§3. Introduction to Complex Numbers – III.
Complex conjugate. Exponential formulas.

• Complex Conjugate.

Now we have learned the mechanism of “square-rooting” of complex numbers. Today I want to move to the direction of quadratic equations. But before that, I need to cover one very important concept associated with complex numbers. It’s called ‘complex conjugate’.

Definition (Complex conjugate).

Let $\alpha \in \mathbb{C}$ be a complex number. So let’s write

$\alpha = a + \sqrt{-1}b \quad \left( a, b \in \mathbb{R} \right)$,

as usual. Define the (complex) conjugate

$\overline{\alpha}$

of $\alpha$ as the following complex number:

$\overline{\alpha} = a - \sqrt{-1}b$.

Stated in other words,

$\overline{a + \sqrt{-1}b} = a - \sqrt{-1}b$. 

Well, this is simple. As I said, this is important. Why can something as innocuous as this possibly be important? Not so fast. Let’s see some examples first.

Example.

(1) \[-1 + \sqrt{-1} \cdot 2 = -1 - \sqrt{-1} \cdot 2.\]

(2) \[10 - \sqrt{-1} \cdot 2 \sqrt{5} = 10 + \sqrt{-1} \cdot 2 \sqrt{5}.\]

(3) \[\sqrt{-1} = -\sqrt{-1}.\]

(4) \[\overline{3} = 3.\]

(5) \[\sqrt{2} = \sqrt{2}.\]

(6) \[\overline{0} = 0.\]

Basic Facts 1.

(1) (involutive property) Let \( \alpha \in \mathbb{C} \) be arbitrary. Then

\[
\overline{\alpha} = \alpha.
\]

(2) Let \( \alpha \in \mathbb{C} \) be arbitrary. Then

(2a) \( \alpha \) is a real number \( \iff \overline{\alpha} = \alpha. \)

(2b) \( \alpha \) is a purely imaginary number \( \iff \overline{\alpha} = -\alpha. \)

Next, let me introduce another, seemingly unrelated, notion.
**Definition (Absolute value).**

Let $\alpha \in \mathbb{C}$ be a complex number. So, once again, let’s write

$$\alpha = a + \sqrt{-1} b \quad (a, \ b \in \mathbb{R}).$$

Define the absolute value $|\alpha|$ of $\alpha$ as the following real number:

$$|\alpha| = \sqrt{a^2 + b^2}.$$

Stated in other words,

$$|a + \sqrt{-1} b| = \sqrt{a^2 + b^2}.$$

Again, let’s see some examples.

**Example.**

(1) $|4 + \sqrt{-1} \cdot 3| = \sqrt{4^2 + 3^2} = 5.$

(2) $|\sqrt{3} + \sqrt{-1} \cdot 2| = \sqrt{(\sqrt{3})^2 + 2^2} = \sqrt{7}.$

(3) $|\log 2 + \sqrt{-1} \cdot \pi| = \sqrt{\left(\log 2\right)^2 + \pi^2}.$

(4) $|\sqrt{-1} \cdot 4| = \sqrt{0^2 + 4^2} = 4.$

(5) $|-8| = \sqrt{(-8)^2 + 0^2} = 8.$

(6) $|0| = \sqrt{0^2 + 0^2} = 0.$
Basic Facts 2.

1. Let \( \alpha \in \mathbb{C} \) be arbitrary. Then

\[
|\alpha| \in \mathbb{R} \quad \text{and} \quad |\alpha| \geq 0.
\]

2. Let \( \alpha \in \mathbb{C} \) be arbitrary. Then

\[
|\alpha| = 0 \iff \alpha = 0.
\]

(2)’ Stated in other words, for an arbitrary complex number \( \alpha \) which is non-zero; \( \alpha \neq 0 \), we have

\[
|\alpha| > 0.
\]

3. If \( \alpha \) is a real number, then \( |\alpha| \) as defined above coincides with the absolute value of \( \alpha \) as a real number. In other words, the notion of the absolute values of complex numbers is an extension of the notion of the absolute values of real numbers. In particular,

\[
\left| \left| \alpha \right| \right| = |\alpha|.
\]

- Okay, we have just learned two concepts, one is the complex conjugate operation, and the other the absolute values. It doesn’t seem like those two are related. The truth is, on the contrary, they are related.

