

Math 290 ELEMENTARY LINEAR ALGEBRA
QUIZ – IX (TAKE HOME)

February 28 (Thu), 2008

Due Date: March 6 (Thu), 2008

Instructor: Yasuyuki Kachi

Line #: 74449 / 82588 .

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[I] (40pts) Solve
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \end{cases}$$
 under the conditions

(*) $a_{11} \neq 0$, $a_{11}a_{22} - a_{12}a_{21} \neq 0$, and

$\Delta = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0$.

(1) Multiply $(1/a_{11})$ to (row 1) of the augmented matrix:

$$\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{b_1}{a_{11}} \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right] .$$

(2) Add $[(-a_{21}) \times (\text{row 1})]$ to (row 2), add $[(-a_{31}) \times (\text{row 1})]$ to (row 3):

$$\rightarrow \left[\begin{array}{cccc} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{13}a_{21}}{a_{11}} & b_2 - \frac{a_{21}b_1}{a_{11}} \\ 0 & a_{32} - \frac{a_{12}a_{31}}{a_{11}} & a_{33} - \frac{a_{13}a_{31}}{a_{11}} & b_3 - \frac{a_{31}b_1}{a_{11}} \end{array} \right] .$$

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([I] continued)

(3) Agree that the resulting matrix in (2) is the same as

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} & \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}} & \frac{a_{11}b_2 - b_1a_{21}}{a_{11}} \\ 0 & \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}} & \frac{a_{11}a_{33} - a_{13}a_{31}}{a_{11}} & \frac{a_{11}b_3 - b_1a_{31}}{a_{11}} \end{bmatrix} .$$

Multiply a_{11} to (row 2), and also multiply a_{11} to (row 3):

$$\rightarrow \begin{bmatrix} \boxed{1} & \frac{\boxed{a_{12}}}{\boxed{a_{11}}} & \frac{\boxed{a_{13}}}{\boxed{a_{11}}} & \frac{\boxed{b_1}}{\boxed{a_{11}}} \\ \boxed{0} & \boxed{a_{11}a_{22} - a_{12}a_{21}} & \boxed{\phantom{a_{11}a_{22} - a_{12}a_{21}}} & \boxed{\phantom{a_{11}a_{22} - a_{12}a_{21}}} \\ \boxed{0} & \boxed{a_{11}a_{32} - a_{12}a_{31}} & \boxed{\phantom{a_{11}a_{32} - a_{12}a_{31}}} & \boxed{\phantom{a_{11}a_{32} - a_{12}a_{31}}} \end{bmatrix} .$$

(4) Multiply $1/(a_{11}a_{22} - a_{12}a_{21})$ to (row 2):

$$\rightarrow \begin{bmatrix} \boxed{1} & \frac{\boxed{a_{12}}}{\boxed{a_{11}}} & \frac{\boxed{a_{13}}}{\boxed{a_{11}}} & \frac{\boxed{b_1}}{\boxed{a_{11}}} \\ \boxed{0} & \boxed{1} & \frac{\boxed{\phantom{a_{11}a_{22} - a_{12}a_{21}}}}{\boxed{a_{11}a_{22} - a_{12}a_{21}}} & \frac{\boxed{\phantom{a_{11}a_{22} - a_{12}a_{21}}}}{\boxed{a_{11}a_{22} - a_{12}a_{21}}} \\ \boxed{0} & \boxed{a_{11}a_{32} - a_{12}a_{31}} & \boxed{\phantom{a_{11}a_{32} - a_{12}a_{31}}} & \boxed{\phantom{a_{11}a_{32} - a_{12}a_{31}}} \end{bmatrix} .$$

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([I] continued)

(5) Agree that the resulting matrix in (4) is the same as

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 1 & \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} & a_{11}b_3 - b_1a_{31} \end{bmatrix} .$$

Add $\left[-(a_{11}a_{32} - a_{12}a_{31}) \times (\text{row } 2) \right]$ to (row 3):

$$\rightarrow \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 1 & \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \\ 0 & 0 & \boxed{\text{(a)}} & \boxed{\text{(b)}} \end{bmatrix} .$$

Agree

$$\boxed{\text{(a)}}$$

$$= - \left(a_{11}a_{32} - a_{12}a_{31} \right) \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}} + \left(a_{11}a_{33} - a_{13}a_{31} \right) ,$$

and

$$\boxed{\text{(b)}}$$

$$= - \left(a_{11}a_{32} - a_{12}a_{31} \right) \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} + \left(a_{11}b_3 - b_1a_{31} \right) .$$

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([I] continued)

