Math 290 ELEMENTARY LINEAR ALGEBRA

STUDY GUIDE FOR MIDTERM EXAM – I B

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- **Section 2.2: Properties on matrix operations.**

[1] First check on the Textbook light blue portion on pages 60, 61, 62, 65, 66, 67 .

Then study the following package as a more detailed version of these :

- From Progress Check – IV :

○ Be able to form the product AB for two marices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Also be able to form AB for marices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

○ Follow explanation on page 1. Agree that, when A and B are both in size $n \times n$, for $n = 4, 5, \cdots$, we are able to form AB, and it is a matrix in size $n \times n$. Understand that, writing out the formula for AB is in theory possible for any large n, although it consumes more space for a larger n.

○ Agree that in general AB and BA are not the same. Have a glance at Example 1, page 2:

Example 1. For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

○ Agree the following usage: We say $\underline{\ A \text{ and } B \text{ commute, if } AB = BA\ }$.

Be aware that, A and B $\underline{\text{ rarely }}$ commute.

○ Understand Formula 1, and Example 2:

Formula 1. If A and B are of form

$$A = \begin{bmatrix} * & b \\ c & * \end{bmatrix}, \quad B = \begin{bmatrix} * & q \\ r & * \end{bmatrix},$$

where $b \neq 0$, $c \neq 0$, $q \neq 0$, $r \neq 0$, and moreover if

$$\frac{b}{c} \neq \frac{q}{r},$$

then A and B $\underline{\text{ never }}$ commute.

Example 2. Let $A = \begin{bmatrix} * & 1 \\ 2 & * \end{bmatrix}$, and $B = \begin{bmatrix} * & 2 \\ 3 & * \end{bmatrix}$, where each $*$ is filled by any scalar. Then we have $AB \neq BA$, because $\dfrac{1}{2} \neq \dfrac{2}{3}$.

○ Check on Example 3, page 4:

Example 3. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}$. Then $AB \neq BA$. Indeed,

$$AB = \begin{bmatrix} * & 1 \cdot 5 + 2 \cdot 6 \\ 0 & * \end{bmatrix}, \qquad BA = \begin{bmatrix} * & 2 \cdot 4 + 3 \cdot 5 \\ 0 & * \end{bmatrix}.$$

We have $1 \cdot 5 + 2 \cdot 6 \neq 2 \cdot 4 + 3 \cdot 5$.

○ Check on Example 4, page 5:

Example 4. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $AB \neq BA$.

- Move on to page 6. Agree that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

satisfy $tA = B$ for some scalar t, then we say A and B are <u>proportionate</u>. In other words, we say

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix}$$

are proportionate. Understand that in this case, A and B commute: $AB = BA$.

- For example, agree that

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

are proportionate, because $2A = B$. Agree that A and B commute: $AB = BA$.

- Move on to page 7. Agree that if $A = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then A and B commute: $AB = BA$. Agree that in this case, AB and BA both equal O.

- For example, for

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- Move on to page 8. Agree that if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (<u>the identity matrix</u>), then A and B commute: $AB = BA$. Agree that in this case, AB and BA both equal B.

○ For example, for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

○ Agree that if $A = tI = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$ for some scalar t $\left(\underline{\text{a scalar matrix}}\right)$, then A and B commute: $AB = BA$. Agree that in this case, AB and BA both equal tB.

○ For example, for

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} -1 & -3 \\ -5 & -7 \end{bmatrix}, \quad BA = \begin{bmatrix} -1 & -3 \\ -5 & -7 \end{bmatrix}.$$

○ Move on to the bottom of page 8. Agree that both the zero matrix $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are scalar matrices.

○ Follow Example 10, page 9: For

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad B = \begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} ap & 0 \\ 0 & bs \end{bmatrix}, \quad BA = \begin{bmatrix} ap & 0 \\ 0 & ds \end{bmatrix}.$$

In particular, $AB = BA$.

○ For example, for

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix}, \quad BA = \begin{bmatrix} -3 & 0 \\ 0 & -8 \end{bmatrix}.$$

○ Follow pages 10–13 for the equivalent counterparts of the above, for the 3×3 case. In particular, agree that we define

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We call O <u>the zero matrix</u> in size 3×3, and I <u>the identity matrix</u> in size 3.