Formula 1. Let \( \alpha \in \mathbb{C} \) be arbitrary. Then

\[
\alpha \overline{\alpha} = |\alpha|^2.
\]

Exercise 1. Prove Formula 1.
[Answer]: Proof. Let $\alpha = a + \sqrt{-1} b \ (a, b \in \mathbb{R})$ as usual.

Then

$$\alpha \overline{\alpha} = \left( a + \sqrt{-1} b \right) \left( a - \sqrt{-1} b \right)$$

$$= \left( a + \sqrt{-1} b \right) \left( a + \sqrt{-1} (-b) \right)$$

$$= \left( a - b \right) + \sqrt{-1} \left( a \cdot (-b) + b \cdot a \right)$$

$$= \left( a^2 - b^2 \right) + \sqrt{-1} \left( -ab + ba \right)$$

$$= \left( a^2 + b^2 \right) + \sqrt{-1} \cdot 0 = a^2 + b^2.$$  

This is nothing else but $|\alpha|^2$. □

Formula 1 immediately yields the following:

**Corollary 1.** Let $\alpha \in \mathbb{C}$ be arbitrary, which is non-zero; $\alpha \neq 0$. Then

$$\alpha^{-1} = \frac{\overline{\alpha}}{|\alpha|^2}.$$  

- Remember, we covered reciprocating complex numbers. How I did it was I just gave you a formula to memorize for the reciprocal of $\alpha = a + \sqrt{-1} b$ (with $a$, $b \in \mathbb{R}$). It looked slightly involved. Can you recite it? Yes, it is

$$\alpha^{-1} = \frac{a}{a^2 + b^2} - \sqrt{-1} \cdot \frac{b}{a^2 + b^2}.$$  

At that time, this didn’t look too natural to you. Now, after seeing Corollary 1 above, this formula becomes completely natural to you, right? The two highlighted boxes above are one and the same. Agree?
Corollary 2. \ Let \( \alpha \in \mathbb{C} \). Suppose \(|\alpha| = 1\). Then
\[
\alpha^{-1} = \overline{\alpha}.
\]

Formula 2. \ Let \( \alpha, \beta \in \mathbb{C} \) be arbitrary. Then

(1a) \[
\overline{\alpha \beta} = \bar{\alpha} \bar{\beta}, \quad |\alpha \beta| = |\alpha| |\beta|.
\]

(1b) In particular, suppose \( \alpha = a \) is a real number. Then
\[
\overline{a \beta} = a \overline{\beta}, \quad |a \beta| = \begin{cases} a |\beta| & (a \geq 0), \\ -a |\beta| & (a \leq 0). \end{cases}
\]

(2) \[
|\alpha| = |\bar{\alpha}|.
\]

(3) Suppose \( \alpha \neq 0 \). Then
\[
\left( \frac{1}{\overline{\alpha}} \right)^{-1} = \overline{\alpha^{-1}}, \quad |\alpha^{-1}| = |\alpha|^{-1}.
\]

Before we proceed further, I want to introduce one notation, for our convenience. From now on, we allow ourselves to write
\[
\sqrt{-2}, \quad \sqrt{-3}, \quad \sqrt{-5}, \quad \sqrt{-6}, \quad \sqrt{-7}, \quad \sqrt{-8}, \quad \text{etc.}
\]

These mean precisely as follows:
\[
\sqrt{-2} = \sqrt{-1} \sqrt{2}, \quad \sqrt{-3} = \sqrt{-1} \sqrt{3}, \quad \sqrt{-5} = \sqrt{-1} \sqrt{5},
\]
\[
\sqrt{-6} = \sqrt{-1} \sqrt{6}, \quad \sqrt{-7} = \sqrt{-1} \sqrt{7}, \quad \sqrt{-8} = \sqrt{-1} \sqrt{8}, \quad \text{etc.}
\]

More generally, let \( a \) be a positive real number: \( a > 0 \). Then \( \sqrt{-a} \) always means \( \sqrt{-1} \cdot \sqrt{a} \).
• Caution.

We have to be very careful about this way of writing. In fact, how much is

$$\sqrt{-2} \sqrt{-3}$$?

If you say the answer is $\sqrt{6}$, because $\sqrt{a} \sqrt{b}$ must equal $\sqrt{ab}$, and in this case $a = -2$ and $b = -3$, so $ab = 6$, hence $\sqrt{-2} \sqrt{-3}$ must equal $\sqrt{6}$, then you are wrong. Indeed, the correct answer is $-\sqrt{6}$. Repeat:

$$\sqrt{-2} \sqrt{-3} = -\sqrt{6}.$$ (Can you explain why?) Also, $\sqrt{\frac{1}{-2}}$ and $\frac{1}{\sqrt{-2}}$ are not the same.

