Math 590  LINEAR ALGEBRA

POLICY ON COMPUTER SOFTWARE USE

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- With regard to homework assignment, I would like to remind you:

You may rely on a mathematical software such as Maple or Mathematica as a checking device. Knowledge of how to use those softwares is not a part of the prerequisites for this class.

Unless otherwise stated, officially, in your homework papers, you are obligated to offer a solution which does not resort to a computer software. In other words, even though you are never discouraged to use one of those softwares to double-check the accuracy of your answers once you have worked out your solution, on your paper which you submit, you must pretend that you did not use those softwares.

Still, in many cases they can tell us answers right way. So, some may argue that those of you who are proficient on those softwares are given a decisive advantage. I assign problems and grade papers in such a way that your level of proficiency on computer software use minimally affects your grade.

A concrete example: Suppose I assign the following problem:

* Problem.  "Let

$$A = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} \begin{bmatrix} a & -c & -b & -d \\ c & a & -d & b \\ b & d & a & -c \\ d & -b & c & a \end{bmatrix} = \begin{bmatrix} a^2-d^2-2bc & -a(b+c)+d(b-c) & -a(b+c)+d(b-c) & -2ad-b^2+c^2 \\ a(b+c)+d(b-c) & a^2+d^2 & -b^2-c^2 & a(b-c)-d(b+c) \\ a(b+c)+d(b-c) & -b^2-c^2 & a^2+d^2 & a(b-c)-d(b+c) \\ 2ad-b^2+c^2 & -a(b-c)-d(b+c) & -a(b-c)-d(b+c) & a^2-d^2+2bc \end{bmatrix}. $$

Observe that \[
\begin{bmatrix}
0 \\
1 \\
-1 \\
0
\end{bmatrix}
\]
is one of its eigenvectors. Use this fact, if necessary, to concretely write out the characteristic polynomial of \( A \).”

• Suppose you submit a paper that goes as follows:

* **Solution**.

“ I used Maple (or Mathematica) and input the above matrix. I ordered

‘ Evaluate \( \det (\lambda I - A) \). ’

Then it gave me

\[
\chi_A(\lambda) = \lambda^4 - 4a^2 \lambda^3 \\
+ 2 \left( a^2 + b^2 + c^2 + d^2 \right) \left( 3a^2 - b^2 - c^2 - d^2 \right) \lambda^2 \\
- 4a^2 \left( a^2 + b^2 + c^2 + d^2 \right)^2 \lambda + \left( a^2 + b^2 + c^2 + d^2 \right)^4.
\]

*End of proof.*

• Well, I cannot give you any credit. On the other hand, if you actually physically calculate \( \det (\lambda I - A) \), using, say, co-factors, and spell out detailed calculation without omitting steps, then simplify the result, and actually come up with the same polynomial as in \( \chi_A(\lambda) \) above, then I will give you a full credit.

However, in this hypothetical assignment, what I really expect from you is not exactly your “tenacity”, your ability to never quit tedious and laborious calculation in the middle*, but rather, your sound knowledge and command on eigenvalues and characteristic polynomials. In particular, what I would really like to see on your paper is as follows:

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*I am not saying I do not value such a quality.*
(i) Your ability to multiply the column vector \[
\begin{bmatrix}
0 \\
1 \\
-1 \\
0
\end{bmatrix}
\] to \(A\) from the right, and your ability to recognize that \(a^2 + b^2 + c^2 + d^2\) is one of the eigenvalues of \(A\).

(I could have been more demanding, namely, I could have let you discover the above eigenvector “by speculation” by yourself. Namely, I could have let you realize that, in the matrix \(A\), subtracting the third column from the second column yields a vector whose first and fourth entries both equal 0, and whose second and third entries are negative of each other.)

