&=\; b_m \end{aligned}$$

is consistent, if and only if

$$\boxed{\boldsymbol{b} \;=\; s_1\,\boldsymbol{a}_1 \;+\; s_2\,\boldsymbol{a}_2 \;+\; \cdots \;+\; s_n\,\boldsymbol{a}_n} \;,$$

for some scalars $s_1,\; s_2,\; \cdots\;,\; s_n$. ”

(2) “ The homogeneous system

$$\begin{aligned} a_{11}\,x_1 \;+\; a_{12}\,x_2 \;+\; \cdots \;+\; a_{1n}\,x_n \;&=\; 0, \\ a_{21}\,x_1 \;+\; a_{22}\,x_2 \;+\; \cdots \;+\; a_{2n}\,x_n \;&=\; 0, \\ \cdots \qquad \cdots \qquad \cdots & \\ a_{m1}x_1 \;+ a_{m2}x_2 \;+ \cdots + a_{mn}x_n \;&=\; 0 \end{aligned}$$

has a non-trivial solution, if and only if

$$\boxed{\boldsymbol{0} \;=\; s_1\,\boldsymbol{a}_1 \;+\; s_2\,\boldsymbol{a}_2 \;+\; \cdots \;+\; s_n\,\boldsymbol{a}_n} \;,$$

for some scalars $s_1,\; s_2,\; \cdots\;,\; s_n,$ not all of which equal $\boxed{0}$. ”

[III] (15pts) Let $A \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ and $B \;=\; \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(1) Find $A^2 - B^2$. For this matter, we first calculate A^2, and B^2 each.

$$A^2 = A\,A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and

$$B^2 = B\,B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Accordingly, we may find $A^2 - B^2$ as

$$A^2 - B^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(2) Find $AB - BA$. For this matter, we first calculate AB, and BA each.

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Accordingly, we may find $AB - BA$ as

$$AB - BA = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(3) Find $\Big(A - B\Big)\Big(A + B\Big)$. For this matter, notice

$$\Big(A - B\Big)\Big(A + B\Big) = A^2 + AB - BA - B^2$$

$$= \Big(A^2 - B^2\Big) + \Big(AB - BA\Big).$$

In other words, the answer for (3) is obtained by simply adding the answer for (1) and the answer for (2).

$$\Big(A - B\Big)\Big(A + B\Big) = \Big(A^2 - B^2\Big) + \Big(AB - BA\Big)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

⋆ Alternatively, you may multiply the two matrices

$$A - B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

in this order.

(4) Find $2\,BAB - B^2A - AB^2$. For this matter, notice

$$2\,BAB - B^2A - AB^2 = B\Big(AB - BA\Big) - \Big(AB - BA\Big)B.$$

In other words, by setting $C = AB - BA$, we may rewrite the matrix to find as $BC - CB$. We already found C as

$$C = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So all we have to do is calculate BC, and CB each, and then subtract CB from BC. First,

$$BC = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Second,

$$CB = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$BC - CB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

⋆ Agree that this matrix $BC - CB$ equals A.

(5) Find $2\,ABA - BA^2 - A^2B$. For this matter, notice

$$2\,ABA - BA^2 - A^2B = \Big(AB - BA\Big)A - A\Big(AB - BA\Big).$$

In other words, by setting $C = AB - BA$, we may rewrite the matrix to find as $CA - AC$. We have

$$C = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, as in (4), all we have to do is calculate CA, and AC each, and then subtract AC from CA. First,

$$CA = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Second,

$$AC = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Hence

$$CA - AC = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$\star$ Agree that this matrix $CA - AC$ equals B.

(6)* (Extra 10pts with a full justification)

True or false :

" An arbitrary 3×3 skew-symmetric matrix is a linear combination of

A, B and $AB - BA$. "

The answer is "true". Indeed, an arbitrary 3×3 skew-symmetric matrix is written as

$$\begin{bmatrix} 0 & -c & -b \\ c & 0 & a \\ b & -a & 0 \end{bmatrix},$$

using scalars (real numbers) a, b and c. This matrix is rewritten as

$$\begin{bmatrix} 0 & -c & -b \\ c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= a \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In other words, using A, B and $C = AB - BA$ as above, we may write

$$\begin{bmatrix} 0 & -c & -b \\ c & 0 & a \\ b & -a & 0 \end{bmatrix} = a\,A + b\,B + c\,C,$$

or the same to say,

$$\begin{bmatrix} 0 & -c & -b \\ c & 0 & a \\ b & -a & 0 \end{bmatrix} = a\,A + b\,B + c\Big(AB - BA\Big).$$

This shows that an arbitrary 3×3 skew-symmetric matrix is a linear combination of A, B and $C = AB - BA$.