§6. INTRODUCTION TO COMPLEX NUMBERS – VI.
FUNDAMENTAL THEOREM OF ALGEBRA.

• Fundamental Theorem of Algebra.

As announced, today’s focus is Fundamental Theorem of Algebra, along with how it plays a role in geometry. I want to talk about Bézout’s theorem, which is a geometric interpretation (a direct generalization) of Fundamental Theorem of Algebra. First let me begin by the exact statement of Fundamental Theorem of Algebra.

Fundamental Theorem of Algebra \textit{(due to Gauss (1799)).}

Let $d$ be a positive integer. Consider the equation

\[ \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0 = 0, \]

where $\alpha_d, \alpha_{d-1}, \cdots, \alpha_1, \alpha_0$ are all knowns and in $\mathbb{C}$, and $z$ unknown. Assume $\alpha_d \neq 0$. Then the equation has a root $z = \alpha$ which is a complex number.

• I want you to appreciate the content of this theorem. Indeed, we already know that, to any complex $\alpha$ number, its $d$-th root $^d\sqrt{\alpha}$ exists within $\mathbb{C}$. (“Important Idea” in “Review of Lectures – XXXIII”, page 21.) However, that does not suffice to validate the above theorem. Why? Recall, from last lecture, that Abel has proved that there is just no way to write out a concrete formula for a general algebraic equation $(\ast)$ as above of degree $d \geq 5$ only using radicals. That’s why whether to any algebraic equation with $\mathbb{C}$-coefficients there is a root within $\mathbb{C}$ is not at all self-evident. Last time I mentioned that it was Gauss who proved this theorem, in 1799. His proof is actually very indirect. His proof does not actually help one to write out roots of a given algebraic equation concretely.
Before moving on to the proof, let me talk a little bit about what benefits the Theorem may bring us. There are actually a few dozens, but the most fundamental consequence of the Theorem is ‘factorizability’. It is the factorizability that ultimately leads to Bézout’s theorem.

- **Consequence of Fundamental Theorem of Algebra — factorizability.**

As a starter, do you remember the ‘Division Principle’ (‘Synthetic Division’) from ‘Algebra 101’? I know this is a little out of the blue. But can you recall?

**Example (Synthetic Division).** Suppose you want to divide

\[ f(x) = x^3 + 2x^2 - 4x + 3 \]

by \( x + 5 \). Then do

\[
\begin{array}{c|ccc}
1 & -3 & 11 \\
\hline
1 & 5 & & \\
1 & 2 & -4 & 3 \\
\hline
-3 & -4 \\
-3 & -15 \\
\hline
11 & 3 \\
11 & 55 \\
\hline
-52
\end{array}
\]

From this you conclude

\[ f(x) = x^3 + 2x^2 - 4x + 3 = (x + 5) \left( x^2 - 3x + 11 \right) + (-52). \]

However, the above does not give you a factorization of the given polynomial \( f(x) \). Indeed, \( f(-5) \) is non-zero. It is \(-52\). The Division Principle basically says that if you find the right number, call it \( s \), and substitute \( x = s \) in \( f(x) \) and get 0 as a result, as in \( f(s) = 0 \), then \( f(x) \) factors as \( (x - s) \) times another polynomial.
Let’s do another example:

**Example (Synthetic Division).** Suppose you want to divide

\[ f(x) = x^3 - 5x^2 + 4x + 6 \]

by \( x - 3 \). Then do

\[
\begin{array}{ccc|ccc}
1 & -2 & -2 & 1 & -5 & 4 & 6 \\
1 & -3 & 1 & -3 & \\
-2 & 4 & \\
-2 & 6 & \\
-2 & 6 & \\
0 & \\
\end{array}
\]

From this you conclude

\[ f(x) = x^3 - 5x^2 + 4x + 6 = (x - 3) \left( x^2 - 2x - 2 \right). \]

This time, the above indeed gives you a factorization of the given polynomial \( f(x) \), in the sense there is no residue. This is no surprise. Indeed, \( f(3) = 0 \):

\[ f(3) = 3^3 - 5 \cdot 3^2 + 4 \cdot 3 + 6 = 27 - 45 + 12 + 6 = 0. \]

What the Division Principle asserts in this case is that, even before you actually perform the above synthetic division, you know beforehand that \( f(x) \) has a factor \( (x - 3) \), by virtue of \( f(3) = 0 \). And you can indeed confirm that \( f(x) \) has the factor \( (x - 3) \) by performing the synthetic division, and know the quotient.
You remember all this, right? By the way, the above synthetic division works when you divide one polynomial \( f(x) \) with another polynomial, not necessarily of the form \((x - s)\). For now we won’t need to worry about that.

- The precise mathematical formulation of this principle is as follows:

Let \( a_d, a_{d-1}, \ldots, a_1, a_0 \) be real numbers (constants). Form

\[
f(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0.
\]

Assume \( a_d \neq 0 \). Agree that

\[
f(x) = 0
\]

signifies an algebraic equation with real number coefficients. Note that this **may or may not** have roots in \( \mathbb{R} \).*

**Division principle (over \( \mathbb{R} \)).**

Now, in the above setting, suppose \( f(x) = 0 \) has a root \( s_1 \in \mathbb{R} \). Thus \( f(s_1) = 0 \). Then there exist \( b_{d-1}, b_{d-2}, \ldots, b_1, b_0 \in \mathbb{R} \), with \( b_{d-1} \neq 0 \), such that

\[
f(x) = (x - s_1) \left( b_{d-1}x^{d-1} + b_{d-2}x^{d-2} + \cdots + b_1x + b_0 \right).
\]

(Note, in fact, that \( b_{d-1} \) equals \( a_d \).)

---

*Thanks to ‘Intermediate Value Theorem’ in calculus, if \( d \) is odd, then \( f(x) = 0 \) indeed has at least one root in \( \mathbb{R} \).
Next, in the above, let’s take a look at the underlined part, which I called \( g(x) \).

Agree that

\[
g(x) = 0
\]

signifies an algebraic equation with real number coefficients. Once again, this last equation may or may not have roots in \( \mathbb{R} \).

Now, in that setting, suppose \( g(x) = 0 \) has a root \( s_2 \in \mathbb{R} \). Thus, \( g(s_2) = 0 \). Then, the same ‘Division Principle’ applies to \( g(x) \), instead of \( f(x) \).

So, there exist \( c_{d-2}, c_{d-3}, \ldots, c_1, c_0 \in \mathbb{R} \), with \( c_{d-2} \neq 0 \), such that

\[
g(x) = (x - s_2) \left( c_{d-2} x^{d-2} + c_{d-3} x^{d-3} + \cdots + c_1 x + c_0 \right).
\]

(Note, in fact, that \( c_{d-2} \) equals \( b_{d-1} \).)

Accordingly, the original

\[
f(x) = (x - s_1) g(x)
\]

becomes

\[
f(x) = (x - s_1) (x - s_2) \left( c_{d-2} x^{d-2} + c_{d-3} x^{d-3} + \cdots + c_1 x + c_0 \right).
\]
Next, in the above, let’s take a look at the underlined part, which I called $h(x)$. Agree that

$$h(x) = 0$$

signifies an algebraic equation with real number coefficients. Once again, this last equation may or may not have roots in $\mathbb{R}$.

Now, in that setting, suppose $h(x) = 0$ has a root $s_3 \in \mathbb{R}$. Thus, $h(s_3) = 0$. Then, the same ‘Division Principle’ applies to $h(x)$, and so on so forth.

We may continue this procedure until we get an expression

$$f(x) = (x - s_1) \cdots (x - s_{d-r}) \left( p_r x^r + p_{r-1} x^{r-1} + \cdots + p_1 x + p_0 \right),$$

$$\xi(x)$$

with $p_r, p_{r-1}, \cdots, p_1, p_0 \in \mathbb{R}$; $p_r \neq 0$, and moreover, the equation

$$\xi(x) = 0$$

has no real roots.

Sometimes, but not always, we can go all the way:

$$f(x) = (x - s_1) \cdots (x - s_{d-1}) \left( x - s_d \right) \cdot p_0,$$

$p_0 \in \mathbb{R}$; $p_0 \neq 0$. This is feasible sometimes. But not always.
Example. \[ f(x) = x^3 + 6x^2 + 11x + 6 \]
factors as \[ (x - 1)(x - 2)(x - 3). \]

Example. \[ f(x) = 2x^4 - 16x^3 + 42x^2 - 44x + 16 \]
factors as \[ 2(x - 1)^2(x - 2)(x - 4). \]

Example. \[ f(x) = 3x^4 + 12x^3 + 12x^2 - 27 \]
factors as \[ 3(x - 1)(x + 3)(x^2 + 2x + 3). \]
\[ x^2 + 2x + 3 \] does not have a factor of the form \( (x - s) \), with \( s \in \mathbb{R} \).

Example. \[ f(x) = x^4 + 2x^3 + x^2 + 1 \]
does not have a factor of the form \( (x - s) \), with \( s \in \mathbb{R} \).