(ii) Your knowledge of the general fact that the characteristic polynomial for a \(4 \times 4\) matrix \(A\) takes the form

\[
\chi_A(\lambda) = \lambda^4 - \left(\begin{array}{c}
\text{Tr } A
\end{array}\right) \lambda^3
+ \left(\begin{array}{c}
*
\end{array}\right) \lambda^2
- \left(\begin{array}{c}
\text{Tr } (\text{adj } A)
\end{array}\right) \lambda
+ \left(\begin{array}{c}
\text{det } A
\end{array}\right).
\]

(iii) Your ability to calculate \(\text{Tr } A\) (straightforward) and \(\text{det } A\) (less straightforward). Your ability to apply the well-known formula that \(\text{det } A\) equals

\[
\left(\begin{array}{c}
\text{det } \begin{bmatrix}
a & -b & -c & -d \\
b & a & d & -c \\
c & -d & a & b \\
d & c & -b & a
\end{bmatrix}
\end{array}\right)
\left(\begin{array}{c}
\text{det } \begin{bmatrix}
a & -c & -b & -d \\
c & a & -d & b \\
b & d & a & -c \\
d & -b & c & a
\end{bmatrix}
\end{array}\right),
\]

along with your ability to calculate each of the above two factors. Your “memory” that there is one convenient formula for this purpose which was covered in class.
(iv) Your “memory” that, in the class, it was mentioned that, in the matrix

\[ B_1 = \begin{bmatrix}
a & -b & -c & -d \\
b & a & d & -c \\
c & -d & a & b \\
d & c & -b & a
\end{bmatrix}, \]

you negate each of \( b, c \) and \( d \), thus obtain

\[ \begin{bmatrix}
a & b & c & d \\
-b & a & -d & c \\
-c & d & a & -b \\
-d & -c & b & a
\end{bmatrix}, \]

then you multiply these two matrices (in either order) and the result is

\[ \begin{bmatrix}
a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\
0 & a^2+b^2+c^2+d^2 & 0 & 0 \\
0 & 0 & a^2+b^2+c^2+d^2 & 0 \\
0 & 0 & 0 & a^2+b^2+c^2+d^2
\end{bmatrix}. \]

Your ability to “suspect” that, you do the same for the matrix

\[ B_2 = \begin{bmatrix}
a & -c & -b & -d \\
c & a & -d & b \\
b & d & a & -c \\
d & -b & c & a
\end{bmatrix}, \]

instead of \( B_1 \), and get the same result. Your ability to logically conclude that, in view of \( A = B_1 B_2 \), the adjoint matrix of \( A \) is simply \( \left( a^2 + b^2 + c^2 + d^2 \right)^2 \) times the matrix obtained by negating each of \( b, c \) and \( d \) in \( A \). Your thoroughness to note that, for this to work, the commutativity of the two matrices \( B_1 \) and \( B_2 \) is required. Your memory that this was actually covered in class. Your ability to realize that, this way, \( \chi_A(\lambda) \) is decided except the \( \lambda^2 \) term.

(v) Your ability to decide the \( \lambda^2 \) term of \( \chi_A(\lambda) \) simply by means of substituting \( \lambda = a^2 + b^2 + c^2 + d^2 \) and equating the resulting quantity with 0.
Again, the above is only one way to solve the problem (also there are many minor variations of the above solution).

If you follow the above steps, then the actual amount of calculation is minimal, and only a painless kind of calculation is involved. But the true benefit of working out this problem as in the above method is this way you get a sense of “symmetry” the matrix $A$ carries. (Only a small part of symmetry is the fact that $\chi_A(\lambda) = 0$ is stable under the substitution $\lambda = (a^2 + b^2 + c^2 + d^2)^2 / \mu$.)

Another example.

Problem. “Evaluate
\[
\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} \frac{d z \, d y \, d x}{\sqrt{(1+x-y-z)(1-x+y-z)(1-x-y+z)}}.
\]
”

Suppose you submit a paper that goes:

Solution.

“ I used Maple (\Mathematica) and input the above triple integral.
I ordered ‘Evaluate’. It showed ‘overflow’. Thus, the integral is divergent.
End of proof. ”

The integral is actually convergent. (There is a way to perform calculation by hand. Otherwise I would not assign such a problem.) I actually tried this one on both Maple and Mathematica. Maple gave me an error message. Mathematica could not finish calculating after fifteen or so minutes so I quit and I do not know what would have happened if I let it run for longer minutes.*

* I did not say that, therefore, those softwares are not intelligently designed. I am just modestly pointing out that, those softwares may not necessarily do everything for you.