We simplify (a) and (b) each. First,

(a)

$$\begin{aligned}
 &= \frac{-(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{23} - a_{13}a_{21})}{a_{11}a_{22} - a_{12}a_{21}} + \frac{(a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{33} - a_{13}a_{31})}{a_{11}a_{22} - a_{12}a_{21}} \\
 &= \frac{-a_{11}^2 \square \square + a_{11}a_{13} \square \square + a_{11}a_{12} \square \square - a_{12}a_{13} \square \square}{a_{11}a_{22} - a_{12}a_{21}} \\
 &+ \frac{a_{11}^2 \square \square - a_{11}a_{13} \square \square - a_{11}a_{12} \square \square + a_{12}a_{13} \square \square}{a_{11}a_{22} - a_{12}a_{21}}.
 \end{aligned}$$

(6) Agree that the above is

$$\begin{aligned}
 \text{(a)} &= \frac{-**** + **** + **** - a_{12}a_{13}a_{21}a_{31}}{a_{11}a_{22} - a_{12}a_{21}} \\
 &+ \frac{**** - **** - **** + a_{12}a_{13}a_{21}a_{31}}{a_{11}a_{22} - a_{12}a_{21}},
 \end{aligned}$$

which is a sum of two fractions with a common denominator. Therefore the two terms $-a_{12}a_{13}a_{21}a_{31}$ and $+a_{12}a_{13}a_{21}a_{31}$ in the numerators cancel out. Thus,

$$\begin{aligned}
 \text{(a)} &= \frac{-a_{11}^2 \square \square + a_{11}a_{13} \square \square + a_{11}a_{12} \square \square}{a_{11}a_{22} - a_{12}a_{21}} \\
 &+ \frac{a_{11}^2 \square \square - a_{11}a_{13} \square \square - a_{11}a_{12} \square \square}{a_{11}a_{22} - a_{12}a_{21}}.
 \end{aligned}$$

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([I] continued)

(7) Agree that we have obtained

$$\begin{aligned} & \boxed{(a)} \\ &= \frac{-a_{11}^2 \boxed{a_{23}} \boxed{a_{32}} + a_{11}a_{13} \boxed{a_{21}} \boxed{a_{32}} + a_{11}a_{12} \boxed{a_{23}} \boxed{a_{31}}}{a_{11}a_{22} - a_{12}a_{21}} \\ &+ \frac{a_{11}^2 \boxed{a_{22}} \boxed{a_{33}} - a_{11}a_{13} \boxed{a_{22}} \boxed{a_{31}} - a_{11}a_{12} \boxed{a_{21}} \boxed{a_{33}}}{a_{11}a_{22} - a_{12}a_{21}}. \end{aligned}$$

Now, each term in the numerators in each fraction has factor a_{11} . Thus

$$\begin{aligned} & \boxed{(a)} \\ &= \frac{-a_{11}^2 \boxed{a_{23}} \boxed{a_{32}} + a_{11}a_{13} \boxed{a_{21}} \boxed{a_{32}} + a_{11}a_{12} \boxed{a_{23}} \boxed{a_{31}}}{a_{11}a_{22} - a_{12}a_{21}} \\ &+ \frac{a_{11}^2 \boxed{a_{22}} \boxed{a_{33}} - a_{11}a_{13} \boxed{a_{22}} \boxed{a_{31}} - a_{11}a_{12} \boxed{a_{21}} \boxed{a_{33}}}{a_{11}a_{22} - a_{12}a_{21}} \\ &= \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \left(-a_{11} \boxed{} \boxed{} + a_{13} \boxed{} \boxed{} + a_{12} \boxed{} \boxed{} \right) \\ &+ \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \left(a_{11} \boxed{} \boxed{} - a_{13} \boxed{} \boxed{} - a_{12} \boxed{} \boxed{} \right). \end{aligned}$$

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([I] continued)

(8) Agree that we have obtained

(a)

$$= \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \left(-a_{11} \boxed{a_{23}} \boxed{a_{32}} + a_{13} \boxed{a_{21}} \boxed{a_{32}} + a_{12} \boxed{a_{23}} \boxed{a_{31}} \right) \\ + \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \left(a_{11} \boxed{a_{22}} \boxed{a_{33}} - a_{13} \boxed{a_{22}} \boxed{a_{31}} - a_{12} \boxed{a_{21}} \boxed{a_{33}} \right).$$

We re-order terms and

(a)

$$= \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \cdot \\ \left(\underbrace{a_{11}a_{22} \boxed{} - a_{11} \boxed{} a_{32} - a_{12}a_{21} \boxed{} + a_{12} \boxed{} a_{31} + \boxed{} a_{21}a_{32} - \boxed{} a_{22}a_{31}}_{\Delta} \right) \\ = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \Delta ,$$

where

$\Delta =$

_____ .

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([I] continued)