Example. \[ f(x) = -x^7 + x^6 + x^5 - x^4 + x^3 - x^2 - x + 1 \]
factors as \[ -(x - 1)^3(x + 1)^2(x^2 + 1). \]
\[ x^2 + 1 \] does not have a factor of the form \( (x - s) \), with \( s \in \mathbb{R} \).
Example. \( f(x) = x^{15} - x^{14} - x^{13} + x^{12} - x^{11} + x^{10} + x^9 - x^8 \\
- x^7 + x^6 + x^5 - x^4 + x^3 - x^2 - x + 1 \)

factors as

\[
\left( x - 1 \right)^4 \left( x + 1 \right)^3 \left( x^8 + 2x^6 + 2x^4 + 2x^2 + 1 \right). 
\]

\( x^8 + 2x^6 + 2x^4 + 2x^2 + 1 \) does not have a factor of the form \( (x - s) \), with \( s \in \mathbb{R} \).

- Note that, in the above half-dozen examples I deliberately chose polynomials whose real roots are all integers. As I emphasized last time, due to Abel’s work, for a polynomial of degree at least 5 with randomly assigned coefficients, there is no formula to write out its roots. Also as we have practiced, some special type of polynomials have roots which are written using the trigonometric arguments. The following couple of examples are essentially the same as “Review of Lectures – XXXV”, page 10, ‘Fact 1’, ‘Fact 2’ (see also ‘Gauss sum’, page 28–31 of the present notes):

Example. \( f(x) = 32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 \)

factors as

\[
\left( x - \cos \frac{2\pi}{11} \right) \left( x - \cos \frac{4\pi}{11} \right) \left( x - \cos \frac{6\pi}{11} \right) \left( x - \cos \frac{8\pi}{11} \right) \left( x - \cos \frac{10\pi}{11} \right). 
\]

Example. \( f(x) = 32x^5 - 16\sqrt{11}x^4 + 12\sqrt{11}x^2 - 22x + \sqrt{11} \)

factors as

\[
\left( x - \sin \frac{2\pi}{11} \right) \left( x + \sin \frac{4\pi}{11} \right) \left( x - \sin \frac{6\pi}{11} \right) \left( x - \sin \frac{8\pi}{11} \right) \left( x - \sin \frac{10\pi}{11} \right). 
\]
Below is the case $a = 2$ of the left-hand side of the equation (1) on page 8, “Review of Lectures – XXXV”:

**Example.** \( f(x) = x^5 + 1250x^3 + 7500x^2 + 25000x + 18750 \)

factors as

\[
\left(x + 1 + \left(1 - 2^{\frac{1}{5}}\right)^5\right) \\
\cdot \left(x^4 - \left(5 \cdot 2^{\frac{1}{5}} - 10 \cdot 2^{\frac{3}{5}} + 10 \cdot 2^{\frac{7}{5}} - 5 \cdot 2^{\frac{1}{5}}\right)x^3\right) \\
+ \left(200 \cdot 2^{\frac{1}{5}} - 50 \cdot 2^{\frac{3}{5}} - 175 \cdot 2^{\frac{7}{5}} + 400 \cdot 2^{\frac{1}{5}} + 750\right)x^2 \\
+ \left(1750 \cdot 2^{\frac{1}{5}} + 625 \cdot 2^{\frac{3}{5}} - 1000 \cdot 2^{\frac{7}{5}} + 500 \cdot 2^{\frac{1}{5}} + 3000\right)x \\
+ \left(3125 \cdot 2^{\frac{1}{5}} + 2500 \cdot 2^{\frac{3}{5}} + 1250 \cdot 2^{\frac{7}{5}} + 2500 \cdot 2^{\frac{1}{5}} + 5000\right). \\
\]

The part

\[
x^4 - \left(5 \cdot 2^{\frac{1}{5}} - 10 \cdot 2^{\frac{3}{5}} + 10 \cdot 2^{\frac{7}{5}} - 5 \cdot 2^{\frac{1}{5}}\right)x^3 \\
+ \left(200 \cdot 2^{\frac{1}{5}} - 50 \cdot 2^{\frac{3}{5}} - 175 \cdot 2^{\frac{7}{5}} + 400 \cdot 2^{\frac{1}{5}} + 750\right)x^2 \\
+ \left(1750 \cdot 2^{\frac{1}{5}} + 625 \cdot 2^{\frac{3}{5}} - 1000 \cdot 2^{\frac{7}{5}} + 500 \cdot 2^{\frac{1}{5}} + 3000\right)x \\
+ \left(3125 \cdot 2^{\frac{1}{5}} + 2500 \cdot 2^{\frac{3}{5}} + 1250 \cdot 2^{\frac{7}{5}} + 2500 \cdot 2^{\frac{1}{5}} + 5000\right)
\]

does not have a factor of the form \( (x - s) \), with \( s \in \mathbb{R} \).
Below is the case $a = 1$ of the left-hand side of the equation (2) on page 8, "Review of Lectures – XXXV":

**Example.**

$$f(x) = x^5 + x + 1$$

factors as

$$
\left( x - b \right) \left( x^4 + b x^3 + b^2 x^2 + b^3 x + b^4 + 1 \right),
$$

where $b$ is the unique real root of the equation $f(x) = 0$. There is no way to concretely write out $b$ using integers and radicals.

$$x^4 + b x^3 + b^2 x^2 + b^3 x + b^4 + 1 \quad \text{(with the same } b \in \mathbb{R} \text{)}$$

does not have a factor of the form $\left( x - s \right)$, with $s \in \mathbb{R}$.

In case, calculation (synthetic division):

<table>
<thead>
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<th>1</th>
<th>$b$</th>
<th>$b^2$</th>
<th>$b^3$</th>
<th>$b^4 + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-b$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>$-b$</td>
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<tr>
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</tr>
<tr>
<td>1</td>
<td>$-b$</td>
<td></td>
<td>$b^4 + 1$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$-b$</td>
<td></td>
<td>$b^4 + 1$</td>
<td>$-b^5 - b$</td>
</tr>
<tr>
<td>1</td>
<td>$-b$</td>
<td></td>
<td>$b^5 + b + 1 = f(b) = 0$</td>
<td></td>
</tr>
</tbody>
</table>
• Now, the above ‘Division Principle’ was a statement over \( \mathbb{R} \). We are interested in knowing if the same result over \( \mathbb{C} \) holds true. Does that make sense? Yes, the makes perfect sense, indeed, so far we relied only on the following aspect of \( \mathbb{R} \): Namely, that \( \mathbb{R} \) is a field. This is important, so let me highlight:

> “The synthetic division uses addition, subtraction, multiplication and division of real numbers, and nothing else. In short, the division principle pertains to the field axioms aspect of \( \mathbb{R} \), nothing else.”

So, in other words, the same is applicable to any field, not just \( \mathbb{R} \). Below we are specifically interested in the case of the complex number field \( \mathbb{C} \). (Remember, we have adopted ‘the complex number field’ as the official name for \( \mathbb{C} \).) Let me highlight:

• Let \( \alpha_d, \alpha_{d-1}, \ldots, \alpha_1, \alpha_0 \) be complex numbers (constants). Form

\[
f(z) = \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0.
\]

Assume \( \alpha_d \neq 0 \). Agree that

\[
f(z) = 0
\]

signifies an algebraic equation.

**Division principle (over \( \mathbb{C} \)).**

In the above setting, suppose \( f(x) = 0 \) has a root \( \sigma_1 \in \mathbb{C} \). Thus \( f(\sigma_1) = 0 \). Then there exist \( \beta_{d-1}, \beta_{d-2}, \ldots, \beta_1, \beta_0 \in \mathbb{C} \), with \( \beta_{d-1} \neq 0 \), such that

\[
f(z) = (x - \sigma_1)(\beta_{d-1} z^{d-1} + \beta_{d-2} z^{d-2} + \cdots + \beta_1 z + \beta_0).
\]

(Note, in fact, that \( \beta_{d-1} \) equals \( \alpha_d \).)
Now, in that setting, suppose \( g(z) = 0 \) has a root \( \sigma_2 \in \mathbb{C} \). Thus, \( g(\sigma_2) = 0 \). Then, the same ‘Division Principle’ applies to \( g(z) \), instead of \( f(z) \). So, there exist \( \gamma_{d-2}, \gamma_{d-3}, \cdots, \gamma_1, \gamma_0 \in \mathbb{C} \), with \( \gamma_{d-2} \neq 0 \), such that

\[
g(z) = (z - \sigma_2) \left( \frac{\gamma_{d-2} z^{d-2} + \gamma_{d-3} z^{d-3} + \cdots + \gamma_1 z + \gamma_0}{h(z)} \right).
\]

(Note, in fact, that \( \gamma_{d-2} \) equals \( \beta_{d-1} \).) Accordingly, \( f(z) \) becomes

\[
f(z) = (x - \sigma_1) (x - \sigma_2) \left( \frac{\gamma_{d-2} z^{d-2} + \gamma_{d-3} z^{d-3} + \cdots + \gamma_1 z + \gamma_0}{h(z)} \right).
\]

