Math 290 ELEMENTARY LINEAR ALGEBRA

SOLUTION FOR QUIZ – VIII (03/11)

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[I] (8pts) (1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called <u>non-singular</u>, if

$$ad - bc \neq 0.$$

If A is non-singular, define

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(2) For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, A <u>is non-singular</u>. Indeed,

$$1 \cdot 4 - 2 \cdot 3 = -2 \neq 0.$$

Moreover,

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$$

[II] (12pts) (1) For $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$, its determinant $\Delta = \det A$ is defined as

$$\begin{aligned} \Delta = \; & aqz - ary - bpz \\ & + brx + cpy - cqx. \end{aligned}$$

We may use this definition to calculate the determinant for $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$:

$$\begin{aligned}\Delta &= 2\cdot 2\cdot 1 - 2\cdot 1\cdot 0 - 1\cdot 3\cdot 1 \\ &\quad + 1\cdot 1\cdot 1 + (-1)\cdot 3\cdot 0 - (-1)\cdot 2\cdot 1 \\ &= 4 - 0 - 3 + 1 + 0 - (-2) \\ &= 4.\end{aligned}$$

Since we found $\Delta \neq 0$, A is non-singular.

(2) For $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$, its inverse A^{-1} is found as

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} +\begin{vmatrix} q & r \\ y & z \end{vmatrix} & -\begin{vmatrix} b & c \\ y & z \end{vmatrix} & +\begin{vmatrix} b & c \\ q & r \end{vmatrix} \\ -\begin{vmatrix} p & r \\ x & z \end{vmatrix} & +\begin{vmatrix} a & c \\ x & z \end{vmatrix} & -\begin{vmatrix} a & c \\ p & r \end{vmatrix} \\ +\begin{vmatrix} p & q \\ x & y \end{vmatrix} & -\begin{vmatrix} a & b \\ x & y \end{vmatrix} & +\begin{vmatrix} a & b \\ p & q \end{vmatrix} \end{bmatrix},$$

<u>provided $\Delta \neq 0$</u>. For $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (the same matrix as (1)), we found the value of Δ to be 4, which is not equal to 0. Thus

$$A^{-1} = \frac{1}{4}\begin{bmatrix} +\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} \\ +\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -1 & 3 \\ -2 & 3 & -5 \\ -2 & 1 & 1 \end{bmatrix}.$$

[III] (20pts) We simplify the same 4×4 determinant in two ways ((a), (b) below):

(a)
$$\begin{vmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ p & q & t & 0 \\ r & s & 0 & t \end{vmatrix} = \begin{vmatrix} \boxed{1} & \boxed{0} & \boxed{a} & \boxed{b} \\ \boxed{0} & \boxed{1} & \boxed{c} & \boxed{d} \\ \boxed{0} & \boxed{q} & \boxed{t-ap} & \boxed{-bp} \\ \boxed{0} & \boxed{s} & \boxed{-ar} & \boxed{t-br} \end{vmatrix}$$

$\Big[$Add $(-p)$ times (row 1) to (row 3); add $(-r)$ times (row 1) to (row 4)$\Big]$

$$= \begin{vmatrix} \boxed{1} & \boxed{c} & \boxed{d} \\ \boxed{q} & \boxed{t-ap} & \boxed{-bp} \\ \boxed{s} & \boxed{-ar} & \boxed{t-br} \end{vmatrix}$$

$\Big[$Co-factor with respect to (column 1)$\Big]$

$$= \begin{vmatrix} \boxed{1} & \boxed{c} & \boxed{d} \\ \boxed{0} & \boxed{t-ap-cq} & \boxed{-bp-dq} \\ \boxed{0} & \boxed{-ar-cs} & \boxed{t-br-ds} \end{vmatrix}.$$

$\Big[$Add $(-q)$ times (row 1) to (row 2); add $(-s)$ times (row 1) to (row 3)$\Big]$

$$= \begin{vmatrix} \boxed{t-ap-cq} & \boxed{-bp-dq} \\ \boxed{-ar-cs} & \boxed{t-br-ds} \end{vmatrix}.$$

