

Math 290 ELEMENTARY LINEAR ALGEBRA
SOLUTION FOR QUIZ – VI (02/12)

February 19 (Tue), 2008

Instructor: Yasuyuki Kachi

Line #: 74449 / 82588.

[I] (10pts) Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(1) There is no way to express \mathbf{b} as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

Justification for (1). Indeed, if $\mathbf{b} = s_1 \mathbf{a}_1 + s_2 \mathbf{a}_2 + s_3 \mathbf{a}_3$, then

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = s_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

that is,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 - s_2 \\ s_2 - s_3 \\ -s_1 + s_3 \end{bmatrix}.$$

Add up the three entries on each side, and $3 = 0$, a contradiction.

(2) For the above $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, define

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}.$$

This means

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

(3) For the above A , \mathbf{b} , and the unknown column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the system of linear equations $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(4) Let $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. For the above \mathbf{a}_1 , \mathbf{a}_2 , \mathbf{a}_3 , trivially,

$$\mathbf{0} = 0\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3.$$

(5) There is a way to make

$$\mathbf{0} = s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + s_3\mathbf{a}_3,$$

where $s_1 \neq 0$, or $s_2 \neq 0$, or $s_3 \neq 0$. For example,

$$\mathbf{0} = 1 \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3.$$

(6) For the above A , $\mathbf{0}$, and the unknown column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \boxed{1} \\ \boxed{1} \\ \boxed{1} \end{bmatrix}.$$

[II] (10pts) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(1)
$$A^T = \begin{bmatrix} \boxed{a} & \boxed{c} \\ \boxed{b} & \boxed{d} \end{bmatrix}.$$

$$(2) \quad AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix},$$

$$A^T A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix}.$$

(3) True or false :

“ Both AA^T and $A^T A$ are symmetric. ”

The answer is “true”. Indeed, by (2), the two entries off the main diagonal in AA^T are both $ac + bd$. Hence AA^T is symmetric. Similarly, by (2), the two entries off the main diagonal in $A^T A$ are both $ab + cd$. Hence $A^T A$ is symmetric.

(4) True or false :

“ $AA^T = A^T A$ always. ”

The answer is “false”. Indeed, for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we have $A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Accordingly,

$$AA^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$A^T A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we have $AA^T \neq A^T A$.

(5)* (Extra 20pts with a full justification)

True or false :

“ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AA^T = A^T A$ holds if and only if

either

$$(i) \quad A = A^T, \quad \text{or} \quad (ii) \quad AA^T = (ad - bc)I. ”$$

The answer is “true”. Justification is an extra credit problem, **due March 6th (Thu)**. Note the following:

★ You must provide a proof for both implications :

[Implication (A)] :

“ Either the condition (i) or the condition (ii) for A implies $AA^T = A^T A$ ”

(worth 10 pts), and

[Implication (B)] :

“ $AA^T = A^T A$ implies either the condition (i) or the condition (ii) for A ”

(worth 10 pts).

★ Use the following fact, if necessary:

“ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $AA^T = (ad - bc)I$, and moreover if

$ad - bc \neq 0$, then $A^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. ”

★ Note that, separately from the above, you will have to characterize A satisfying $AA^T = O$. You may prove that, if you allow only real numbers for entries of A , then the condition $AA^T = O$ implies $A = O$. In particular, $A^T A = O$, and hence $AA^T = A^T A$.

The subtlety of the matter is, if, on the contrary, you allow complex numbers for entries of A , then the same condition $AA^T = O$ does not imply $A = O$. For example, consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{-1} \\ 1 & \sqrt{-1} \end{bmatrix}$. Then $AA^T = O$. In this example, actually $A^T A \neq O$. In particular, $AA^T \neq A^T A$. This example offers a counterexample to the statement (5), but not within the real number system.

Thus, when you solve this problem (5), it is important to agree that, a , b , c and d are all assumed to be real numbers.