Next, suppose \( h(z) = 0 \) has a root \( \sigma_3 \in \mathbb{C} \). Thus, \( h(\sigma_3) = 0 \). Then the same ‘Division Principle’ applies to \( h(z) \), and so on so forth. So far this is entirely parallel to the case of \( \mathbb{R} \). We may continue this procedure until we reach

\[
f(z) = (z - \sigma_1) \cdots (z - \sigma_{d-r}) \left( \frac{\rho_r z^r + \rho_{r-1} z^{r-1} + \cdots + \rho_1 z + \rho_0}{\xi(z)} \right),
\]

with \( \rho_r, \rho_{r-1}, \cdots, \rho_1, \rho_0 \in \mathbb{C} ; \rho_r \neq 0 \), and moreover, the equation

\[
\xi(z) = 0
\]

has no complex roots. But wait a second..
Can we recall the content of Fundamental Theorem of Algebra? Yes, according to Fundamental Theorem of Algebra, the equation
\[ \rho_r z^r + \cdots + \rho_1 z + \rho_0 = 0 \]
always has complex roots. So, this means that, (unlike the case over \( \mathbb{R} \)) we can go all the way:
\[ f(z) = (z - \sigma_1) \cdots (z - \sigma_{d-1})(z - \sigma_d) \cdot \rho_0, \]
\(\rho_0 \in \mathbb{C}; \ \rho_0 \neq 0\). We can always attain this. This way, we obtain the following ‘neat’ theorem, which is a stronger form of ‘Fundamental Theorem of Algebra’:

**Factorization Theorem (Fundamental Theorem of Algebra – II)**

Let \( \alpha_d, \ \alpha_{d-1}, \ \cdots, \ \alpha_1, \ \alpha_0 \in \mathbb{C} \) be complex number constants. Assume \( \alpha_d \neq 0 \). Then
\[ \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0 \]
always factors as
\[ \alpha (z - \sigma_1)(z - \sigma_2) \cdots (z - \sigma_d), \]
where \( \alpha = \alpha_d \), in particular, \( \alpha \neq 0 \), and \( \sigma_1, \ \sigma_2, \ \cdots, \ \sigma_d \in \mathbb{C} \).

- In the above, the set \( \{ \sigma_1, \sigma_2, \cdots, \sigma_d \} \) equals the entire set of roots of the algebraic equation
\[ \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0 = 0. \]
• Multiplicity of roots. Factorization Theorem paraphrased.

In the above, sometimes \( \sigma_i \) and \( \sigma_j \) for \( i \neq j \) coincide. Taking such a possibility into account, we may rewrite

\[
\alpha \left( z - \sigma_1 \right) \left( z - \sigma_2 \right) \cdots \left( z - \sigma_d \right)
\]

as

\[
\alpha \left( z - \lambda_1 \right)^{\mu_1} \left( z - \lambda_2 \right)^{\mu_2} \cdots \left( z - \lambda_r \right)^{\mu_r},
\]

where \( \lambda_1, \lambda_2, \cdots, \lambda_r \in \mathbb{C} \) are mutually distinct, and \( \mu_1, \mu_2, \cdots, \mu_r \) are all positive integers. We may accordingly paraphrase the above theorem:

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**Factorization Theorem – paraphrased.**

Let \( \alpha_d, \alpha_{d-1}, \cdots, \alpha_1, \alpha_0 \in \mathbb{C} \) be complex number constants. Assume \( \alpha_d \neq 0 \). Then

\[
\alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0
\]

always factors as

\[
\alpha \left( z - \lambda_1 \right)^{\mu_1} \left( z - \lambda_2 \right)^{\mu_2} \cdots \left( z - \lambda_r \right)^{\mu_r},
\]

where \( \alpha = \alpha_d \), in particular, \( \alpha \neq 0 \), \( \lambda_1, \lambda_2, \cdots, \lambda_r \in \mathbb{C} \) are mutually distinct, also \( \mu_1, \mu_2, \cdots, \mu_r \), are positive integers.
• In the above, the set \( \{ \lambda_1, \lambda_2, \cdots, \lambda_r \} \) equals the entire set of roots of the algebraic equation

\[
\alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0 = 0.
\]

Definition (Multiplicity).

In the above paraphrased version of the ‘Factorizability Theorem’, the positive integer \( \mu_j \) is called the multiplicity of the root \( z = \lambda_j \) of the equation

\[
\alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0 = 0.
\]

Remark. The following is self-evident, but important. In the above, \( r \leq d \), and moreover,

\[
\sum_{j=1}^{r} \mu_j = d.
\]

To phrase it verbally:

"The sum of the multiplicities of all the roots of an algebraic equation equals the degree of that equation."

• Application to algebraic equations with real coefficients.

We may apply this to algebraic equations with real number coefficients. As we have already seen, an algebraic equation with real number coefficients does not always factor into a product of quantities of the form \( (x - s) \) with \( s \in \mathbb{R} \) (linear factors). Nevertheless, we can utilize Fundamental Theorem of Algebra and prove that it factors into a product of linear factors and factors of the form \( (x^2 + px + q) \) with \( p, q \in \mathbb{R} \) (quadratic factors). A more precise statement is as follows:
Corollary (Factorization Theorem over $\mathbb{R}$).

Let $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathbb{R}$ be real number constants. Assume $a_d \neq 0$. Then

$$a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

always factors as

$$a \left( x - s_1 \right)^{\mu_1} \left( x - s_2 \right)^{\mu_2} \cdots \left( x - s_m \right)^{\mu_m} \cdot \left( x^2 + p_1 x + q_1 \right)^{\nu_1} \cdots \left( x^2 + p_\ell x + q_\ell \right)^{\nu_\ell}$$

where

- $a = a_d$, in particular, $a \neq 0$,
- $s_1, s_2, \ldots, s_m \in \mathbb{R}$ are mutually distinct,
- $\left( p_1, q_1 \right), \ldots, \left( p_\ell, q_\ell \right) \in \mathbb{R}^2$ are mutually distinct, $p_j^2 - 4 q_j < 0$,
- $\mu_1, \mu_2, \ldots, \mu_m; \nu_1, \nu_2, \ldots, \nu_\ell$ are positive integers.

**Proof.** Regard $a_d, a_{d-1}, \ldots, a_1$ and $a_0$, which are real numbers, as complex numbers. Thus we may apply Factorization Theorem to

$$a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

It factors as

$$a \left( x - \lambda_1 \right)^{\mu_1} \left( x - \lambda_2 \right)^{\mu_2} \cdots \left( x - \lambda_r \right)^{\mu_r},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are mutually distinct complex numbers, and $\mu_1, \mu_2, \ldots, \mu_r$ are positive integers. Among $\lambda_1, \lambda_2, \ldots, \lambda_r$, some are real, some are not.
Let us rewrite the real ones among those as $s_1$, $s_2$, \ldots, $s_m$, and the rest as $\sigma_1$, $\sigma_2$, \ldots, $\sigma_{d-m}$. Clearly the substitution of $x = \sigma_1$ into $a_dx^d + a_1x + a_0$ yields $0$:

$$a_d\sigma_1^d + a_{d-1}\sigma_1^{d-1} + \cdots + a_1\sigma_1 + a_0 = 0.$$ 

Take the complex conjugate of the both sides:

$$a_d\overline{\sigma_1^d} + a_{d-1}\overline{\sigma_1^{d-1}} + \cdots + a_1\overline{\sigma_1} + a_0 = 0.$$ 

Taking $a_d, a_{d-1}, \cdots, a_1, a_0 \in \mathbb{R}$ into account, we may rewrite this as

$$a_d\overline{\sigma_1^d} + a_{d-1}\overline{\sigma_1^{d-1}} + \cdots + a_1\overline{\sigma_1} + a_0 = 0.$$ 

Since the set

$$\{ \lambda_1, \lambda_2, \cdots, \lambda_r \} = \{ s_1, s_2, \cdots, s_m \} \cup \{ \sigma_1, \sigma_2, \cdots, \sigma_{d-m} \}$$

equals the entire set of roots of the algebraic equation

$$a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0 = 0,$$

$\overline{\sigma_1}$ belongs to this set. Since $\sigma_1$ is non-real by assumption, it follows that $\overline{\sigma_1}$ is one of $\sigma_2, \cdots, \sigma_{d-m}$. This shows that the set

$$\{ \sigma_1, \sigma_2, \cdots, \sigma_{d-m} \}$$

consists of mutually conjugate pairs of non-real complex numbers. In particular, this set consists of an even number of elements. By re-ordering if necessary, we can assume that $\sigma_1$ and $\sigma_2$ are mutually conjugate; $\sigma_3$ and $\sigma_4$ are mutually conjugate, and so on. Recall that each of the sum and the product of a mutually conjugate pair of complex numbers is real. Thus

$$\left( x - \sigma_1 \right) \left( x - \sigma_2 \right) = x^2 - \left( \sigma_1 + \sigma_2 \right)x + \sigma_1\sigma_2$$

is of the form $x^2 + p_1x + q_1$, using some $p_1, q_1 \in \mathbb{R}$. Clearly $p_1^2 - 4q_1 < 0$.