○ Agree that for

$$A = O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

we have $AB = O$, $BA = O$. In particular, $AB = BA$.

○ For example, for

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

○ Agree that for

$$A = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

we have $AB = B$, $BA = B$. In particular, $AB = BA$.

○ For example, for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix},$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}.$$

○ Follow Example 15, page 13:

Example 15. Let

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}.$$

Then

$$AB = BA = \begin{bmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}.$$

In particular, $AB = BA$.

∘ For example, if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

then

$$AB = BA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}.$$

∘ Move on to page 15 "Case of larger size matrices" . Agree with the following definitions for O and I:

$$O = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

(the zero matrix), sometimes denoted $O_{m \times n}$, and

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

(the identity matrix), sometimes denoted I_n.

∘ Observe

$$\begin{bmatrix} t & 0 & \cdots & 0 & 0 \\ 0 & t & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t & 0 \\ 0 & 0 & \cdots & 0 & t \end{bmatrix} = t \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = tI,$$

for a scalar t. tI is called a scalar matrix.

○ Check out Formula 4, page 17:

$$AO = O, \quad OB = O, \quad AI = A, \quad IB = B,$$
$$A\Big(tI\Big) = tA, \quad \Big(tI\Big)B = tB, \quad 1A = A$$

(assuming that the matrices are in right sizes so that the products are defined).

○ Move on to page 18 "Associativity Law" . Follow explanation on page 18: $\Big(AB\Big)C$ and $A\Big(BC\Big)$ are always equal, for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \qquad C = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

○ Undestand Formula 5, page 19: $\Big(AB\Big)C = A\Big(BC\Big)$ is true, as long as A, B, C are in right sizes so that AB and BC are defined.

○ Understand that, more generally, Formula 6, page 20:

$$\Big(\big(AB\big)C\Big)D = \Big(A\big(BC\big)\Big)D = A\Big(\big(BC\big)D\Big)$$
$$= A\Big(B\big(CD\big)\Big) = \Big(AB\Big)\Big(CD\Big)$$

is true, as long as A, B, C, D are in right sizes so that AB, BC and CD are defined.

○ Understand that, the k-uple consecutive product $A_1 A_2 A_3 \cdots A_{k-1} A_k$ makes sense:

$$A_1 A_2 A_3 \cdots A_{k-1} A_k = \Bigg(\Big(\big(\cdot\cdot\big((A_1 A_2) A_3\big) \cdots\big) A_{k-2}\Big) A_{k-1}\Bigg) A_k.$$

Understand that, by virtue of Associativity Law, we may write the same as

$$
\begin{aligned}
A_1\,A_2\,A_3\,\cdots A_{k-1}A_k &= \Big(\big(\,A_1\,A_2\,\cdots\,A_{k-3}\,\big)\,A_{k-2}\Big)\Big(\,A_{k-1}\,A_k\,\Big) \\
&= \Big(\big(\,A_1\,A_2\,\cdots\,A_{k-4}\big)\,A_{k-3}\Big)\Big(\,A_{k-2}\,A_{k-1}\,A_k\,\Big) \\
&= \Big(\big(\,A_1\,A_2\,\cdots\,A_{k-5}\big)\,A_{k-4}\Big)\big(\,A_{k-3}\,A_{k-2}\,A_{k-1}\,A_k\,\big) \\
&= \cdots \\
&= A_1\left(\,A_2\,\Big(\,A_3\,\big(\,\cdots\,\big(\,A_{k-2}\,(A_{k-1}\,A_k)\big)\,\cdot\cdot\big)\Big)\right).
\end{aligned}
$$

○ The bottom line is, you may calculate the consecutive product $A_1\,A_2\,\cdots A_{k-1}A_k$ any way you want, as long as you keep the order of matrices A_1, A_2, $\cdots$, A_{k-1}, A_k from left to right and do not permute them.