Indeed, in this case, it is wrong to assume that $\sqrt{\frac{1}{c}}$ and $\frac{1}{\sqrt{c}}$ are equal. The correct identity is

$$\sqrt{\frac{1}{-2}} = \frac{-1}{\sqrt{-2}}.$$ 

• Now I want to talk a little more about complex numbers having absolute value 1. Such complex numbers are not too special, in the following sense:

**Basic Fact 3.** Let $\alpha \in \mathbb{C}$ be arbitrary, with $\alpha \neq 0$. Recall that $|\alpha| \neq 0$ holds, thus $\beta = \frac{\alpha}{|\alpha|}$ is well-defined. Then $\beta$ satisfies $|\beta| = 1$. Repeat:

$$\beta = \frac{\alpha}{|\alpha|} \implies |\beta| = 1.$$
Exercise 2. Prove Basic Fact 3.

[Answer] :  Proof. We may rewrite \( \beta = \frac{\alpha}{|\alpha|} \) as

\[
\beta = |\alpha|^{-1} \alpha.
\]

By Formula 2, (3), \( |\alpha|^{-1} \) equals \( |\alpha^{-1}| \). So

\[
\beta = |\alpha^{-1}| \alpha.
\]

Accordingly,

\[
|\beta| = | |\alpha^{-1}| \alpha| |
= |\alpha^{-1}| |\alpha| \quad \text{(by virtue of Formula 2 (1b))}
= |\alpha^{-1} \alpha| \quad \text{(by virtue of Formula 2 (1a))}
= |1| = 1. \quad \Box
\]

Okay. So, the next question. Can you imagine what kind of complex numbers have the absolute value 1? Yes, 1 and \(-1\), of course. But is that all? What else? Yes, \( \sqrt{-1} \) and \(- \sqrt{-1}\), of course. Oh, wait a second. Since the absolute value of \( a + \sqrt{-1} b \) with \( a, b \in \mathbb{R} \) is \( \sqrt{a^2 + b^2} \), basically all complex numbers \( a + \sqrt{-1} b \) satisfying \( a^2 + b^2 = 1 \) has absolute value 1. Now, from calculus, \( (\cos \theta)^2 + (\sin \theta)^2 = 1 \). So, basically, any complex number of the form \( \cos \theta + \sqrt{-1} \sin \theta \), where \( \theta \) is some real number, has absolute value 1. Now, conversely, once again from calculus, if two real numbers \( a \) and \( b \) satisfy \( a^2 + b^2 = 1 \) then there exists a real number \( \theta \) such that \( a = \cos \theta \) and \( b = \sin \theta \). So, in short:
Basic Fact 4.

(1) Let \( \theta \) be an arbitrary real number. Then the complex number
\[
\beta = \cos \theta + \sqrt{-1} \sin \theta
\]
has absolute value 1: \( |\beta| = 1 \). To paraphrase:
\[
| \cos \theta + \sqrt{-1} \sin \theta | = 1.
\]

(2) Conversely, let \( \beta \) be an arbitrary complex number satisfying
\[
|\beta| = 1.
\]
Then there exists \( \theta \in \mathbb{R} \) such that
\[
\beta = \cos \theta + \sqrt{-1} \sin \theta.
\]

• The following may look weird at first. But it is customary in mathematics. In fact, adopting this definition (notation) makes our life easier. (Trust me.)

Definition (Notation). Let \( \theta \in \mathbb{R} \). Write
\[
\exp(\sqrt{-1} \theta), \quad \text{or alternatively,} \quad e^{\sqrt{-1} \theta},
\]

to mean the complex number \( \cos \theta + \sqrt{-1} \sin \theta \). Thus
\[
\exp(\sqrt{-1} \theta) = e^{\sqrt{-1} \theta} = \cos \theta + \sqrt{-1} \sin \theta.
\]
Using this definition, Basic Fact 3 above is paraphrased as follows:

**Basic Fact 4 paraphrased.**

(1) Let $\theta$ be an arbitrary real number; $\theta \in \mathbb{R}$. Then the complex number

$$\beta = \exp \left( \sqrt{-1} \theta \right) = e^{\sqrt{-1} \theta}$$

has absolute value 1: $|\beta| = 1$. To paraphrase:

$$|\exp \left( \sqrt{-1} \theta \right)| = 1.$$  

Or, the same to say,

$$|e^{\sqrt{-1} \theta}| = 1.$$  

(2) Conversely, let $\beta$ be an arbitrary complex number satisfying

$$|\beta| = 1.$$  

Then there exists $\theta \in \mathbb{R}$ such that

$$\beta = \exp \left( \sqrt{-1} \theta \right) = e^{\sqrt{-1} \theta}.$$  

Exercise 4. (1) Recall that $|\sqrt{-1}| = 1$. Find one $\theta \in \mathbb{R}$ such that

$$\sqrt{-1} = \exp\left(\sqrt{-1}\theta\right).$$

(2) Verify $\left|\frac{-1 + \sqrt{-3}}{2}\right| = 1$. Find one $\theta \in \mathbb{R}$ such that

$$\frac{-1 + \sqrt{-3}}{2} = \exp\left(\sqrt{-1}\theta\right).$$

(3) Verify $\left|\frac{1 + \sqrt{-1}}{\sqrt{2}}\right| = 1$. Find one $\theta \in \mathbb{R}$ such that

$$\frac{1 + \sqrt{-1}}{\sqrt{2}} = \exp\left(\sqrt{-1}\theta\right).$$

(4) Verify $\left|\frac{\sqrt{3} + \sqrt{-1}}{2}\right| = 1$. Find one $\theta \in \mathbb{R}$ such that

$$\frac{\sqrt{3} + \sqrt{-1}}{2} = \exp\left(\sqrt{-1}\theta\right).$$

(5) Verify

$$\left|\frac{\sqrt{2} + \sqrt{2} + \sqrt{-2 + \sqrt{2}}}{2}\right| = 1.$$

Find one $\theta \in \mathbb{R}$ such that

$$\frac{\sqrt{2} + \sqrt{2} + \sqrt{-2 + \sqrt{2}}}{2} = \exp\left(\sqrt{-1}\theta\right).$$

[Note]: $-2 + \sqrt{2} < 0$, so $\sqrt{-2 + \sqrt{2}}$ is a purely imaginary number.
(6) Verify
\[ \left| \frac{\sqrt{2 + \sqrt{3}} + \sqrt{-2 + \sqrt{3}}}{2} \right| = 1. \]

Find one \( \theta \in \mathbb{R} \) such that
\[ \frac{\sqrt{2 + \sqrt{3}} + \sqrt{-2 + \sqrt{3}}}{2} = \exp \left( \sqrt{-1} \theta \right). \]

[Note]: \(-2 + \sqrt{3} < 0\), so \(\sqrt{-2 + \sqrt{3}}\) is a purely imaginary number.

(7) Verify
\[ \left| \frac{\left( -1 + \sqrt{5} \right) + \sqrt{-10 - 2\sqrt{5}}}{4} \right| = 1. \]

Find one \( \theta \in \mathbb{R} \) such that
\[ \frac{\left( -1 + \sqrt{5} \right) + \sqrt{-10 - 2\sqrt{5}}}{4} = \exp \left( \sqrt{-1} \theta \right). \]

[Note]: \(-10 - 2\sqrt{5} < 0\), so \(\sqrt{-10 - 2\sqrt{5}}\) is a purely imaginary number.

(8) Verify
\[ \left| \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}} + \sqrt{-2 + \sqrt{2 + \sqrt{2}}}}{2} \right| = 1. \]

Find one \( \theta \in \mathbb{R} \) such that
\[ \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}} + \sqrt{-2 + \sqrt{2 + \sqrt{2}}}}{2} = \exp \left( \sqrt{-1} \theta \right). \]

[Note]: \(-2 + \sqrt{2 + \sqrt{2}} < 0\) so \(\sqrt{-2 + \sqrt{2 + \sqrt{2}}}\) is a purely imaginary number.
(9) (optional) Verify

\[
\frac{-(1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}}) + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}}{16}
\]

\[
+ \sqrt{\frac{16}{\left(136 - 8\sqrt{17} + (6 - 2\sqrt{17})\sqrt{34 - 2\sqrt{17} + 8\sqrt{34 + 2\sqrt{17}} + 4 (1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}})\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}})}}}
\]

= 1.

Use computer software to make a guess on one \(\theta \in \mathbb{R}\) such that

\[
\frac{-(1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}}) + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}}{16}
\]

\[
+ \sqrt{\frac{16}{\left(136 - 8\sqrt{17} + (6 - 2\sqrt{17})\sqrt{34 - 2\sqrt{17} + 8\sqrt{34 + 2\sqrt{17}} + 4 (1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}})\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}})}}}
\]

= \exp \left(\sqrt{-1}\theta \right).