The same goes for the rest of the pairs $(\sigma_3, \sigma_4), \cdots, (\sigma_{2\ell-1}, \sigma_{2\ell})$. \hfill \Box
Paraphrase of Factorization Theorem over $\mathbb{R}$.

Let $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathbb{R}$ be real number constants. Assume $a_d \neq 0$. Then

$$a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

always factors as

$$a (x - s_1)^{\mu_1} (x - s_2)^{\mu_2} \cdots (x - s_m)^{\mu_m}$$

$$\cdot (x - \sigma_1)^{\nu_1} (x - \overline{\sigma_1})^{\nu_1}$$

$$\cdot (x - \sigma_2)^{\nu_2} (x - \overline{\sigma_2})^{\nu_2}$$

$$\cdots$$

$$\cdot (x - \sigma_{\ell})^{\nu_{\ell}} (x - \overline{\sigma_{\ell}})^{\nu_{\ell}}$$

where

- $a = a_d$, in particular, $a \neq 0$,
- $s_1, s_2, \ldots, s_m \in \mathbb{R}$ are mutually distinct,
- $\sigma_1, \sigma_2, \ldots, \sigma_{\ell} \in \mathbb{C}$ are mutually distinct, $\sigma_1, \sigma_2, \ldots, \sigma_{\ell} \notin \mathbb{R}$, none of them are mutually conjugate,
- $\mu_1, \mu_2, \ldots, \mu_m; \nu_1, \nu_2, \ldots, \nu_{\ell}$ are positive integers.

The above two statements are corollaries of ‘Factorization Theorem over $\mathbb{C}$’, and ‘Factorization Theorem over $\mathbb{C}$’ is a corollary of Fundamental Theorem of Algebra. So everything I am covering today comes out of Fundamental Theorem of Algebra. We are yet to prove Fundamental Theorem of Algebra. We will do that today.
Below let me state one immediate corollary of Factorization Theorem above, about an odd degree equation. This is what you already know from calculus, namely, a result obtained as an application of ‘Intermediate Value Theorem’. But now we can reproduce the same result as an application of ‘Fundamental Theorem of Algebra’ instead of ‘Intermediate Value Theorem’:

**Corollary.** Let \( a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathbb{R} \) be real number constants. Assume \( a_d \neq 0 \). Assume \( d \) is odd. Then the equation

\[
a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 = 0
\]

has a real root \( x = s \).

In short:

"Any algebraic equation with real number coefficients of an odd degree has at least one real root."

Bear in mind, though, that this does not mean you can concretely write out the real root using the coefficients of the equation and radicals. (See Example in page 10.)

I want to revisit some previous examples. For that, I have to rely on one concept:

**Notation (Primitive root of unity).** Let \( r \) be an integer, \( r \geq 2 \). Write

\[
\zeta_r = e^{\frac{-2\pi i}{r}}.
\]

In other words,

\[
\zeta_r = \left( \cos \frac{2\pi}{r} \right) + \sqrt{-1} \left( \sin \frac{2\pi}{r} \right).
\]

\( \zeta_r \) is called the primitive \( r \)-th root of unity.
Basic Facts. \hspace{1cm} (1a) \hspace{1cm} \zeta_r^r = 1.

(1b) \hspace{1cm} \zeta_r^j \neq 1 \hspace{1cm} (j = 1, 2, 3, \ldots, r - 1).

(2) \hspace{1cm} |\zeta_r| = 1. \hspace{1cm} \hspace{1cm} \hspace{1cm} (3) \hspace{1cm} \zeta_{r^k} = \zeta_r \hspace{1cm} (k = 1, 2, 3, \ldots).

(4) \hspace{1cm} \zeta_r^j = \frac{\zeta_r^{r-j}}{\zeta_r^j} \hspace{1cm} (j = 1, 2, 3, \ldots, r - 1).

Example. For \( r = 7 \), we have

\[
\zeta_7 = e^{\frac{2\pi}{7}} = \left( \cos \frac{2\pi}{7} \right) + \sqrt{-1} \left( \sin \frac{2\pi}{7} \right).
\]

Recall (from “Review of Lectures – XXXIV”, page 34, Example 8) that \( \zeta_7 \) is a root of

\[
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 = 0.
\]

A quick refresher how to see this. First, by (1) of ‘Facts’ above, \( z = \zeta_7 \) is a root of

\[
1 - z^7 = 0.
\]

Use the fact that \( 1 - z^7 \) factors as

\[
(1 - z) \left( 1 + z + z^2 + z^3 + z^4 + z^5 + z^6 \right),
\]

to rewrite the same equation \( 1 - z^7 = 0 \) as

\[
(1 - z) \left( 1 + z + z^2 + z^3 + z^4 + z^5 + z^6 \right) = 0.
\]

As claimed above, \( z = \zeta_7 \) is a root of this equation. Hence

\[
(1 - \zeta_7) \left( 1 + \zeta_7 + \zeta_7^2 + \zeta_7^3 + \zeta_7^4 + \zeta_7^5 + \zeta_7^6 \right) = 0.
\]

Here, clearly \( 1 - \zeta_7 \) is not equal to 0. Hence

\[
1 + \zeta_7 + \zeta_7^2 + \zeta_7^3 + \zeta_7^4 + \zeta_7^5 + \zeta_7^6 = 0.
\]
This shows that $z = \zeta_7$ is indeed a root of

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 = 0.$$ 

- As a matter of fact, we can run the same argument as above and can prove that each of $\zeta_7^2$, $\zeta_7^3$, $\zeta_7^4$, $\zeta_7^5$, and $\zeta_7^6$ is a root of

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 = 0.$$ 

Indeed,

$$\left( \zeta_7^2 \right)^r = \zeta_7^{2r} = \left( \zeta_7^7 \right)^2 = 1^2 = 1,$n$$

$$\left( \zeta_7^3 \right)^r = \zeta_7^{3r} = \left( \zeta_7^7 \right)^3 = 1^3 = 1,$n$$

$$\vdots$$

Since the equation $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 = 0$ is of degree 6, the list

$$\{ \zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6 \}$$

exhausts the entire set of roots of the same equation. To conclude:

**Example.**

$$f(z) = 1 + z + z^2 + z^3 + z^4 + z^5 + z^6$$

factors as

$$\left( 1 - \zeta_7 \right) \left( 1 - \zeta_7^2 \right) \left( 1 - \zeta_7^3 \right) \left( 1 - \zeta_7^4 \right) \left( 1 - \zeta_7^5 \right) \left( 1 - \zeta_7^6 \right).$$

In particular, each of the six roots $\zeta_7^2$, $\zeta_7^3$, $\zeta_7^4$, $\zeta_7^5$, and $\zeta_7^6$ of

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 = 0$$

has multiplicity 1.
I want to freely use the above notation $\zeta_r$ from now on until the end of the semester. This is extremely convenient. Now, in the above I demonstrated some facts for the case $r = 7$. It is self-evident how to extrapolate this and establish the same facts for the case $r$ is an arbitrary integer, $r \geq 2$. Let me highlight:

**Basic Facts (about $\zeta_r$ and the equation $1 + z + z^2 + \cdots + z^{r-1} = 0$).**

Let $r$ be an arbitrary integer, $r \geq 3$. Then

(1a) The list

\[
\left\{ \zeta_r, \ z_r^2, \ z_r^3, \ \cdots, \ z_r^{r-1} \right\}
\]

exhausts the entire set of roots of the equation

\[
1 + z + z^2 + \cdots + z^{r-1} = 0.
\]

Each root has multiplicity 1.

(1b) $f(z) = 1 + z + z^2 + z^3 + \cdots + z^{r-1}$

factors as

\[
(z - \zeta_r)(z - \zeta_r^2)(z - \zeta_r^3) \cdots (z - z_r^{r-1}).
\]

(2a) The list

\[
\left\{ 1, \ z_r, \ z_r^2, \ z_r^3, \ \cdots, \ z_r^{r-1} \right\}
\]

exhausts the entire set of roots of the equation

\[
z^r - 1 = 0.
\]

Each root has multiplicity 1.
(2b) \[ f(z) = z^r - 1 \]

factors as

\[ (z - 1) (z - \zeta_r) (z - \zeta_r^2) (z - \zeta_r^3) \cdots (z - \zeta_r^{r-1}) \].

**Example.** For \( r = 4 \), we have

\[ \zeta_4 = \sqrt{-1}. \]

Indeed,

\[ \zeta_4 = e^{\pi \frac{i}{2}} = \left( \cos \frac{\pi}{2} \right) + \sqrt{-1} \left( \sin \frac{\pi}{2} \right) = 0 + \sqrt{-1} \cdot 1 = \sqrt{-1}. \]

Thus

\[ \zeta_4^2 = -1, \quad \text{and} \quad \zeta_4^3 = -\sqrt{-1}. \]

As for factorizations, we have

\[ 1 + z + z^2 + z^3 = (z - \sqrt{-1})(z + 1)(z + \sqrt{-1}) \]

\[ = (z - \zeta_4)(z - \zeta_4^2)(z - \zeta_4^3), \]

\[ z^4 - 1 = (z - 1)(z - \sqrt{-1})(z + 1)(z + \sqrt{-1}) \]

\[ = (z - 1)(z - \zeta_4)(z - \zeta_4^2)(z - \zeta_4^3). \]
Below we calculate each of $\zeta_3$, $\zeta_6$, $\zeta_5$ and $\zeta_8$, and their powers, without the knowledge on the special values of 'sin' and 'cos' except the knowledge of over which intervals 'sin' and 'cos' are positive (negative).