$\Big[$Co-factor with respect to (column 1)$\Big]$

(b) $$\begin{vmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ p & q & t & 0 \\ r & s & 0 & t \end{vmatrix} = \begin{vmatrix} \boxed{1-(br/t)} & \boxed{-(bs/t)} & \boxed{a} & \boxed{0} \\ \boxed{-(dr/t)} & \boxed{1-(ds/t)} & \boxed{c} & \boxed{0} \\ \boxed{p} & \boxed{q} & \boxed{t} & \boxed{0} \\ \boxed{r} & \boxed{s} & \boxed{0} & \boxed{t} \end{vmatrix}$$

$\Big[$Add $(-b/t)$ times (row 4) to (row 1); add $(-d/t)$ times (row 4) to (row 2)$\Big]$

$$= \quad t \begin{vmatrix} \boxed{1-(br/t)} & \boxed{-(bs/t)} & \boxed{a} \\ \boxed{-(dr/t)} & \boxed{1-(ds/t)} & \boxed{c} \\ \boxed{p} & \boxed{q} & \boxed{t} \end{vmatrix}$$

$\Big[$Co-factor with respect to (column 4)$\Big]$

$$= \quad t \begin{vmatrix} \boxed{1-(br/t)-(ap/t)} & \boxed{-(bs/t)-(aq/t)} & \boxed{0} \\ \boxed{-(dr/t)-(cp/t)} & \boxed{1-(ds/t)-(cq/t)} & \boxed{0} \\ \boxed{p} & \boxed{q} & \boxed{t} \end{vmatrix}$$

$\Big[$Add $(-a/t)$ times (row 3) to (row 1); add $(-c/t)$ times (row 3) to (row 2)$\Big]$

$$= \quad t^2 \begin{vmatrix} \boxed{1-(br/t)-(ap/t)} & \boxed{-(bs/t)-(aq/t)} \\ \boxed{-(dr/t)-(cp/t)} & \boxed{1-(ds/t)-(cq/t)} \end{vmatrix}$$

$\Big[$Co-factor with respect to (column 3)$\Big]$

$$= \begin{vmatrix} \boxed{t-ap-br} & \boxed{-aq-bs} \\ \boxed{-cp-dr} & \boxed{t-cq-ds} \end{vmatrix} .$$

The result in (a) and the result in (b) represent the same determinant

$$\begin{vmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ p & q & t & 0 \\ r & s & 0 & t \end{vmatrix} .$$

Thus the results for (a) and (b) must be equal:

$$(*) \quad \begin{vmatrix} t - ap - cq & - bp - dq \\ - ar - cs & t - br - ds \end{vmatrix} = \begin{vmatrix} t - ap - br & - aq - bs \\ - cp - dr & t - cq - ds \end{vmatrix} .$$

Notice that <u>the left-hand side</u> of this identity $(*)$ is the determinant of

$$\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{bmatrix} ,$$

or, the same to say,

$$\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

Similarly, notice that <u>the right-hand side</u> of this identity $(*)$ is the determinant of

$$\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} ,$$

or, the same to say,

$$\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} .$$

Hence, we may paraphrase this identity $(*)$ as in

$$\det\left(\begin{bmatrix} \boxed{t} & \boxed{0} \\ \boxed{0} & \boxed{t} \end{bmatrix} - \begin{bmatrix} \boxed{p} & \boxed{q} \\ \boxed{r} & \boxed{s} \end{bmatrix} \begin{bmatrix} \boxed{a} & \boxed{b} \\ \boxed{c} & \boxed{d} \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} \boxed{t} & \boxed{0} \\ \boxed{0} & \boxed{t} \end{bmatrix} - \begin{bmatrix} \boxed{a} & \boxed{b} \\ \boxed{c} & \boxed{d} \end{bmatrix} \begin{bmatrix} \boxed{p} & \boxed{q} \\ \boxed{r} & \boxed{s} \end{bmatrix} \right) .$$

Equivalently,

$$\det\Big(t\,I - B\,A \Big) = \det\Big(t\,I - A\,B \Big) ,$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

⋆ **Extra Credit Problem (Due March 25th (Tue).)**

Generalize the above argument to prove that, for arbitrary two $n \times n$ matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix},$$

and a scalar t, prove

$$\det\Big(t\,I - B\,A \Big) = \det\Big(t\,I - A\,B \Big) .$$

Write your answer in a separate sheet.