Math 290 ELEMENTARY LINEAR ALGEBRA

STUDY GUIDE FOR MIDTERM EXAM – I C

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- **Sections 2.3, 2.4: The inverse of a matrix, Elementary matrices.**

[1] First check on the Textbook light blue portion on pages 72, 74, 78, 80, 81, 85, 87, 89, 90. Then study the following package as a more detailed version of these :

- From Progress Check – V :

○ Before everything, understand the goal – build a theory of "the matrix analog" of the familiar scalar equation $ax = b$. Recall that, provided $a \neq 0$, we may solve $ax = b$ as $x = \dfrac{b}{a}$. By the matrix analog of this we mean

$$AX = B,$$

where A is a square $n \times n$ matrix, and B a (not necessarily square) matrix. Agree that this latter equation looks like

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1r} \\ x_{21} & x_{22} & \cdots & x_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nr} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}.$$

Agree that the entries of X are variables $\big(= \text{unknowns}\big)$. Agree that X and B are in the same size. One representative case is when X and B are column vectors. In such a case, the equation is written as $A\boldsymbol{x} = \boldsymbol{b}$, or

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Agree that our more specific goal is to introduce a matrix A^{-1} (having the same size as A) such that we may solve $AX = B$ as

$$X = A^{-1}B,$$

whenever possible. One big part of our assignment is to decide precisely when this is feasible, namely, decide precisely when A^{-1} makes sense.

○ Move on to Primitive Example 1, page 2. Agree that $AX = B$ has no solution, when $A = O$, $B \neq O$ (obvious).

○ Check on Examples 2–3, page 2. Agree that $AX = B$ may still have no solution, even when $A \neq O$ (less obvious):

Example 2. For $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has no solution $\boldsymbol{x}$.

Example 3. For $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the equation $AX = B$ has no solution X.

○ Understand, and memorize, Key Example 4, page 3.

Key Example 4 (the case $B = I$). For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with

(∗) $$ad - bc = 0,$$

the equation $AX = I$ has no solution X.

○ Understand, and memorize, Example 5, page 4.

Example 5. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, with

$$(**) \qquad a\,d - b\,c = 0, \qquad \text{and} \qquad p\,s - q\,r \neq 0,$$

the equation $AX = B$ has no solution X.

○ Be able to apply Example 5. Be able to show that each of the following equations (all of which are of form $AX = B$) has no solution:

- $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$,
- $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
- $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$,
- $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
- $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

○ Understand, and memorize, Key Formula 1, page 6.

(Key) Formula 1. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying

$$(*) \qquad a\,d - b\,c \neq 0,$$

and for any $\boldsymbol{b} = \begin{bmatrix} e \\ f \end{bmatrix}$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution

$$\boldsymbol{x} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}.$$

○ Formula 1 paraphrased: Provided $ad - bc \neq 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

if and only if

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}.$$

○ Understand, and memorize, Formula 3, page 8.

Formula 3. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying

$$(*) \qquad ad - bc \neq 0,$$

the equation $AX = I$ has a unique solution

$$X = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

○ Formula 3 paraphrased: Provided $ad - bc \neq 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

○ Understand, and memorize, Definition, page 8: We define

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$.

Repeat:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

provided $ad - bc \neq 0$.

○ Agree that we do not define $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 0$.

○ Follow Example 6, page 9. Be able to show

- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is non-singular,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}.$$

- $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ is non-singular,

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}.$$

- $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$ is singular, $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}^{-1}$ is undefined.

○ Take a glance at the collection of Examples and Formulas, page 10:

Example 4 paraphrased. For a 2×2 singular matrix A, the equation $AX = I$ has no solution X.

Example 5 paraphrased. For a 2×2 singular matrix A, and for a 2×2 non-singular matrix B, the equation $AX = B$ has no solution X.