Example. For $r = 3$, we have

$$\zeta_3 = \frac{-1 + \sqrt{-3}}{2}, \quad \text{and} \quad \zeta_3^2 = \frac{-1 - \sqrt{-3}}{2}.$$ 

Indeed, as stated above, $\zeta_3$ and $\zeta_3^2$ are the roots of the equation

$$1 + z + z^2 = 0.$$ 

This is a quadratic equation, and it is solved as

$$z = \frac{-1 + \sqrt{-3}}{2}, \quad \frac{-1 - \sqrt{-3}}{2}.$$ 

Of these, one is $\zeta_3$, one is $\zeta_3^2$. The imaginary part of $\zeta_3$ is $\sin \frac{2\pi}{3}$, which is positive. Hence the claim.

As for factorizations, we have

$$1 + z + z^2 = (z - \zeta_3)(z - \zeta_3^2),$$

$$z^3 - 1 = (z - 1)(z - \zeta_3)(z - \zeta_3^2).$$

Example. For $r = 6$, we have

$$\zeta_6 = \frac{1 + \sqrt{-3}}{2}, \quad \zeta_6^2 = \frac{-1 + \sqrt{-3}}{2}, \quad \zeta_6^3 = -1,$$

$$\zeta_6^4 = \frac{-1 - \sqrt{-3}}{2}, \quad \zeta_6^5 = \frac{1 - \sqrt{-3}}{2}. $$

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Indeed, as stated above, $\zeta_6$ is a root of the equation

$$1 - z^6 = 0.$$ 

This factors as

$$\left(1 - z^3\right)\left(1 + z^3\right) = 0.$$ 

We already know the three roots of $1 - z^3 = 0$, namely, $1, \zeta_3$ and $\zeta_3^2$. It is self-evident that the three roots of $1 + z^3 = 0$ are the negatives of the three roots of $1 - z^3 = 0$, which are $-1, -\zeta_3$ and $-\zeta_3^2$. In short, the complete list of the roots of $1 - z^6 = 0$ is

$$\{1, -1, \zeta_3, -\zeta_3, \zeta_3^2, -\zeta_3^2\}.$$ 

This set coincides with

$$\{1, \zeta_6, \zeta_6^2, \zeta_6^3, \zeta_6^4, \zeta_6^5\}.$$ 

To identify $\zeta_6$, note that the real and the imaginary parts of $\zeta_6$, namely, $\cos\frac{\pi}{3}$ and $\sin\frac{\pi}{3}$, are both positive. The only number among $1, -1, \zeta_3, -\zeta_3, \zeta_3^2$ and $-\zeta_3^2$ satisfying this property is $-\zeta_3^2 = \frac{1 + \sqrt{-3}}{2}$. In short,

$$\zeta_6 = -\zeta_3^2 = \frac{1 + \sqrt{-3}}{2}.$$ 

We also have

$$\zeta_6^2 = \zeta_3 = \frac{-1 + \sqrt{-3}}{2}, \quad \zeta_6^3 = -1,$$

$$\zeta_6^4 = \zeta_3^2 = \frac{-1 - \sqrt{-3}}{2}, \quad \zeta_6^5 = -\zeta_3 = \frac{1 - \sqrt{-3}}{2}.$$
As for factorizations, we have

\[ 1 + z + z^2 + z^3 + z^4 + z^5 = (z - \zeta_6)(z - \zeta_6^2)(z - \zeta_6^3)(z - \zeta_6^4)(z - \zeta_6^5), \]
\[ z^6 - 1 = (z - 1)(z - \zeta_6)(z - \zeta_6^2)(z - \zeta_6^3)(z - \zeta_6^4)(z - \zeta_6^5). \]

**Example.** For \( r = 5 \), we have

\[ \zeta_5 = \frac{-1 + \sqrt{5} + \sqrt{-
\left(10 + 2\sqrt{5}\right)}}{4}, \]
\[ \zeta_5^2 = \frac{-1 - \sqrt{5} + \sqrt{-\left(10 - 2\sqrt{5}\right)}}{4}, \]
\[ \zeta_5^3 = \frac{-1 - \sqrt{5} - \sqrt{-\left(10 - 2\sqrt{5}\right)}}{4}, \]
\[ \zeta_5^4 = \frac{-1 + \sqrt{5} - \sqrt{-\left(10 + 2\sqrt{5}\right)}}{4}. \]

Note that both \( 10 + 2\sqrt{5} \) and \( 10 - 2\sqrt{5} \) are positive. Thus
\[ \sqrt{-\left(10 + 2\sqrt{5}\right)} \quad \text{and} \quad \sqrt{-\left(10 - 2\sqrt{5}\right)} \]
are purely imaginary.
The best way to derive these that I know of is to rely on Gauss sum, which I briefly mentioned (in “Review of Lectures – XXXV”, page 12). I will state a general result (of Gauss) later, but specifically for the case of \( r = 5 \), it is

\[
\zeta_5 + \zeta_5^4 = \frac{-1 + \sqrt{5}}{2}.
\]

Since \( \zeta_5 \) and \( \zeta_5^4 \) are mutually conjugate (part (3) of ‘Facts’ on page 20), this identity reads

\[
\text{Re} \zeta_5 = \frac{-1 + \sqrt{5}}{4},
\]

Since \( \left| \zeta_5 \right| = 1 \) (part (2) of ‘Facts’ on page 20), the imaginary part of \( \zeta_5 \) is calculated as

\[
\text{Im} \zeta_5 = \pm \sqrt{1 - \left( \frac{-1 + \sqrt{5}}{4} \right)^2}
\]

\[
= \pm \frac{\sqrt{10 + 2\sqrt{5}}}{4}.
\]

Here, \( \text{Im} \zeta_5 = \sin \frac{2\pi}{5} \) is positive. Hence \( \text{Im} \zeta_5 = \frac{\sqrt{10 + 2\sqrt{5}}}{4} \).

As for factorizations, we have

\[
1 + z + z^2 + z^3 + z^4 = (z - \zeta_5)(z - \zeta_5^2)(z - \zeta_5^3)(z - \zeta_5^4),
\]

\[
z^5 - 1 = (z - 1)(z - \zeta_5)(z - \zeta_5^2)(z - \zeta_5^3)(z - \zeta_5^4).
\]
Example. For \( r = 8 \), we have

\[
\begin{align*}
\zeta_8 &= \frac{1 + \sqrt{-1}}{\sqrt{2}}, & \zeta_8^2 &= \sqrt{-1}, \\
\zeta_8^3 &= \frac{-1 + \sqrt{-1}}{\sqrt{2}}, & \zeta_8^4 &= -1, \\
\zeta_8^5 &= \frac{-1 - \sqrt{-1}}{\sqrt{2}}, & \zeta_8^6 &= -\sqrt{-1}, \\
\zeta_8^7 &= \frac{1 - \sqrt{-1}}{\sqrt{2}}.
\end{align*}
\]

Indeed, as stated above, \( \zeta_8 \) is a root of the equation

\[ 1 - z^8 = 0. \]

This factors as

\[
\left( 1 - z^4 \right) \left( 1 + z^4 \right) = 0.
\]

\( z = \zeta_8 \) is a root of this equation. Hence \( \left( 1 - \zeta_8^4 \right) \left( 1 + \zeta_8^4 \right) = 0. \) Since \( \zeta_8^4 \) equals \( \zeta_2 = -1, \) the first factor \( \left( 1 - \zeta_8^4 \right) \) equals \( 1 + 1 = 2. \) Hence

\[
1 + \zeta_8^4 = 0.
\]

In other words, \( z = \zeta_8 \) is a root of \( 1 + z^4 = 0. \) Here, let’s artificially write \( 1 + z^4 \) as

\[
1 + z^4 = \left( 1 + 2 z^2 + z^4 \right) - 2 z^2
\]

\[
= \left( 1 + z^2 \right)^2 - \left( \sqrt{2} z \right)^2 = \left( z^2 - \sqrt{2} z + 1 \right) \left( z^2 + \sqrt{2} z + 1 \right).
\]
Hence \( z = \zeta_8 \) is a root of
\[
\left(z^2 - \sqrt{2}z + 1\right)\left(z^2 + \sqrt{2}z + 1\right) = 0.
\]
So \( z = \zeta_8 \) is a root of either
\[
z^2 - \sqrt{2}z + 1 = 0, \quad \text{or} \quad z^2 + \sqrt{2}z + 1 = 0.
\]
Solve these two quadratic equations using quadratic formula:
\[
z = \frac{1 \pm \sqrt{-1}}{\sqrt{2}}, \quad \text{and} \quad z = \frac{-1 \pm \sqrt{-1}}{\sqrt{2}}.
\]
\( \zeta_8 \) is one of these four roots. To identify \( \zeta_8 \), the real and the imaginary parts of \( \zeta_8 \), namely, \( \cos \frac{\pi}{4} \) and \( \sin \frac{\pi}{4} \), are both positive. The only number among the above four satisfying this property is \( \frac{1 + \sqrt{-1}}{\sqrt{2}} \). Hence \( \zeta_8 = \frac{1 + \sqrt{-1}}{\sqrt{2}} \).