○ Understand that, for a square matrix A and for $k = 1, 2, 3, 4, \cdots$, the k-th power A^k is defined as $A_1\,A_2\cdots A_{k-1}A_k$, where A_1, A_2, $\cdots$, A_{k-1}, A_k are all equal to A. For example, $A^2 = A\,A$, $A^3 = A\,A\,A$, and so on. Understand that, A^k is expressed in many ways:

$$
\begin{aligned}
\underbrace{A\,A\,A\cdots A\,A\,A}_{k} &= \Big(\underbrace{A\,A\,\cdots\,A\,A\,A}_{k-1}\Big)\,A \\
&= \Big(\underbrace{A\,A\,\cdots\,A\,A}_{k-2}\Big)\Big(A\,A\Big) \\
&= \Big(\underbrace{A\,A\,\cdots\,A}_{k-3}\Big)\Big(A\,A\,A\Big) \\
&= \quad\cdots\qquad\cdots \\
&= \Big(A\,A\Big)\Big(\underbrace{A\,A\,\cdots\,A\,A}_{k-2}\Big) \\
&= A\,\Big(\underbrace{A\,A\,A\,\cdots\,A\,A}_{k-1}\Big)\,.
\end{aligned}
$$

○ Agree that the above is rephrased as

$$A^k = A^i A^j \quad \text{for} \quad k = i + j.$$

○ Also agree that the following is true:

$$A^k = \left(A^i\right)^j \quad \text{for} \quad k = i\,j.$$

○ Agree that, for example,

$$A^{100} = A^{53} A^{47}, \qquad A^{100} = \left(A^5\right)^{20}$$

are true.

○ Be able to calculate powers A^k for a concrete $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in some representative cases.

○ For example, for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,

$$A^2 = A\,A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$A^3 = A^2\,A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},$$

$$A^4 = A^3\,A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix},$$

$$A^5 = A^4\,A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix},$$

and so on. We conclude

$$A^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

○ In particular,

$$A^{100} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{100} = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}.$$

- For $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$$B^2 = B\,B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I,$$

$$B^3 = B^2\,B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -B,$$

$$B^4 = B^3\,B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

We conclude

$$B^k = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = B, \quad \text{for} \quad k = 1, 5, 9, 13, 17, \cdots ,$$

$$B^k = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I, \quad \text{for} \quad k = 2, 6, 10, 14, 18, \cdots ,$$

$$B^k = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -B, \quad \text{for} \quad k = 3, 7, 11, 15, 19, \cdots ,$$

$$B^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad \text{for} \quad k = 4, 8, 12, 16, 20, \cdots .$$

- Understand Examples 18–19, page 23:

Example 18. Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then

$$A^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}.$$

Example 19. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. Then

$$A^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}.$$

- For example, for $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$,

$$A^k = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^k = \begin{bmatrix} 2^k & 0 \\ 0 & \left(-1\right)^k \end{bmatrix}.$$

- Also for $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,

$$A^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}.$$

- Move on to page 24. Agree that the identities (1–14) in Formula 9:

(1) $$A + B = B + A,$$
(2) $$A + \left(B + C\right) = \left(A + B\right) + C,$$
(3) $$A\left(B + C\right) = AB + AC,$$
(4) $$\left(A + B\right)C = AC + BC,$$
(5) $$t\left(B + C\right) = tB + tC,$$
(6) $$\left(t + u\right)A = tA + uA,$$
(7) $$t\left(uA\right) = \left(tu\right)A,$$
(8) $$t\left(AB\right) = \left(tA\right)B,$$
(9) $$t\left(AB\right) = A\left(tB\right),$$
(10) $$1\,A = A.$$
(11) $$A + O = A,$$
(12) $$A - A = O,$$
(13) $$0\,A = O,$$
(14) $$t\,O = O$$

are true.

○ Understand Examples 20–21, page 25, as applications of Formula 9:

Example 20. $\left(tA\right)\left(uB + vC\right) = \left(tu\right)AB + \left(tv\right)AC.$

In particular, $A\left(B - C\right) = AB - AC.$

Example 21. $\left(A + B\right)\left(A - B\right) = A^2 - AB + BA - B^2.$

○ Agree that, in Example 21 above, we are not allowed to further simplify

$$A^2 - AB + BA - B^2$$

as

$$A^2 - B^2.$$

Indeed, AB need not equal BA.