Formula 1 paraphrased. For a 2×2 <u>non-singular</u> matrix A, and for any column vector $\boldsymbol{b}$ of length 2, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

In other words, provided A is a non-singular 2×2 matrix, $A\boldsymbol{x} = \boldsymbol{b}$ if and only if $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Formula 2 paraphrased. For a 2×2 <u>non-singular</u> matrix A, and for any $2 \times n$ matrix B, the equation $AX = B$ has a unique solution

$$X = A^{-1}B.$$

In other words, provided A is a non-singular 2×2 matrix, $AX = B$ if and only if $X = A^{-1}B$.

Formula 3 paraphrased. For a 2×2 <u>non-singular</u> matrix A, the equation $AX = I$ has a unique solution

$$X = A^{-1}.$$

In other words, provided A is a non-singular 2×2 matrix, $AX = I$ if and only if $X = A^{-1}$.

○ If all of the above statements look trivial to you, then proceed. If not, then make an effort to digest these statements. Once you have convinced yourself of the validity of each of the above statements, proceed.

○ Check on Formula 4, page 11:

Formula 4. For a 2×2 matrix A, A is non-singular if and only if A^T is non-singular. Moreover, if A is non-singular, then

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}.$$

◦ Check on the collection of examples and formulas (Examples 7–8, Formulas 5–7) on page 12 and compare them with the previous ones:

Example 7 (= the dual of Example 4). For a 2×2 singular matrix A, the equation $XA = I$ has no solution X.

Example 8 (= the dual of Example 5). For a 2×2 singular matrix A, and for a 2×2 non-singular matrix B, the equation $XA = B$ has no solution X.

Formula 5 (= the dual of Formula 1). For a 2×2 non-singular matrix A, and for any row vector $\boldsymbol{b}$ of length 2, the equation $\boldsymbol{x}A = \boldsymbol{b}$ has a unique solution

$$\boldsymbol{x} = \boldsymbol{b}A^{-1}.$$

In other words, provided A is a non-singular 2×2 matrix, $\boldsymbol{x}A = \boldsymbol{b}$ if and only if $\boldsymbol{x} = \boldsymbol{b}A^{-1}$.

Formula 6 (= the dual of Formula 2). For a 2×2 non-singular matrix A, and for any $n \times 2$ matrix B, the equation $X A = B$ has a unique solution

$$X = BA^{-1}.$$

In other words, provided A is a non-singular 2×2 matrix, $XA = B$ if and only if $X = BA^{-1}$.

Formula 7 (= the dual of Formula 3). For a 2×2 non-singular matrix A, the equation $XA = I$ has a unique solution

$$X = A^{-1}.$$

In other words, provided A is a non-singular 2×2 matrix, $XA = I$ if and only if $X = A^{-1}$.

◦ You do not have to memorize the above five statements. Instead, convince yourself that the above listed are precisely the "dual" of Examples 4–5, Formulas 1–3.

○ Now move on to page 14. Agree on Formula 8:

Formula 8. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Assume A is <u>non-singular</u>. Then

$$A\,A^{-1} \;=\; I, \qquad \text{and} \qquad A^{-1}\,A \;=\; I.$$

○ Agree on Formulas 10–12, pages 15–16:

Formula 10. Let A and B be two 2×2 matrices, not necessarily non-singular. Assume $AB = I$. Then A, B are both non-singular,

$$B \;=\; A^{-1}, \qquad A \;=\; B^{-1}.$$

Formula 11. The identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is non-singular,

$$I^{-1} \;=\; I.$$

Formula 12. Let A be a 2×2 non-singular matrix. Then A^{-1} is non-singular,

$$\left(A^{-1}\right)^{-1} \;=\; A.$$

○ Now you are on page 16. Understand Example 9 (which you <u>do not have to memorize</u>):

Example 9. For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, let

$$\Delta \;=\; a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

Assume

$$(*) \qquad \Delta \;=\; 0,$$

Then neither $AX = I$ nor $XA = I$ has a solution X.

○ Instead, agree that Example 9 is a precise 3×3 analog of Examples 4, 7.

○ Just know that Δ will be called <u>the determinant</u> of A.