As for factorizations, we have
\[
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7
\]
\[
= (z - \zeta_8)(z - \zeta_8^2)(z - \zeta_8^3)(z + 1)(z - \zeta_8^4)(z + \sqrt{-1})(z - \zeta_8^7)
\]
\[
= (z - \zeta_8)(z - \zeta_8^2)(z - \zeta_8^3)(z - \zeta_8^4)(z - \zeta_8^5)(z - \zeta_8^6)(z - \zeta_8^7),
\]
\[
z^8 - 1 = (z - 1)(z - \zeta_8)(z - \zeta_8^2)(z - \zeta_8^3)(z + 1)(z - \zeta_8^4)(z + \sqrt{-1})(z - \zeta_8^7)
\]
\[
= (z - 1)(z - \zeta_8)(z - \zeta_8^2)(z - \zeta_8^3)(z - \zeta_8^4)(z - \zeta_8^5)(z - \zeta_8^6)(z - \zeta_8^7).
\]
• Gauss Sum.

As I announced, I will state the result concerning the Gauss sum, without giving a proof, and only for the case \( r \) is a prime number. (The same result for the case \( r \) is a non-prime is technical.) Here, let me recap the definition of a prime number:

**Definition (Prime numbers).**

A prime number \( p \) is an integer \( p \geq 2 \) such that there is no pair of integers \( a, b \) with \( a \geq 2, b \geq 2 \) such that \( ab \) equals \( p \).

Thus,

\[
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, \ldots
\]

are prime numbers, whereas

\[
1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 49, 50, 51, 52, \ldots
\]

are not prime numbers.

**Definition (Congruence of integers).**

Let \( a, b \) and \( r \) be integers, \( r \geq 2 \). We say \( a \) is congruent to \( b \) modulo \( r \), if \( a - b \) is divisible by \( r \).

5 is congruent to 1 modulo 4, because \( 5 - 1 = 4 \) is divisible by 4.

8 is congruent to 1 modulo 7, because \( 8 - 1 = 7 \) is divisible by 7.

29 is congruent to 5 modulo 8, because \( 29 - 5 = 24 \) is divisible by 8.

32 is not congruent to 2 modulo 4, because \( 32 - 2 = 30 \) is not divisible by 4.

15 is not congruent to 1 modulo 6, because \( 15 - 1 = 14 \) is not divisible by 6.
Definition (Gauss sum). Let $p$ be a prime number. Assume $p \geq 3$. Let $j$ be an integer, such that $1 \leq j \leq p - 1$. Define $\chi_p(j)$ as follows:

$$
\chi_p(j) = \begin{cases} 
1 & (j \text{ is congruent to } k^2 \text{ modulo } p \text{ for some integer } k), \\
-1 & (\text{otherwise}).
\end{cases}
$$

(This $\chi_p$ is called the Legendre’s character). Thus

- $\chi_3(1) = 1. \quad \chi_3(2) = -1.$
- $\chi_5(1) = 1. \quad \chi_5(2) = -1. \quad \chi_5(3) = -1. \quad \chi_5(4) = 1.$
- $\chi_7(1) = 1. \quad \chi_7(2) = 1. \quad \chi_7(3) = -1. \quad \chi_7(4) = 1. \quad \chi_7(5) = -1. \quad \chi_7(6) = -1.$
- $\chi_{11}(1) = 1. \quad \chi_{11}(2) = -1. \quad \chi_{11}(3) = 1. \quad \chi_{11}(4) = 1. \quad \chi_{11}(5) = 1. \quad \chi_{11}(6) = -1. \quad \chi_{11}(7) = -1. \quad \chi_{11}(8) = -1. \quad \chi_{11}(9) = 1. \quad \chi_{11}(10) = -1.$

Though we will not prove it, it turns out that the number of $j$ with $1 \leq j \leq p - 1$ and such that $\chi_p(j)$ equals 1 is exactly one half of $p - 1$.

Using $\chi_p(j)$, we define the Gauss sum as

$$
\sum_{j=1}^{p-1} \chi_p(j) \zeta_p^j.
$$
Theorem (Gauss Sum). Let $p$ be a prime number. Assume $p \geq 3$.

(1) Suppose $p$ is congruent to 1 modulo 4:

$$p = 5, 13, 17, 29, 37, 41, 53, 57, \ldots.$$  

Then

$$\sum_{j=1}^{p-1} \chi_p(j) \zeta_j^p = \sqrt{p}.$$  

(2) Suppose $p$ is congruent to 3 modulo 4:

$$p = 3, 7, 11, 19, 23, 31, 43, 47, \ldots.$$  

Then

$$\sum_{j=1}^{p-1} \chi_p(j) \zeta_j^p = \sqrt{-p}.$$  

Corollary (Gauss Sum). Let $p$ be a prime number. Assume $p \geq 3$.

(1) Suppose $p$ is congruent to 1 modulo 4. Then

$$\sum_{j=1}^{\frac{p-1}{2}} \zeta_j^2 = \frac{-1 + \sqrt{p}}{2}.$$  

(2) Suppose $p$ is congruent to 3 modulo 4. Then

$$\sum_{j=1}^{\frac{p-1}{2}} \zeta_j^2 = \frac{-1 + \sqrt{-p}}{2}.$$
To spell out the content of Theorem for small \( p \):

\[
\zeta_3 - \zeta_3^2 = \sqrt{-3}.
\]

\[
\zeta_5 - \zeta_5^2 - \zeta_5^3 + \zeta_5^4 = \sqrt{5}.
\]

\[
\zeta_7 + \zeta_7^2 - \zeta_7^3 + \zeta_7^4 - \zeta_7^5 - \zeta_7^6 = \sqrt{-7}.
\]

\[
\zeta_{11} - \zeta_{11}^2 + \zeta_{11}^3 + \zeta_{11}^4 - \zeta_{11}^5 - \zeta_{11}^6 - \zeta_{11}^7 - \zeta_{11}^8 + \zeta_{11}^9 - \zeta_{11}^{10} = \sqrt{-11}.
\]

\[
\zeta_{13} - \zeta_{13}^2 + \zeta_{13}^3 + \zeta_{13}^4 - \zeta_{13}^5 - \zeta_{13}^6 - \zeta_{13}^7 - \zeta_{13}^8 + \zeta_{13}^9 + \zeta_{13}^{10} - \zeta_{13}^{11} + \zeta_{13}^{12} = \sqrt{13}.
\]

\[
\zeta_{17} + \zeta_{17}^2 - \zeta_{17}^3 + \zeta_{17}^4 - \zeta_{17}^5 - \zeta_{17}^6 - \zeta_{17}^7 + \zeta_{17}^8 + \zeta_{17}^9 - \zeta_{17}^{10} - \zeta_{17}^{11} - \zeta_{17}^{12} + \zeta_{17}^{13} - \zeta_{17}^{14} + \zeta_{17}^{15} + \zeta_{17}^{16} = \sqrt{17}.
\]

\[
\zeta_3 = \frac{-1 + \sqrt{-3}}{2}.
\]

\[
\zeta_5 + \zeta_5^4 = \frac{-1 + \sqrt{5}}{2}.
\]

\[
\zeta_7 + \zeta_7^2 + \zeta_7^4 = \frac{-1 + \sqrt{-7}}{2}.
\]

\[
\zeta_{11} + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9 = \frac{-1 + \sqrt{-11}}{2}.
\]

\[
\zeta_{13} + \zeta_{13}^3 + \zeta_{13}^4 + \zeta_{13}^5 + \zeta_{13}^9 + \zeta_{13}^{10} + \zeta_{13}^{12} = \frac{-1 + \sqrt{13}}{2}.
\]

\[
\zeta_{17} + \zeta_{17}^2 + \zeta_{17}^4 + \zeta_{17}^8 + \zeta_{17}^9 + \zeta_{17}^{13} + \zeta_{17}^{15} + \zeta_{17}^{16} = \frac{-1 + \sqrt{17}}{2}.
\]
• Let’s go back to Example in page 9. What I said earlier is

\[ f(x) = x^5 + 1250x^3 + 7500x^2 + 25000x + 18750 \]

factors as

\[
\left( x + 1 + \left( 1 - 2^{\frac{1}{5}} \right)^5 \right) \\
\quad \cdot \left( x^4 - \left( 5 \cdot 2^{\frac{2}{5}} - 10 \cdot 2^{\frac{3}{5}} + 10 \cdot 2^{\frac{4}{5}} - 5 \cdot 2^{\frac{1}{5}} \right) x^3 \\
\quad + \left( 200 \cdot 2^{\frac{4}{5}} - 50 \cdot 2^{\frac{6}{5}} - 175 \cdot 2^{\frac{7}{5}} + 400 \cdot 2^{\frac{1}{5}} + 750 \right) x^2 \\
\quad + \left( 1750 \cdot 2^{\frac{8}{5}} + 625 \cdot 2^{\frac{9}{5}} - 1000 \cdot 2^{\frac{10}{5}} + 500 \cdot 2^{\frac{7}{5}} + 3000 \right) x \\
\quad + \left( 3125 \cdot 2^{\frac{13}{5}} + 2500 \cdot 2^{\frac{14}{5}} + 1250 \cdot 2^{\frac{16}{5}} + 2500 \cdot 2^{\frac{18}{5}} + 5000 \right) \right).
\]