[Note]:

\[-\left(136 - 8\sqrt{17} + (6 - 2\sqrt{17})\sqrt{34 - 2\sqrt{17} + 8\sqrt{34 + 2\sqrt{17}} + 4 (1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}})\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}})}\right) < 0\]

so

\[
\sqrt{\frac{16}{\left(136 - 8\sqrt{17} + (6 - 2\sqrt{17})\sqrt{34 - 2\sqrt{17} + 8\sqrt{34 + 2\sqrt{17}} + 4 (1 - \sqrt{17} - \sqrt{34 - 2\sqrt{17}})\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}})}}}
\]

is a purely imaginary number.
[Answers] : (1) \( \theta = \frac{\pi}{2} \). Indeed,

\[
\cos \frac{\pi}{2} = 0, \quad \sin \frac{\pi}{2} = 1.
\]

(2) \( \theta = \frac{\pi}{3} \). Indeed,

\[
\cos \frac{\pi}{3} = \frac{1}{2}, \quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.
\]

(3) \( \theta = \frac{\pi}{4} \). Indeed,

\[
\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.
\]

(4) \( \theta = \frac{\pi}{6} \). Indeed,

\[
\cos \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \sin \frac{\pi}{3} = \frac{1}{2}.
\]

(5) \( \theta = \frac{\pi}{8} \). Indeed,

\[
\cos \frac{\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2}, \quad \sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}.
\]

(6) \( \theta = \frac{\pi}{12} \). Indeed,

\[
\cos \frac{\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2}, \quad \sin \frac{\pi}{12} = \frac{\sqrt{2 - \sqrt{3}}}{2}.
\]

(7) \( \theta = \frac{2\pi}{5} \). Indeed,

\[
\cos \frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}, \quad \sin \frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.
\]

(8) \( \theta = \frac{\pi}{16} \). Indeed,

\[
\cos \frac{\pi}{16} = \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2}, \quad \sin \frac{\pi}{16} = \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{2}.
\]
\[ \theta = \frac{2 \pi}{17}. \] Indeed,

\[
\cos \frac{2 \pi}{17} = \frac{-(1 - \sqrt{17} - \sqrt{34} - 2 \sqrt{17}) + 2 \sqrt{17 + 3 \sqrt{17} - \sqrt{34} - 2 \sqrt{17} - 2 \sqrt{34 + 2 \sqrt{17}}}}{16},
\]

\[
\sin \frac{2 \pi}{17} = \frac{\sqrt{136 - 8 \sqrt{17} + 6 - 2 \sqrt{17} \sqrt{34 - 2 \sqrt{17} + 8 \sqrt{34} + 2 \sqrt{17} + 4 \left(1 - \sqrt{17} - 5 \sqrt{17} - \sqrt{34} - 2 \sqrt{17} - 2 \sqrt{34 + 2 \sqrt{17}}\right)}}{16}.
\]

* Question : How to verify that the absolute value of the complex number in (9) equals 1, possibly using computer software?

* Answer : Expand

\[
\left( -(1 - \sqrt{17} - \sqrt{34} - 2 \sqrt{17}) + 2 \sqrt{17 + 3 \sqrt{17} - \sqrt{34} - 2 \sqrt{17} - 2 \sqrt{34 + 2 \sqrt{17}} \right)^2
\]

and see if

\[
120 - \left( -8 \sqrt{17} + 6 - 2 \sqrt{17} \sqrt{34 - 2 \sqrt{17} + 8 \sqrt{34} + 2 \sqrt{17} + 4 \left(1 - \sqrt{17} - 5 \sqrt{17} - \sqrt{34} - 2 \sqrt{17} - 2 \sqrt{34 + 2 \sqrt{17}}\right) \right)
\]

comes out. (Here, 120 is 16^2 - 136.) Do not do it in the decimal answer mode. Do it in the exact answer mode. You can actually do this calculation by hand.

* Question : How to make a guess on \( \theta = \frac{2 \pi}{17} \) in (9) using computer software?