○ Understand Formula 13, page 17 (which you do not have to memorize):

Formula 13. For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ satisfying

$$\Delta = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$\neq 0,$$

and for any $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution

$$\boldsymbol{x} = \frac{1}{\Delta}\begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} - a_{13}a_{22} \\ -a_{21}a_{33} + a_{23}a_{31} & a_{11}a_{33} - a_{13}a_{31} & -a_{11}a_{23} + a_{13}a_{21} \\ a_{21}a_{32} - a_{22}a_{31} & -a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

○ Formula 13 paraphrased: Provided $\Delta \neq 0$,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} - a_{13}a_{22} \\ -a_{21}a_{33} + a_{23}a_{31} & a_{11}a_{33} - a_{13}a_{31} & -a_{11}a_{23} + a_{13}a_{21} \\ a_{21}a_{32} - a_{22}a_{31} & -a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

○ Instead, agree that Formula 13 is a precise 3×3 analog of Key Formula 1.

○ Now move on to page 23. Understand Definition there: We define

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} - a_{13}a_{22} \\ -a_{21}a_{33} + a_{23}a_{31} & a_{11}a_{33} - a_{13}a_{31} & -a_{11}a_{23} + a_{13}a_{21} \\ a_{21}a_{32} - a_{22}a_{31} & -a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix},$$

for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ with $\Delta \neq 0$.

○ Agree that by setting

$$\begin{aligned}
C_{11} &= a_{22}a_{33} - a_{23}a_{32}, & C_{12} &= -a_{12}a_{33} + a_{13}a_{32}, & C_{13} &= a_{12}a_{23} - a_{13}a_{22}, \\
C_{21} &= -a_{21}a_{33} + a_{23}a_{31}, & C_{22} &= a_{11}a_{33} - a_{13}a_{31}, & C_{23} &= -a_{11}a_{23} + a_{13}a_{21}, \\
C_{31} &= a_{21}a_{32} - a_{22}a_{31}, & C_{32} &= -a_{11}a_{32} + a_{12}a_{31}, & C_{33} &= a_{11}a_{22} - a_{12}a_{21},
\end{aligned}$$

$$A^{-1} = \begin{bmatrix} \frac{C_{11}}{\Delta} & \frac{C_{12}}{\Delta} & \frac{C_{13}}{\Delta} \\ \frac{C_{21}}{\Delta} & \frac{C_{22}}{\Delta} & \frac{C_{23}}{\Delta} \\ \frac{C_{31}}{\Delta} & \frac{C_{32}}{\Delta} & \frac{C_{33}}{\Delta} \end{bmatrix}.$$

○ Only when you feel comfortable to do so, memorize this latter definition of A^{-1}, which you will find of practical use. There will be an alternative method to find A^{-1} though, which we discuss later. If you opt to memorize the above formula, then you will have acquired two methods, instead of one, to calculate A^{-1} for a 3×3 matrix A.

○ Take a look at Formula 14, pages 24–25:

Formula 14 (The case A is in size 3×3).

(1) For a 3×3 non-singular matrix A, and for any $3 \times n$ matrix B, the equation $AX = B$ has a unique solution

$$X = A^{-1}B.$$

In other words, provided A is a non-singular 3×3 matrix, $AX = B$ if and only if $X = A^{-1}B$.

(2) For a 3×3 non-singular matrix A, and for any $n \times 3$ matrix B, the equation $XA = B$ has a unique solution

$$X = BA^{-1}.$$

In other words, provided A is a non-singular 3×3 matrix, $XA = B$ if and only if $X = BA^{-1}$.

(3) For a 3×3 non-singular matrix A,

$$A\,A^{-1} = I, \qquad \text{and} \qquad A^{-1}\,A = I.$$

(4) Let A and B be two 3×3 matrices, not necessarily non-singular. Assume $AB = I$. Then A, B are both non-singular,

$$B = A^{-1}, \qquad A = B^{-1}.$$

(5) The identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is non-singular, $I^{-1} = I$.

(6) Let A be a 3×3 non-singular matrix. Then A^{-1} is non-singular,

$$\left(A^{-1}\right)^{-1} = A.$$

○ Again, make an effort to digest the statements in Formula 14. The difference between Formula 14 and its 2×2 equivalent couonterpart which we have encountered earlier is that the definition of non-singularity of A in the present 3×3 case uses Δ, and Δ looks complex. Understand that this is the reality, and we must live with it. Nevertheless, there is some good news (below).