Within \( \mathbb{C} \), this factors into linear pieces as follows:

\[
\left( x + 1 + \left( 1 - 2^{\frac{1}{5}} \right)^5 \right) \\
\quad \cdot \left( x + 1 + \left( 1 - \zeta_5 \cdot 2^{\frac{1}{5}} \right)^5 \right) \\
\quad \cdot \left( x + 1 + \left( 1 - \zeta_5^2 \cdot 2^{\frac{1}{5}} \right)^5 \right) \\
\quad \cdot \left( x + 1 + \left( 1 - \zeta_5^3 \cdot 2^{\frac{1}{5}} \right)^5 \right) \\
\quad \cdot \left( x + 1 + \left( 1 - \zeta_5^4 \cdot 2^{\frac{1}{5}} \right)^5 \right).
\]

If you say this kind of a factorization is always feasible for any quintic, unfortunately, that is not the case. As I emphasized earlier, this is a very special quintic. So I throw another example:
• Let’s go back to Example in page 10. What I said earlier is

\[ f(x) = x^5 + x + 1 \]

factors as

\[ (x - b) \left( x^4 + bx^3 + b^2 x^2 + b^3 x + b^4 + 1 \right), \]

where \( b \) is the unique real root of the equation \( f(x) = 0 \). There is no way to concretely write out \( b \) using integers and radicals.

Now, the question is, can we factor the part

\[ \left( x^4 + bx^3 + b^2 x^2 + b^3 x + b^4 + 1 \right) \]

into linear pieces, using \( b \)? Or, equivalently, can you write out the roots of

\[ x^4 + bx^3 + b^2 x^2 + b^3 x + b^4 + 1 = 0 \]

concretely using \( b \), and radicals? In theory, yes you can, because this is a quartic (= degree 4) polynomial, so Ferrari’s formula (which I briefly mentioned in “Review of Lectures - XXXV”, page 8) is applicable. However, if you physically perform it, the answer is probably too complicated. On the other hand, you might suggest that by Factorization Theorem over \( \mathbb{R} \) this quartic polynomial factors as

\[ (x^2 + p_1 x + q_1) (x^2 + p_2 x + q_2) \]

using some \( p_1, q_1, p_2, q_2 \in \mathbb{R} \), so one may expand this and equate this with \( x^4 + bx^3 + b^2 x^2 + b^3 x + b^4 + 1 \), and obtain a system of equations

\[
\begin{cases}
  p_1 + p_2 = b, \\
  p_1 p_2 + q_1 + q_2 = b^2, \\
  p_1 q_2 + q_1 p_2 = b^3, \\
  q_1 q_2 = b^4 + 1.
\end{cases}
\]

Good point. You can eliminate \( p_1, p_2 \) and \( q_2 \) and get a single equation on \( q_1 \). However, it is a sextic (= degree 6) and I don’t see a reason one can solve it for \( q_1 \) and write out each of \( p_1, q_1, p_2 \) and \( q_2 \) in terms of \( b \) in a simple fashion.
• **Definition (Multiplicity of a real root of an equation over \( \mathbb{R} \)).**

Back to Factorization Theorem over \( \mathbb{R} \) and its paraphrase (page 16 and page 18). In those statements, the positive integer \( \mu_j \) is called the multiplicity of the real root \( x = s_j \) of the equation

\[
ad x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 = 0.
\]

So, this is simply the multiplicity of the root \( x = s_j \) regarded as a complex root.

**Remark.** The following is also self-evident, but important. In the statement of Factorization Theorem over \( \mathbb{R} \) and its paraphrase (page 16 and page 18),

\[
\sum_{j=1}^{m} \mu_j \leq d.
\]

To phrase it verbally:

"The sum of the multiplicities of the real roots of an algebraic equation over \( \mathbb{R} \) is less than or equal to the degree of that equation.""

• **How do you visualize all this? The case over \( \mathbb{R} \).**

Geometrically, in \( \mathbb{R}^2 \), which I want to call \( \mathbb{A}^2 \), with the \((x, y)\)-coordinate system, the graph

\[
C = \left\{ (x, y) \in \mathbb{A}^2 \mid y = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \right\}
\]

\((a_d \neq 0)\) and the \(x\)-axis intersect at \(m\) distinct points, where \(m \leq d\). The sum of the multiplicities at those \(m\) points is less than or equal to \(d\). Let me repeat this, so as to make it a little more impressionable:
In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph ' $y = \left[ \text{a degree } d \text{ polynomial in } x \right]' intersects with the $x$-axis at at most $d$ number of points, even if you count the multiplicity.

Let’s take a look at the examples below:

**Example** (degree $d = 2$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^2 - 1$$

intersects with the $x$-axis at two distinct points. The multiplicity of each intersection point is 1. The total multiplicity (intersection number) $= 1 + 1 = 2 \ (= d)$.

**Example** (degree $d = 2$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^2$$

intersects with the $x$-axis at a single point. The multiplicity of that intersection point is 2. The total multiplicity (intersection number) $= 2 \ (= d)$.

**Example** (degree $d = 2$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^2 + 1$$

does not intersect with the $x$-axis. The total multiplicity (intersection number) $= 0 \ (< d)$.
Example (degree $d = 3$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

\[ y = x^3 - x \]

intersects with the $x$-axis at three distinct points. (To see this, simply factor $x^3 - x$.) The multiplicity of each intersection point is 1. (This is also from the factorization of $x^3 - x$.) The total multiplicity (intersection number) $= 1 + 1 + 1 = 3 \ (= d)$.

Example (degree $d = 3$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

\[ y = x^3 \]

intersects with the $x$-axis at a single point. The multiplicity of that intersection point is 3. The total multiplicity (intersection number) $= 3 \ (= d)$.

Example (degree $d = 3$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

\[ y = x^3 + x^2 \]

intersects with the $x$-axis at two distinct points. (To see this, simply factor $x^3 + x^2$.) The multiplicity of one intersection point is 2, and the multiplicity of the other intersection point is 1. (This is also from the factorization of $x^3 + x^2$.) The total multiplicity (intersection number) $= 2 + 1 = 3 \ (= d)$.

Example (degree $d = 3$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

\[ y = x^3 + x - 2 \]

intersects with the $x$-axis at a single point. (To see this, simply differentiate $x^3 + x - 2$.) The multiplicity of that intersection point is 1. The total multiplicity (intersection number) $= 1 \ (< d)$. 

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Example (degree $d = 4$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^4 - 2x^2 + 1$$

intersects with the $x$-axis at two distinct points. (To see this, simply factor $x^4 - 2x^2 + 1$.) The multiplicity of each of the intersection points is 2. The total multiplicity (intersection number) $= 2 + 2 = 4$ ($= d$).

Example (degree $d = 4$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^4 - x^3$$

intersects with the $x$-axis at two distinct points. (To see this, simply factor $x^4 - x^3$.) The multiplicity of one of the intersection points is 1, the other is 3. The total multiplicity (intersection number) $= 1 + 3 = 4$ ($= d$).

Example (degree $d = 4$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^4$$

intersects with the $x$-axis at a single point. The multiplicity of that intersection point is 4. The total multiplicity (intersection number) $= 4$ ($= d$).

Example (degree $d = 4$). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^4 - 1$$

intersects with the $x$-axis at two distinct points. (To see this, simply factor $x^4 - 1$.) The multiplicity of each of the intersection points is 1. The total multiplicity (intersection number) $= 1 + 1 = 2$ ($< d$).
Example (degree = 4). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^4 + x^2$$

intersects with the $x$-axis at a single point. (To see this, simply factor $x^4 + x^2$.) The multiplicity of that intersection point is 2. The total multiplicity (intersection number) = 2 ($< d$).

Example (degree = 4). In $\mathbb{A}^2_{(x,y)}$ over $\mathbb{R}$, the graph

$$y = x^4 + 1$$

does not intersect with the $x$-axis. The total multiplicity (intersection number) = 0 ($< d$).

• In each of the above thirteen examples, the total intersection number of the graph $y = f(x)$ and the $x$-axis either equals $d$ or strictly less than $d$, where $d$ is the degree of $f(x)$. This is consistent with what Corollary in page 16 (‘Factorization Theorem over $\mathbb{R}$’) states. On the other hand, in five out of those thirteen, the total intersection number of the graph $y = f(x)$ and the $x$-axis is strictly less than $d$. The following is important:

• An important idea.