* Answer : Type in

\[
\arccos \frac{-(1 - \sqrt{17} - \sqrt{34} - 2 \sqrt{17}) + 2 \sqrt{17 + 3 \sqrt{17} - \sqrt{34} - 2 \sqrt{17} - 2 \sqrt{34 + 2 \sqrt{17}}}}{16}.
\]
This time, do it in the decimal answer mode. You will get something like

\[ 0.369599135716462633485462... \]

Now, still in the decimal answer mode, divide \( \pi \) by this number:

\[
\frac{\pi}{0.369599135716462633485462}
\]

and you will get something like \( 8.5000000000000000000001... \).

You can imagine that the appearance of the non-zero digit after twenty some straight 0s following 5 right under the decimal point is because of the rounding of the arccosine value above. So you can imagine that the exact value for

\[
\pi \arccos \left( \frac{1 - \sqrt{17} - \sqrt{34 - 2 \sqrt{17}} + 2 \sqrt{17 + 3 \sqrt{17} - \sqrt{34 - 2 \sqrt{17}} - 2 \sqrt{34 + 2 \sqrt{17}}}}{16} \right)
\]

should be \( \frac{17}{2} \). This way you have pulled a ‘hypothetical’ answer for \( \theta \) as \( \frac{2 \pi}{17} \).

Now, once you have done that, I do not believe that your computer software is willing to give you the exact value of \( \cos \frac{2 \pi}{17} \) as

\[
\frac{-(1 - \sqrt{17} - \sqrt{34 - 2 \sqrt{17}} + 2 \sqrt{17 + 3 \sqrt{17} - \sqrt{34 - 2 \sqrt{17}} - 2 \sqrt{34 + 2 \sqrt{17}}}}{16}
\]

(Mine isn’t.) So, instead, go with the best alternative, or ‘Plan B’: Do each of \( \cos \frac{2 \pi}{17} \) and

\[
\frac{-(1 - \sqrt{17} - \sqrt{34 - 2 \sqrt{17}} + 2 \sqrt{17 + 3 \sqrt{17} - \sqrt{34 - 2 \sqrt{17}} - 2 \sqrt{34 + 2 \sqrt{17}}}}{16}
\]

and see that their decimal answers match no matter how far you go. (Your computer software should allow you to set up how many digits you want for the answers in the decimal answer mode, like 100, or 500, or 1000. Mine does.)
• **Addition Formula = Exponential Formula.**

In the previous lecture I alluded that multiplications of complex numbers has to do with the addition formulas for ‘trigs’:

\[
\begin{align*}
\cos(\theta + \phi) &= \left(\cos\theta\right)\left(\cos\phi\right) - \left(\sin\theta\right)\left(\sin\phi\right), \\
\sin(\theta + \phi) &= \left(\sin\theta\right)\left(\cos\phi\right) + \left(\cos\theta\right)\left(\sin\phi\right),
\end{align*}
\]

but without making it precise. This is the place where I can make it precise. Let \(\alpha\) and \(\beta\) be two complex numbers whose absolute values both equal 1:

\[|\alpha| = 1, \quad |\beta| = 1.\]

By Basic Fact 4 above, there exist real numbers \(\theta\) and \(\phi\) such that

\[\alpha = e^{\sqrt{-1}\theta}, \quad \beta = e^{\sqrt{-1}\phi}.\]

(This time around I used \(e\)-to-the-power notation over the ‘exp’ notation.) Now, in this situation, what can we say about \(\alpha \beta\)? Can we describe \(\alpha \beta\) in terms of \(\theta\) and \(\phi\)? Well, of course

\[\alpha \beta = e^{\sqrt{-1}\theta} e^{\sqrt{-1}\phi}.\]

But this is just rewriting of \(\alpha \beta\). Guess what? We do know that

\[e^a e^b = e^{a+b}\]

is true, provided both \(a\) and \(b\) are real numbers. This is called the **exponential formula** from calculus. Now, if we could just pretend for a second that \(\sqrt{-1}\theta\) and \(\sqrt{-1}\phi\) are real numbers, which they actually aren’t, then we could go like

\[e^{\sqrt{-1}\theta} e^{\sqrt{-1}\phi} = e^{\sqrt{-1}(\theta+\phi)}.\]
Now, we should be aware that this argument is not legitimate, because we have ‘extrapolated’ the exponential formula, that is, we have applied the exponential formula to the case where it is not applicable. But that does not mean that the outcome of our extrapolation is incorrect. It can be correct. So, why don’t we try to see if it is correct? But how? Can you do it? Yes you can. Indeed, you use the original definition of $e^{\sqrt{-1} \theta}$. Remember, $e^{\sqrt{-1} \theta}$ means $\cos \theta + \sqrt{-1} \sin \theta$. So,