○ Move on to page 26. Completely digest Formula 15 which gives another recipé to calculate A^{-1} for a 3×3 matrix A. This one does not rely on Δ :

Formula 15. For a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, set

$$\left[\, A \,\middle|\, I \,\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix}.$$

This is a 3×6 matrix.

(1) If the reduced row echelon form of $\left[\begin{array}{c|c} A & I \end{array}\right]$ takes a form

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix},$$

then A is singular.

(2) If the reduced row echelon form of $\left[\begin{array}{c|c} A & I \end{array}\right]$ takes a form

$$\left[\begin{array}{c|c} I & B \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{bmatrix},$$

then A is non-singular, and moreover

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

is the inverse of A: $B = A^{-1}$.

- ○ Check out Example 10, pages 27–28.

Example 10. $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ is non-singular.

$$A^{-1} = \begin{bmatrix} -4/3 & -5/3 & 1 \\ -4/3 & -8/3 & 1 \\ 1/3 & 2/3 & 0 \end{bmatrix}.$$

- ○ The way to deduce it is as follows: Firstly, form

$$\begin{bmatrix} -2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}.$$

Secondly, apply Gaussian elimination to reduce it to a reduced row echelon form:

$$\left[\begin{array}{ccc|ccc} -2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/3 & -5/3 & 1 \\ 0 & 1 & 0 & -4/3 & -8/3 & 1 \\ 0 & 0 & 1 & 1/3 & 2/3 & 0 \end{array}\right].$$

Thirdly, confirm that the left-half portion of the resulting reduced row echelon form is I, thus read off its right-half portion, and it gives you A^{-1}.

- Alternatively, if you have memorized the formula:

$$A^{-1} = \begin{bmatrix} \dfrac{C_{11}}{\Delta} & \dfrac{C_{12}}{\Delta} & \dfrac{C_{13}}{\Delta} \\ \dfrac{C_{21}}{\Delta} & \dfrac{C_{22}}{\Delta} & \dfrac{C_{23}}{\Delta} \\ \dfrac{C_{31}}{\Delta} & \dfrac{C_{32}}{\Delta} & \dfrac{C_{33}}{\Delta} \end{bmatrix},$$

where

$$\begin{aligned} C_{11} &= a_{22}a_{33} - a_{23}a_{32}, & C_{12} &= -a_{12}a_{33} + a_{13}a_{32}, & C_{13} &= a_{12}a_{23} - a_{13}a_{22}, \\ C_{21} &= -a_{21}a_{33} + a_{23}a_{31}, & C_{22} &= a_{11}a_{33} - a_{13}a_{31}, & C_{23} &= -a_{11}a_{23} + a_{13}a_{21}, \\ C_{31} &= a_{21}a_{32} - a_{22}a_{31}, & C_{32} &= -a_{11}a_{32} + a_{12}a_{31}, & C_{33} &= a_{11}a_{22} - a_{12}a_{21}, \end{aligned}$$

then we may proceed as follows:

Firstly, calculate Δ:

$$\begin{aligned} \Delta &= (-2) \cdot (-1) \cdot 4 - (-2) \cdot 0 \cdot 1 - 2 \cdot 1 \cdot 4 \\ &\quad + 2 \cdot 0 \cdot 0 + 3 \cdot 1 \cdot 1 - 3 \cdot (-1) \cdot 0 \\ &= 8 - 0 - 8 + 0 + 3 - 0 \\ &= 3. \end{aligned}$$

