Math 290 ELEMENTARY LINEAR ALGEBRA

SOLUTION FOR QUIZ – VII (02/12)

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[I] (10pts) (1) For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \\ A^3 &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1 + 2 \cdot 1 & 2 \cdot 1 + 2 \cdot 1 \\ 2 \cdot 1 + 2 \cdot 1 & 2 \cdot 1 + 2 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 2^2 & 2^2 \\ 2^2 & 2^2 \end{bmatrix}. \end{aligned}$$

(2) For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ as in (1), we may find

$$\begin{aligned} A^4 &= A^3 \, A \\ &= \begin{bmatrix} 2^2 & 2^2 \\ 2^2 & 2^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^2 \cdot 1 + 2^2 \cdot 1 & 2^2 \cdot 1 + 2^2 \cdot 1 \\ 2^2 \cdot 1 + 2^2 \cdot 1 & 2^2 \cdot 1 + 2^2 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 2^2 & 2 \cdot 2^2 \\ 2 \cdot 2^2 & 2 \cdot 2^2 \end{bmatrix} = \begin{bmatrix} 2^3 & 2^3 \\ 2^3 & 2^3 \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
A^5 &= A^4 \, A \\
&= \begin{bmatrix} 2^3 & 2^3 \\ 2^3 & 2^3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2^3 \cdot 1 + 2^3 \cdot 1 & 2^3 \cdot 1 + 2^3 \cdot 1 \\ 2^3 \cdot 1 + 2^3 \cdot 1 & 2^3 \cdot 1 + 2^3 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot 2^3 & 2 \cdot 2^3 \\ 2 \cdot 2^3 & 2 \cdot 2^3 \end{bmatrix} = \begin{bmatrix} 2^4 & 2^4 \\ 2^4 & 2^4 \end{bmatrix},
\end{aligned}
$$

and so on. More generally, for $\underline{\text{ a positive integer }}$ m, we may find

$$
\begin{aligned}
A^m &= \begin{bmatrix} 2^{m-1} & 2^{m-1} \\ 2^{m-1} & 2^{m-1} \end{bmatrix} \\
&= 2^{m-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= 2^{m-1} A.
\end{aligned}
$$

(3) For $f(x) = x^2 - 2x,$ and for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$

$$
\begin{aligned}
f(A) &= A^2 - 2A \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 - 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

[II] (10pts) (1) For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$

$$
A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
$$

$$
\begin{aligned}
&= \begin{bmatrix} 0\cdot 0 \;+\; \big(-1\big)\cdot 1 & 0\cdot\big(-1\big) \;+\; \big(-1\big)\cdot 0 \\ 1\cdot 0 \;+\; 0\cdot 1 & 1\cdot\big(-1\big) \;+\; 0\cdot 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \;=\; -I,
\end{aligned}
$$

$$
\begin{aligned}
A^3 &= A^2 \cdot A \\
&= -I \cdot A \\
&= -A \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
A^4 &= A^2 \cdot A^2 \\
&= \Big(-I\Big)\cdot\Big(-I\Big) \\
&= I \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

(2) For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ as in (1), we have

$$
\begin{aligned}
A^5 &= A^4 \cdot A = I \cdot A = A, \\
A^6 &= A^4 \cdot A^2 = I \cdot \Big(-I\Big) = -I, \\
A^7 &= A^4 \cdot A^3 = I \cdot \Big(-A\Big) = -A, \\
A^8 &= A^4 \cdot A^4 = I \cdot I = I, \\
A^9 &= A^8 \cdot A = I \cdot A = -A, \\
A^{10} &= A^8 \cdot A^2 = I \cdot \Big(-I\Big) = -I, \\
A^{11} &= A^8 \cdot A^3 = I \cdot \Big(-A\Big) = -A, \\
A^{12} &= A^8 \cdot A^4 = I \cdot I = I,
\end{aligned}
$$

and so on.

