Math 290 ELEMENTARY LINEAR ALGEBRA
SOLUTION FOR QUIZ – V (02/07)

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[I] (10pts) Let

$$\boldsymbol{a}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

(1) $\boldsymbol{b} = 4\,\boldsymbol{a}_1 + 2\,\boldsymbol{a}_2 + 3\,\boldsymbol{a}_3.$

(2) For the above $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, define

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix}.$$

This means

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(3) For the above A, $\boldsymbol{b}$, and the unknown column vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ is <u>consistent</u>.

(4) Let $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. For the above $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, <u>trivially</u>,

$$\mathbf{0} = 0\,\boldsymbol{a}_1 + 0\,\boldsymbol{a}_2 + 0\,\boldsymbol{a}_3.$$

(5) There is no way to make

$$\mathbf{0} = s_1\,\boldsymbol{a}_1 + s_2\,\boldsymbol{a}_2 + s_3\,\boldsymbol{a}_3,$$

where $s_1 \neq 0$, or $s_2 \neq 0$, or $s_3 \neq 0$.

Justification for (5). Indeed, if $\mathbf{0} = s_1\,\boldsymbol{a}_1 + s_2\,\boldsymbol{a}_2 + s_3\,\boldsymbol{a}_3$, then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = s_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{s_2} \\ \boxed{s_3} \\ \boxed{s_1} \end{bmatrix}.$$

This reads $s_1 = 0,\ s_2 = 0,\ s_3 = 0.$

(6) For the above A, $\mathbf{0}$, and the unknown column vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the homogeneous system of linear equations $A\boldsymbol{x} = \mathbf{0}$ <u>has only the trivial solution</u>.

[II] (10pts) Let

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad B = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix},$$

$$C = \begin{bmatrix} \cos\Big(\theta + \phi\Big) & -\sin\Big(\theta + \phi\Big) \\ \sin\Big(\theta + \phi\Big) & \cos\Big(\theta + \phi\Big) \end{bmatrix}.$$

(1) We may write out AB in terms of $\sin\theta$, $\cos\theta$, $\sin\phi$ and $\cos\phi$, as

AB

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \big(\cos\theta\big)\big(\cos\phi\big) - \big(\sin\theta\big)\big(\sin\phi\big) & -\Big[\big(\sin\theta\big)\big(\cos\phi\big) + \big(\cos\theta\big)\big(\sin\phi\big)\Big] \\ \big(\sin\theta\big)\big(\cos\phi\big) + \big(\cos\theta\big)\big(\sin\phi\big) & \big(\cos\theta\big)\big(\cos\phi\big) - \big(\sin\theta\big)\big(\sin\phi\big) \end{bmatrix}.$$

(2) We may read off the four entries of $C = AB$, and may write out the consequence as

$$\cos\Big(\theta + \phi\Big) = \Big(\cos\theta\Big)\Big(\cos\phi\Big) - \Big(\sin\theta\Big)\Big(\sin\phi\Big),$$
$$\sin\Big(\theta + \phi\Big) = \Big(\sin\theta\Big)\Big(\cos\phi\Big) + \Big(\cos\theta\Big)\Big(\sin\phi\Big).$$

(3) <u>True or false</u> : " For the above A and B, $AB = BA$ holds. "

The answer is "true". Indeed, AB equals

$$C = \begin{bmatrix} \cos\Big(\theta + \phi\Big) & -\sin\Big(\theta + \phi\Big) \\ \sin\Big(\theta + \phi\Big) & \cos\Big(\theta + \phi\Big) \end{bmatrix}.$$

By interchanging the roles of θ and ϕ, it follows that BA equals

$$\begin{bmatrix} \cos\Big(\phi + \theta\Big) & -\sin\Big(\phi + \theta\Big) \\ \sin\Big(\phi + \theta\Big) & \cos\Big(\phi + \theta\Big) \end{bmatrix}.$$

Since $\phi + \theta = \theta + \phi$, it follows that this last matrix equals C. In short, AB and BA both equal C. In particular, $AB = BA$.