\[
\alpha \beta = e^{\sqrt{-1} \theta} e^{\sqrt{-1} \phi} \\
= \left( \cos \theta + \sqrt{-1} \sin \theta \right) \left( \cos \phi + \sqrt{-1} \sin \phi \right) \\
= \left( \cos \theta \right) \left( \cos \phi \right) - \left( \sin \theta \right) \left( \sin \phi \right) \\
+ \sqrt{-1} \left( \left( \cos \theta \right) \left( \sin \phi \right) + \left( \sin \theta \right) \left( \cos \phi \right) \right).
\]

So far what I did was just simply the multiplication of two complex numbers. But then suddenly let’s go back to the pair of addition formulas for sin and cos as highlighted in the previous page. You realize that you can actually rewrite this last quantity as

\[
\left( \cos (\theta + \phi) \right) + \sqrt{-1} \left( \sin (\theta + \phi) \right).
\]

But wouldn’t this exactly be $e^{\sqrt{-1} (\theta + \phi)}$? Yes indeed. So, we have actually managed to prove the above outcome of extrapolation. So, let me highlight:

**Addition Formula.** Let $\theta$ and $\phi$ be real numbers. Then

\[
e^{\sqrt{-1} \theta} e^{\sqrt{-1} \phi} = e^{\sqrt{-1} (\theta + \phi)}.
\]

Now, this explains why I purposely chose the $e$-to-the-power notation (over ‘exp’ notation). This way our Addition Formula as highlighted above looks exactly parallel to the ‘exponential formula’

\[
e^a e^b = e^{a+b}.
\]
The reason why I call it ‘Addition Formula’ should be clear: Namely, the pair of trigonometric addition formulas (previous page) is now encapsulated in one single formula, which is precisely the above formula. That’s why the above formula deserves to be called ‘Addition Formula’. But I also want to emphasize the parallelism between this Addition Formula and the exponential formula \( e^a \cdot e^b = e^{a+b} \). So in this regard, nothing stops me from calling the above formula as ‘exponential formula’ instead. So, let me highlight it one more time, with attachment of a different name:

**Exponential Formula.** Let \( \theta \) and \( \phi \) be real numbers. Then

\[
e^{\sqrt{-1} \theta} \cdot e^{\sqrt{-1} \phi} = e^{\sqrt{-1} (\theta + \phi)}.
\]

\[\text{An immediate corollary of Addition formula is this:}\]

**Double, Triple, .., \( m \)-uple angle Formula.** Let \( \theta \) be a real number. Then

\[
\left( e^{\sqrt{-1} \theta} \right)^2 = e^{\sqrt{-1} \cdot 2\theta},
\]

\[
\left( e^{\sqrt{-1} \theta} \right)^3 = e^{\sqrt{-1} \cdot 3\theta},
\]

\[
\left( e^{\sqrt{-1} \theta} \right)^4 = e^{\sqrt{-1} \cdot 4\theta},
\]

\[
\left( e^{\sqrt{-1} \theta} \right)^5 = e^{\sqrt{-1} \cdot 5\theta},
\]

\[\vdots \]

More generally, let \( m \) be a positive integer. Then

\[
\left( e^{\sqrt{-1} \theta} \right)^m = e^{\sqrt{-1} \cdot m\theta}.
\]
Now, in the above, you are tempted to say that $m$ does not have to be a positive integer, but can be negative integer (or 0). Namely, you are tempted to claim:

\[
\left( e^{\sqrt{-1} \theta} \right)^{-2} = e^{-\sqrt{-1} \cdot 2\theta},
\]
\[
\left( e^{\sqrt{-1} \theta} \right)^{-3} = e^{-\sqrt{-1} \cdot 3\theta},
\]
\[
\left( e^{\sqrt{-1} \theta} \right)^{-4} = e^{-\sqrt{-1} \cdot 4\theta},
\]
\[
\left( e^{\sqrt{-1} \theta} \right)^{-5} = e^{-\sqrt{-1} \cdot 5\theta},
\]
\[
\vdots \quad \vdots
\]

As for this, yes, these are all true, once the negative exponents for a complex number are appropriately defined. That was actually epnding. So, let’s give the definition of raising a negative (integer) power to a complex number.