We conclude that,

$$A^m \;=\; A \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \text{when} \qquad m \;=\; 5,\, 9,\, 13,\, 17,\, 21,\, \cdots\;,$$

$$A^m \;=\; -\,I \;=\; \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{when} \qquad m \;=\; 6,\, 10,\, 14,\, 18,\, 22,\, \cdots\;,$$

$$A^m \;=\; -\,A \;=\; \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \text{when} \qquad m \;=\; 7,\, 11,\, 15,\, 19,\, 23,\, \cdots\;,$$

$$A^m \;=\; I \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{when} \qquad m \;=\; 8,\, 12,\, 16,\, 20,\, 24,\, \cdots\;.$$

(3) For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ as in (1), and for

$$f\left(x\right) \;=\; 1 \;+\; x \;+\; \frac{1}{2}x^2 \;+\; \frac{1}{3 \cdot 2}x^3 \;+\; \frac{1}{4 \cdot 3 \cdot 2}x^4,$$

$$f\big(tA\big) \;=\; I \;+\; t \cdot A \;+\; \frac{t^2}{2} \cdot A^2 \;+\; \frac{t^3}{3 \cdot 2} \cdot A^3 \;+\; \frac{t^4}{4 \cdot 3 \cdot 2} \cdot A^4$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;+\; \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} \;+\; \frac{1}{2}\begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix}$$

$$+\; \frac{1}{3 \cdot 2}\begin{bmatrix} 0 & t^3 \\ -t^3 & 0 \end{bmatrix} \;+\; \frac{1}{4 \cdot 3 \cdot 2}\begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 & -\,t + \frac{1}{6}t^3 \\ t - \frac{1}{6}t^3 & 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 \end{bmatrix}.$$

⋆ $\left[\,\underline{\text{Note}}\,\right]$: For an arbitrary positive integer m, set

$$f_{4m}\left(x\right) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \cdots + \frac{1}{(4m)\,!}\,x^{4m}$$

$$= \sum_{k=0}^{4m} \frac{1}{k\,!}\,x^k,$$

where

$$k\,! = k\left(k-1\right)\left(k-2\right)\left(k-3\right)\cdots 4\cdot 3\cdot 2$$

$\left(\underline{\text{the factorial of } k}\right)$. Note $1\,! = 0\,! = 1$. For the above $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$$f_{4m}\left(tA\right)$$

$$= \sum_{k=0}^{4m} \frac{t^k}{k\,!}\,A^k$$

$$= \begin{bmatrix} 1-\dfrac{t^2}{2!}+\dfrac{t^4}{4!}-\cdots+\dfrac{t^{4m}}{(4m)\,!} & -t+\dfrac{t^3}{3!}-\dfrac{t^5}{5!}+\cdots+\dfrac{t^{4m-1}}{(4m-1)\,!} \\ t-\dfrac{t^3}{3!}+\dfrac{t^5}{5!}-\cdots-\dfrac{t^{4m-1}}{(4m-1)\,!} & 1-\dfrac{t^2}{2!}+\dfrac{t^4}{4!}-\cdots+\dfrac{t^{4m}}{(4m)\,!} \end{bmatrix}.$$

We have

$$\lim_{m\longrightarrow\infty} f_{4m}\left(x\right) = e^x = \exp x,$$

and

$$\lim_{m\longrightarrow\infty} f_{4m}\left(tA\right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$= \left(\cos t\right) I + \left(\sin t\right) A.$$

[III] (10pts) For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and for

$$f(x) = x^2 - \Big(a + d\Big)x + \Big(ad - bc\Big),$$

$$f(A)$$

$$= A^2 - \Big(a + d\Big)A + \Big(ad - bc\Big)I$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - \Big(a + d\Big)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \Big(ad - bc\Big)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$= \begin{bmatrix} bc - ad & 0 \\ 0 & bc - ad \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

⋆ This fact is a special case (the case of 2 × 2) of the Cayley–Hamilton's theorem.