Secondly, calculate C_{ij}'s:

$$\begin{aligned}
C_{11} &= \big(-1\big)\cdot 4 - 0\cdot 1 = -4,\\
C_{12} &= -2\cdot 4 + 3\cdot 1 = -5,\\
C_{13} &= 2\cdot 0 - 3\cdot\big(-1\big) = 3,\\
C_{21} &= -1\cdot 4 + 0\cdot 0 = -4,\\
C_{22} &= \big(-2\big)\cdot 4 - 3\cdot 0 = -8,\\
C_{23} &= -\big(-2\big)\cdot 0 + 3\cdot 1 = 3,\\
C_{31} &= 1\cdot 1 - \big(-1\big)\cdot 0 = 1,\\
C_{32} &= -\big(-2\big)\cdot 1 + 2\cdot 0 = 2,\\
C_{33} &= \big(-2\big)\cdot\big(-1\big) - 2\cdot 1 = 0.
\end{aligned}$$

Hence

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4/3 & -5/3 & 1 \\ -4/3 & -8/3 & 1 \\ 1/3 & 2/3 & 0 \end{bmatrix}.$$

○ Compare the two results for A^{-1} and agree that they match.

○ Again, for a given matrix A, we may either rely on Formula 15 and go through the Gaussian elimination method to find the non-singularity of A and the inverse A^{-1}, or rely on the definition of Δ to find the non-singularity, then rely on co-factor expression of A^{-1} to find A^{-1} if non-singular. The theory guarantees that you get the same answer.

○ Check out Example 11, page 30.

Example 11. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{bmatrix}$ is singular. A^{-1} is undefined.

○ The way to conclude that A is singular takes exactly the same route as Example 10. Firstly, form

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -10 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{array}\right].$$

Secondly, apply Gaussian elimination to reduce this matrix:

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -10 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right].$$

Confirm that the left-half portion of the resulting matrix $\big($which is not in reduced row echelon form yet, but it does not matter$\big)$ has the bottom row that conists entirely of 0s. Thus, you conclude that A is singular. A^{-1} is undefined.

○ Check out Formula 16, page 33 first, and then Example 12, page 31:

Formula 16. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. Then A is non-singular, if and only if

$$a \neq 0, \qquad b \neq 0, \qquad c \neq 0.$$

If this is the case, then

$$A^{-1} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}.$$

Moreover, the determinant of A is given by $\Delta = a\,b\,c$.

Example 12. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is non-singular,

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}.$$

The determinant of A is $\Delta = 2 \cdot 3 \cdot 5 = 30.$

○ Check out Example 13, page 33:

Example 13. Let $\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a^2 + b^2 + c^2 = 1.$ Then

$$A = I - 2\boldsymbol{u}\boldsymbol{u}^T = \begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix}.$$

Moreover, A satisfies $A^2 = I.$ Accordingly, A is non-singular,

$$A^{-1} = A.$$

○ Make sure you will be able to verify Example 13. Follow the calculation below (a little tedious but straightforward):

$$\begin{aligned} A &= I - 2\boldsymbol{u}\boldsymbol{u}^T \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} \\ &= \begin{bmatrix} a^2 + b^2 + c^2 & 0 & 0 \\ 0 & a^2 + b^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 + c^2 \end{bmatrix} - \begin{bmatrix} 2a^2 & 2ab & 2ac \\ 2ab & 2b^2 & 2bc \\ 2ac & 2bc & 2c^2 \end{bmatrix} \end{aligned}$$

$$\left[\text{ since } 1 = a^2 + b^2 + c^2 \text{ by assumption } \right]$$

$$= \begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix}.$$

○ As for verification of $A^2 = I$, you may proceed as follows:

$$
\begin{aligned}
A^2 &= \left(I - 2\boldsymbol{u}\boldsymbol{u}^T\right)^2 \\
&= \left(I - 2\boldsymbol{u}\boldsymbol{u}^T\right)\left(I - 2\boldsymbol{u}\boldsymbol{u}^T\right) \\
&= I\,I - I\left(2\boldsymbol{u}\boldsymbol{u}^T\right) - \left(2\boldsymbol{u}\boldsymbol{u}^T\right)I + \left(2\boldsymbol{u}\boldsymbol{u}^T\right)\left(2\boldsymbol{u}\boldsymbol{u}^T\right) \\
&= I\,I - I\left(2\boldsymbol{u}\boldsymbol{u}^T\right) - \left(2\boldsymbol{u}\boldsymbol{u}^T\right)I + \left(2\boldsymbol{u}\boldsymbol{u}^T\right)\left(2\boldsymbol{u}\boldsymbol{u}^T\right) \\
&= I - 2\left(\boldsymbol{u}\boldsymbol{u}^T\right) - 2\left(\boldsymbol{u}\boldsymbol{u}^T\right) + 4\left(\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{u}^T\right) \\
&= I - 4\left(\boldsymbol{u}\boldsymbol{u}^T\right) + 4\left(\boldsymbol{u}\left(\boldsymbol{u}^T\boldsymbol{u}\right)\boldsymbol{u}^T\right) \\
&= I - 4\left(\boldsymbol{u}\boldsymbol{u}^T\right) + 4\left(\boldsymbol{u}\,I\,\boldsymbol{u}^T\right)
\end{aligned}
$$

$$\left[\text{ since } \boldsymbol{u}^T\boldsymbol{u} = \left[a^2 + b^2 + c^2\right] = \left[1\right] = I \text{ by assumption } \right]$$

$$
\begin{aligned}
&= I - 4\left(\boldsymbol{u}\boldsymbol{u}^T\right) + 4\left(\boldsymbol{u}\boldsymbol{u}^T\right) \\
&= I.
\end{aligned}
$$

Alternatively, you may calculate A^2 directly as

AA

$$= \begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix} \begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(a^2+b^2+c^2\right)^2 & 0 & 0 \\ 0 & \left(a^2+b^2+c^2\right)^2 & 0 \\ 0 & 0 & \left(a^2+b^2+c^2\right)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(where the detailed calculation is omitted).

○ For example, let

$$A = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}.$$

Realize that this is exactly the matrix A for

$$\boldsymbol{u} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

In particular,

$$A^{-1} = A = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}.$$

○ Now move on to page 34. Have just a glance at the formula for Δ for the general 2×2, 3×3, and 4×4, matrices and recognize that there is some pattern.

○ Read paragraphs on pages 35–36. Agree that we may define the notion of "invertibility", as opposed to "non-singularity", for a square $n \times n$ matrix A. Understand that in this context we will not have to rely on, or make a reference to, "determinants" Δ.

○ Agree on Definition, pages 37–38, on elementary matrices.

$$[\text{Type I}] \qquad E = \begin{bmatrix} 1 & \cdots\cdots & 0 & \cdots & 0 \\ & \ddots & \vdots & & \vdots \\ 0 & \cdots t \cdots & 1 & \cdots & 0 \\ & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots \quad \cdots & 0 & \cdots & 1 \end{bmatrix} (i \qquad \Big(t \text{ is a scalar}\Big),$$

$$\qquad\qquad\qquad \overset{\frown}{j} \qquad \overset{\frown}{i}$$

[Type II] $$E = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ (i \\ \\ (j \\ \\ \\ \end{matrix}, \quad \text{columns } \widehat{i},\ \widehat{j}$$

[Type III] $$E = \begin{bmatrix} 1 & \cdots\cdots & 0 & \cdots & 0 \\ & \ddots & \vdots & & \vdots \\ 0 & \cdots\cdots & t & \cdots & 0 \\ & & \vdots & \ddots & \vdots \\ 0 & \cdots\cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ (i \\ \\ \\ \end{matrix} \qquad \Big(t \text{ is a scalar}, \quad t \neq 0 \Big). \quad \text{column } \widehat{i}$$

○ Be able to decide whether each one of the following is an elementary matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

[Answer]: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are elementary matrices.

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is not.

[2] Review all the homework problems in Sections 2.3, 2.4 (Textbook page 83, 95) .

Review all the examples and exercise problems in Progress Check – V .

Representative problems are

- [II], [X], #9 of Section 2.3 $\big($ = [XI] (2) $\big)$,
  #16 of Section 2.3 $\big($ = Example 12 $\big)$, #53 of Section 2.3 $\big($ = Example 13 $\big)$,
  #5 of Section 2.4, #6 of Section 2.4, #7 of Section 2.4.