**Definition.** Let $\alpha \in \mathbb{C}$ be a complex number, which is non-zero; $\alpha \neq 0$. Define

\[
\alpha^{-2} = \left( \alpha^{-1} \right)^2,
\]
\[
\alpha^{-3} = \left( \alpha^{-1} \right)^3,
\]
\[
\alpha^{-4} = \left( \alpha^{-1} \right)^4,
\]
\[
\alpha^{-5} = \left( \alpha^{-1} \right)^5,
\]

and so on. More generally, for a positive integer $m$, define

\[
\alpha^{-m} = \left( \alpha^{-1} \right)^m.
\]
Exercise 5. Prove

\[ \alpha^{-2} = \left( \alpha^2 \right)^{-1}, \]
\[ \alpha^{-3} = \left( \alpha^3 \right)^{-1}, \]
\[ \alpha^{-4} = \left( \alpha^4 \right)^{-1}, \]
\[ \alpha^{-5} = \left( \alpha^5 \right)^{-1}, \]

and so on. More generally, for a positive integer \( m \), prove

\[ \alpha^{-m} = \left( \alpha^m \right)^{-1}. \]

Now we are ready to state a more general form of Double, Triple, .. , \( m \)-uple angle formula:

**Exponential Formula – II.**

Let \( \theta \) be an arbitrary real number. Also, let \( m \) be an arbitrary integer. Then

\[ \left( e^{\sqrt{-1} \theta} \right)^m = e^{\sqrt{-1} \cdot m \theta}. \]

Needless to say, the above mirrors the (other) ‘exponential formula’

\[ \left( e^a \right)^m = e^{am} \]

for the case \( a \) real, and hence deserves the name ‘exponential formula – II’.
• This pretty much covers all the materials I had in mind for today’s lesson, except I still have one final item left before I quit. Let’s take a look at “Exponential Formula – II” above a little carefully. By setting $\phi = m\theta$, it reads

\[
\left( e^{\sqrt{-1}\frac{\phi}{m}} \right)^m = e^{\sqrt{-1}\phi}.
\]

For example, for $m = 2$, this reads

\[
\left( e^{\sqrt{-1}\frac{\phi}{2}} \right)^2 = e^{\sqrt{-1}\phi}.
\]

What does this tell us? Yes, for a given $\alpha = e^{\sqrt{-1}\phi}$, this tells us one complex number $\xi$ which satisfies $\xi^2 = \alpha$, namely,

\[
\xi = e^{\sqrt{-1}\frac{\phi}{2}}.
\]

Remember, though, that there is another complex number $\xi$ which satisfies $\xi^2 = \alpha$. That is,

\[
\xi = -e^{\sqrt{-1}\frac{\phi}{2}}.
\]

So, this basically tells us how to square-root a complex number $\alpha$ having absolute value 1, and that is as long as you know the value $\phi$ when you write $\alpha$ as $e^{\sqrt{-1}\phi}$. (To the extent that such $\phi$ is always arccos ($\text{Re } \alpha$), or the same arcsin ($\text{Im } \alpha$), or the same arctan $\frac{\text{Im } \alpha}{\text{Re } \alpha}$, we can say we indeed know such $\phi$.)
• Now, finally, suppose your complex number $\alpha$ has absolute value $\neq 1$. Then does the same idea apply? The answer is actually ‘yes’. The following idea is important, so listen:

• Important idea – another insight about square-rooting.

Let $\alpha$ be an arbitrary complex number, which is non-zero; $\alpha \neq 0$. Then one may write $\alpha$ as

$$\alpha = |\alpha| e^{\frac{\alpha}{\sqrt{-1}} \phi}$$

using some real number $\phi$. This is simply because the complex number $\frac{\alpha}{|\alpha|}$ has absolute value 1, and hence is written as $e^{\frac{\alpha}{\sqrt{-1}} \phi}$ using some real number $\phi$. Once you have written $\alpha$ as

$$\alpha = |\alpha| e^{\frac{\alpha}{\sqrt{-1}} \phi},$$

the two square-roots $\xi$ of $\alpha$ are found simply as

$$\xi = \pm \sqrt{|\alpha|} e^{\frac{\alpha}{\sqrt{-1}} \frac{\phi}{2}}.$$ 

The above gives another insight about square-rooting complex numbers. Let’s just keep this in mind throughout the next lecture. Finally, in case you are curious, how can one generalize this picture to the $m$-th rooting, with $m = 3, 4, 5, 6, \cdots$, it requires to cover the so-called “primitive roots of unity”. That’s also coming up.