○ Move on to page 26. Understand the definition of polynomial substitutions. For

$$f\left(x\right) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m,$$

where $a_0, a_1, a_2, \cdots, a_m$ are scalars, and for any square matrix A, agree

$$f\left(A\right) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m.$$

○ Follow Example 22, pages 26–27.

Example 22. For $f\left(x\right) = 1 + \sqrt{2}\,x + x^2$, $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$,

$$\begin{aligned} f\left(A\right) &= I + \sqrt{2}\,A + A^2 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^2 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}. \end{aligned}$$

○ Similarly, for $f(x) = 1 - \sqrt{2}\,x + x^2$, $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$,

$$f(A) = I - \sqrt{2}\,A + A^2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

○ Move on to page 27. Review the definitions of addition $f + g$, the multiplication $f\,g$, and the compositions $f \circ g$, $g \circ f$, of two polynomials f and g. Then understand Formula 10:

Formula 10. Let $f(x)$ and $g(x)$ be polynomials, A a square matrix, and t a scalar. Then

(1) $$\big(f + g\big)(A) = f(A) + g(A),$$

(2) $$\big(f\,g\big)(A) = f(A)\,g(A),$$

(3) $$\big(t\,f\big)(A) = t\ f(A), \qquad \text{and}$$

(4) $$\big(f \circ g\big)(A) = f\big(g(A)\big).$$

○ Understand that, Formula 10 (2) yields the commutativity of $f(A)$ and $g(A)$ for any two polynomials f and g and for any square matrix A.

○ Understand that, Formula 10 (2) recovers the formula

$$\text{“ } A^k = A^i\,A^j \qquad \text{if} \qquad k = i + j \text{ ”}$$

as a special case: $f(x) = x^i$, $g(x) = x^j$, $\big(f\,g\big)(x) = x^{i+j}$.

- Understand that, Formula 10 (4) recovers the formula

$$\text{“ } A^k = \left(A^i\right)^j \qquad \text{if} \qquad k = i\,j \text{ ”}$$

as a special case: $f(x) = x^j$, $g(x) = x^i$, $\left(f \circ g\right)(x) = x^{ij}$.

- Follow Example 24, page 29. Knowing

$$\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} = f(A), \qquad \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix} = g(A),$$

for $f(x) = 2 + x + x^2$, $g(x) = 1 - x^2 + x^4$, $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, we may automatically claim that $\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}$ commute:

$$\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}.$$

- Move on to page 30. Review the definition of <u>the transpose</u> of a matrix. For

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

we define A^T as

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Agree that, the transpose A^T of A is the matrix formed by interchanging rows and columns of A. Or stated in other words, A^T is the matrix whose (i, j) entry matches with the (j, i) entry of A.

○ Follow Example 25, page 30:

Example 25. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

For $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$, we have $A^T = \begin{bmatrix} a & p & x \\ b & q & y \\ c & r & z \end{bmatrix}$.

For $A = \begin{bmatrix} a & b & c & d \\ p & q & r & s \end{bmatrix}$, we have $A^T = \begin{bmatrix} a & p \\ b & q \\ c & r \\ d & s \end{bmatrix}$.

○ Understand Formula 11, page 31:

Formula 11. (1) $\left(A^T\right)^T = A.$

(2) $\left(A + B\right)^T = A^T + B^T.$

(3) $\left(tA\right)^T = t\left(A^T\right).$

(4) $\left(AB\right)^T = B^T A^T.$

(5) $\left(A + A^T\right)^T = A + A^T, \qquad \left(AA^T\right)^T = A\,A^T.$

○ Agree that, in general, $A^T A$ and AA^T need not be equal. Understand Example 26, page 31:

Example 26. For $A = \begin{bmatrix} -7 & 11 & 12 \\ 4 & -3 & 1 \\ 6 & -1 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} -7 & 4 & 6 \\ 11 & -3 & -1 \\ 12 & 1 & 3 \end{bmatrix}$,

$$A^T A = \begin{bmatrix} -7 & 4 & 6 \\ 11 & -3 & -1 \\ 12 & 1 & 3 \end{bmatrix} \begin{bmatrix} -7 & 11 & 12 \\ 4 & -3 & 1 \\ 6 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 101 & -95 & -62 \\ -95 & 131 & 126 \\ -62 & 126 & 154 \end{bmatrix},$$

$$AA^T = \begin{bmatrix} -7 & 11 & 12 \\ 4 & -3 & 1 \\ 6 & -1 & 3 \end{bmatrix} \begin{bmatrix} -7 & 4 & 6 \\ 11 & -3 & -1 \\ 12 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 314 & -49 & -17 \\ -49 & 26 & 30 \\ -17 & 30 & 46 \end{bmatrix}.$$

○ Check out the definition of symmetric and skew-symmetric matrices: A matrix A is called <u>symmetric</u> if $A^T = A$. A matrix A is called <u>skew-symmetric</u> if $A^T = -A$.

○ Agree that, by definition, symmetric and skew-symmetric matrices are in square size.

○ Check out Examples 27–28, page 32:

Example 27. $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $\begin{bmatrix} a & p & q \\ p & b & r \\ q & r & c \end{bmatrix}$ are both symmetric matrices.

Example 28. $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix}$ are both skew-symmetric matrices.

○ Agree that, by virtue of Formula 10 (5), $A + A^T$ and $A^T A$ are both symmetric, for any matrix A.

○ Observe that the two matrices $A^T A$ and AA^T in Example 26 are both symmetric.

○ Agree that, any square matrix A is always written as a sum of a symmetric and skew-symmetric matrices. Indeed, if we set

$$B = \frac{1}{2}\left(A + A^T\right), \qquad C = \frac{1}{2}\left(A - A^T\right),$$

then it follows that $A = B + C$, B is symmetric, and C is skew-symmetric. For a detailed proof, see page 49, solution for exercise problem [XXV].

[2] Review all the homework problems in Section 2.1 (Textbook page 69).

Review all the examples and exercise problems in Progress Check – IV.

Representative problems are

- [V], [VIII], [IX], [XV], #13 (= [XVI] (2)), #18 (= [XVIII]), #19 (= [XIX] (1)), #20 (= [XIX] (2)), #30 (= [XX]), #40 (= [XXI]), #43 (= [XXII] (2)), #55–58 (= [XXIV]), #61 (= [XXV–XXVI]), #63 (= [XXVII]), [XXVIII], [XXX].

[3] In addition to the above list of problems, you should be comfortable doing the following types of problems:

- Let $A = \begin{bmatrix} 1 & -1 \end{bmatrix}$, and $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

(1) Find A^T and X^T.

(2) Find XX^T.

(3) Find AXX^TA^T.

(4) Solve the equation $AXX^TA^T = 0$ in terms of X. Assume that solution matrix X has real numbers as entries.

[Answer]: (1) $A^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $X^T = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$.

(2) $$X X^T = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} x^2 + y^2 & xz + yw \\ xz + yw & z^2 + w^2 \end{bmatrix}.$$

(3) AXX^TA^T

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x^2 + y^2 & xz + yw \\ xz + yw & z^2 + w^2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \left(x^2 + y^2\right) - \left(xz + yw\right) \\ \left(xz + yw\right) - \left(z^2 + w^2\right) \end{bmatrix}$$

$$= \left[\left(x^2 + y^2\right) - \left(xz + yw\right)\right] - \left[\left(xz + yw\right) - \left(z^2 + w^2\right)\right]$$

$$= x^2 + y^2 + z^2 + w^2 - 2xz - 2yw.$$

(4) By (3), the equation $AXX^TA^T = 0$ becomes

$$x^2 + y^2 + z^2 + w^2 - 2xz - 2yw = 0.$$

We may simplify the left side of the equation, and the equation becomes

$$\left(x - z\right)^2 + \left(y - w\right)^2 = 0.$$

Under the condition that x, y, z and w are all real numbers, we may solve the equation as

$$x = z, \qquad y = w.$$

Hence, using two parameters t, s, we may solve the equation $AXX^TA^T = 0$ as $X = \begin{bmatrix} t & s \\ t & s \end{bmatrix}$.