(9) Now agree that we have reduced

$$\boxed{(a)} = - \left(a_{11}a_{32} - a_{12}a_{31} \right) \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}} + \left(a_{11}a_{33} - a_{13}a_{31} \right)$$

from page 3 to

$$\boxed{(a)} = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \Delta,$$

where

$$\begin{aligned} \Delta = & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ & + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

Agree that the only difference between

$$\boxed{(a)} = - \left(a_{11}a_{32} - a_{12}a_{31} \right) \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}} + \left(a_{11}a_{33} - a_{13}a_{31} \right)$$

(= the original form of (a) in page 3) and

$$\boxed{(b)} = - \left(a_{11}a_{32} - a_{12}a_{31} \right) \frac{a_{11}b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} + \left(a_{11}b_3 - b_1 a_{31} \right)$$

(= the original form of (b) in page 3) is that a_{13} , a_{23} , a_{33} in (a) are replaced with b_1 , b_2 , b_3 in (b), respectively. Thus, (b) is obtained by reproducing the bottom half of page 6 with substitutions $(a_{13}, a_{23}, a_{33}) \rightarrow (b_1, b_2, b_3)$:

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([I] continued)

(b)

$$= \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} .$$

$$\left(\underbrace{a_{11}a_{22} \square - a_{11} \square a_{32} - a_{12}a_{21} \square + a_{12} \square a_{31} + \square a_{21}a_{32} - \square a_{22}a_{31}}_{\parallel \Delta_3} \right)$$

$$= \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \Delta_3 ,$$

where

$$\Delta_3 =$$

_____ .

Now agree that we have reduced

$$(b) = - \left(a_{11}a_{32} - a_{12}a_{31} \right) \frac{a_{11}b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} + \left(a_{11}b_3 - b_1 a_{31} \right)$$

from page 3 to

$$(b) = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \Delta_3 ,$$

where

$$\begin{aligned} \Delta_3 = & a_{11}a_{22}b_3 - a_{11}b_2a_{32} - a_{12}a_{21}b_3 \\ & + a_{12}b_2a_{31} + b_1a_{21}a_{32} - b_1a_{22}a_{31} . \end{aligned}$$

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([I] continued)

(10) In sum, agree that the matrix in (5) (in page 3) is the same as

$$\left[\begin{array}{cccc} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 1 & \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \\ 0 & 0 & \frac{a_{11}\Delta}{a_{11}a_{22} - a_{12}a_{21}} & \frac{a_{11}\Delta_3}{a_{11}a_{22} - a_{12}a_{21}} \end{array} \right] .$$

Multiply $\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}\Delta}$ to (row 3):

$$\rightarrow \left[\begin{array}{cccc} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 1 & \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \\ 0 & 0 & 1 & \frac{\square}{\square} \end{array} \right] .$$

(11) Agree that the resulting matrix in (10) is the same as

$$\left[\begin{array}{cccc} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 1 & \frac{a_{11}a_{23} - a_{13}a_{21}}{a_{11}a_{22} - a_{12}a_{21}} & \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \\ 0 & 0 & 1 & \frac{\Delta_3}{\Delta} \end{array} \right] .$$

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([I] continued)

Write out the system of linear equations whose augmented matrix is in the bottom of page 9:

$$\begin{aligned} x_1 + \frac{\boxed{}}{a_{11}} x_2 + \frac{\boxed{}}{a_{11}} x_3 &= \frac{\boxed{}}{a_{11}}, \\ x_2 + \frac{\boxed{}}{a_{11}a_{22} - a_{12}a_{21}} x_3 &= \frac{\boxed{}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_3 &= \frac{\boxed{}}{\boxed{}}. \end{aligned}$$

(12) From (11), agree that the solution for x_3 is

$$x_3 = \frac{\boxed{}}{\boxed{}}.$$

Now, by symmetry, the solution for x_1 and x_2 are

$$\begin{aligned} x_1 &= \frac{\boxed{\Delta_1}}{\boxed{}}, \\ x_2 &= \frac{\boxed{\Delta_2}}{\boxed{}}, \end{aligned}$$

where Δ_1 and Δ_2 are as follows:

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([I] continued)

Δ_1 is obtained by doing substitutions $(a_{11}, a_{21}, a_{31}) \rightarrow (b_1, b_2, b_3)$

in the formula

$$\begin{aligned}\Delta &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.\end{aligned}$$

Thus,

$$\Delta_1 = \underline{\hspace{15em}} .$$

Similarly, Δ_2 is obtained by doing substitutions $(a_{12}, a_{22}, a_{32}) \rightarrow (b_1, b_2, b_3)$

in the formula

$$\begin{aligned}\Delta &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.\end{aligned}$$

Thus,

$$\Delta_2 = \underline{\hspace{15em}} .$$