"There is one school of thought that advocates the idea one should look at those intersections at ‘non-real’ points, because there indeed are intersections at non-real points in such a case, and if you count those then the total intersection number exactly equals $d$."

This is a shift of paradigm. This way of thinking potentially opens up the door to the entirely new world of geometry, that is, geometry over the complex number field $\mathbb{C}$, don’t you feel? Why can’t we adopt $\mathbb{C}$ as an underlying number system, instead of $\mathbb{R}$, and that way we can develop a more organized geometry?"
• What about over \( \mathbb{C} \)?

Being consistent with that philosophy, we look at what the Factorization Theorem over \( \mathbb{C} \) (in page 14) states one more time: The total intersection number of the graph \( y = f(x) \) and the \( x \)-axis always equals \( d \), no matter what \( f(x) \) is, as long as it is a polynomial of degree \( d \). If you want to write it down using mathematical symbols, that’s already in ‘Remark’ in page 15: Under the same notation as the Factorization Theorem over \( \mathbb{C} \) (in page 14),

\[
\sum_{j=1}^{r} \mu_j = d.
\]

To phrase it verbally:

“\begin{center}
The sum of the multiplicities of the complex roots of an algebraic equation over \( \mathbb{C} \) is exactly equal to the degree of that equation.\end{center}”

• How do you visualize all this? The case over \( \mathbb{C} \).

Geometrically, in \( \mathbb{C}^2 \), which I want to call \( \mathbb{A}^2 \), with the \( (x,y) \)-coordinate system,* the graph

\[
C = \left\{ \left( x, y \right) \in \mathbb{A}^2 \left| y = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \cdots + \alpha_1 x + \alpha_0 \right. \right\}
\]

and the \( x \)-axis intersect at \( r \) distinct points, and the sum of the multiplicities at those \( r \) points equals \( d \). To make it a little more impressionable:

‘Visualization’. “\begin{center}
In \( \mathbb{A}^2_{(x,y)} \) over \( \mathbb{C} \), the graph \text{‘} y = \left[ a \text{ degree } d \text{ polynomial in } x \right] \text{’} \text{ intersects with the } x\text{-axis at precisely } d \text{ number of points, if you count the multiplicity. } \end{center}”

---

*I no longer stick with \( z \) for the complex variable. Here \( x \) and \( y \) signify complex variables. 41
• FAQ.

Here, a frequently asked question: How can you possibly ‘visualize’ $\mathbb{C}^2$? Am I not forcing you to see something which one cannot see? Good point. Since $\mathbb{C}$ itself is $\mathbb{R}^2$, and so then $\mathbb{C}^2$ is going to be $\mathbb{R}^4$, so this is four dimensional. One cannot visualize 4-D. Right?

If you say so, yes, you have a valid point. However, remember, earlier in the semester we dealt with some 4-dimensional object residing in the 5-dimensional space. The necessity naturally arose from studying the lines and planes in the 3-dimensional space. In mathematics, the common wisdom is that nothing stops us from considering higher dimensional objects. Later today we deal with complex analytic functions, something more general than $f(z) = \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0$. There we will learn how to visualize the graph of a complex variable function. The above statement, that the total intersection number of the curve “$y = a$ degree $d$ polynomial in $x$” with the $x$-axis exactly equals $d$ is indeed the ‘prototype’ of Bézout’s theorem, which we will address next.

• More benefits of Fundamental Theorem of Algebra.

Now I want to incorporate the perspective of the materials covered in the first half of the semester. Earlier I said there are easily a few dozens of ‘benefits’ Fundamental Theorem of Algebra brings us. ‘Factorization Theorem’ is one, as we have just seen. Now I would like to add the following to the list:

○ [A] **No endless repeat of enlargement of the number system.**

Thanks to Fundamental Theorem of Algebra, there is a guarantee any polynomial factors into linear ones, namely, polynomials of the form $(z - \alpha)$, strictly within $\mathbb{C}$. So we do not have to further keep enlarging the number systems starting with $\mathbb{C}$. Without Fundamental Theorem of Algebra, you need to worry that you would have to keep enlarging the number system endlessly.*

○ [B] **Eigenvalues exist in $\mathbb{C}$ for matrices over $\mathbb{R}$ and over $\mathbb{C}$.**

*There is a notion called ‘the algebraic closure’. If you accept the existence of the algebraic closure of $F$ for any field $F$ (which is standad material in abstract algebra, requires to assume ‘the axiom of choice’), then one may apply it to the case $F = \mathbb{R}$ and obtain a field which satisfies the content of Fundamental Theorem of Algebra. In that sense the potential concern expressed in this paragraph about the necessity to keep extending the number system endlessly is actually futile. However, *a priori* $\mathbb{C}$ may not be the algebraic closure of $\mathbb{R}$. *A posteriori* $\mathbb{C}$ is indeed the algebraic closure of $\mathbb{R}$, and that is exactly the content of (a paraphrase of) Fundamental Theorem of Algebra.*
Square matrices with real or complex number entries all have eigenvalues within \( \mathbb{C} \). (However, Abel’s theorem still applies here. Namely, one may or may not be able to concretely express the eigenvalues and entries of eigenvectors using the entries of the original matrix.) Proving some basic facts about matrices with entries in \( \mathbb{R} \), stating which does not require to refer to \( \mathbb{C} \), has to rely on \( \mathbb{C} \). The only proof that I know of of the fact below uses the decomposition of the complex vectorspace \( \mathbb{C}^n \) into eigenspaces of the matrix \( A \), and a kind of inner product (\( = \) dot product) of vectors called ‘Hermitian inner product’, which involves complex conjugates:

**Fact (on symmetric matrices over \( \mathbb{R} \)).**

A real symmetric matrix \( A \) is diagonalized by a real orthogonal matrix. To be precise: Let \( A \) be an arbitrary \( n \times n \) matrix having entries in \( \mathbb{R} \) which is symmetric, namely, \( A = A^T \). Then there exists a matrix \( P \) with entries in \( \mathbb{R} \) which is orthogonal, namely, \( PP^T = I \), such that \( PAP^T \) is a diagonal matrix.

\( \circ \) [C] **Real points vs. complex points.**

As I already briefly touched in “Review of Lectures – XXXI”, in projective geometry over \( \mathbb{R} \), we had to deal with the subtlety pertaining to the notion of ‘enough real points’. Thanks to the fundamental theorem of algebra, we don’t have to deal with the same difficulty over \( \mathbb{C} \). Any ‘complex algebraic variety’ (non-singular ones are called ‘complex algebraic manifolds’) defined by a system of algebraic equations inside the projective space over the complex numbers always has ‘enough complex points’. This is a blessing. The concept of ‘enough real points’ itself was difficult to digest in the first place. This all disappears. The following exemplifies this:

\( \circ \) [D] **Bézout’s theorem (Intersection theory).**

There is a theorem called Bézout’s theorem. This is a geometric generalization of Fundamental Theorem of Algebra. Bézout’s theorem can be thought of as a 2-dimensional analog of Fundamental Theorem of Algebra, where Fundamental Theorem of Algebra is thought of as the 1-dimensional counterpart of Bézout’s theorem. This is the focus of our next lecture.

- We are getting close to the proof Fundamental Theorem of Algebra. We need one more ingredient. We need to cover some rudiments of analytic functions from complex analysis. So, let’s switch gears.

- **Rudiments of complex analytic functions.**
As a starter, let’s define the exponential function (exponential mapping), as a complex variable function (mapping from \( \mathbb{C} \) to \( \mathbb{C} \)):

**Definition (Exponential mapping).**

(a) Recall (from “Review of Lectures – XXXIII”, page 9) that, for a real number \( \theta \), we have defined

\[
\exp\left(\sqrt{-1}\theta\right) \quad \text{or alternatively,} \quad e^{\sqrt{-1}\theta},
\]

to mean the complex number \( \cos \theta + \sqrt{-1} \sin \theta \). Thus

\[
\exp\left(\sqrt{-1}\theta\right) = e^{\sqrt{-1}\theta} = \cos \theta + \sqrt{-1} \sin \theta.
\]

(b) More generally, for \( z \in \mathbb{C} \), define ‘exp \( z \)’, or alternatively, \( e^{z} \), as follows: First, write \( z \) as \( z = t + \sqrt{-1} \theta \) \((t, \theta \in \mathbb{R})\). Then define

\[
\exp z = e^z = e^t \cdot \left(\cos \theta + \sqrt{-1} \sin \theta\right).
\]

This is natural in view of (a), indeed, using the notation in (a),

\[
e^{t+\sqrt{-1}\theta} = e^t \cdot e^{\sqrt{-1}\theta}.
\]

Agree that, (a) is nothing but the special case \( t = 0 \) in (b). This way we have created a ‘function’, or a ‘mapping’

\[
\exp : \mathbb{C} \longrightarrow \mathbb{C}; \quad \exp z = e^z.
\]

**Formula (The Exponential Laws).**
(E1) \( \exp 0 = 1. \)

(E2) \( \exp (z + w) = (\exp z)(\exp w). \)

(E3) \( \exp z \neq 0, \) and \( \exp (-z) = (\exp z)^{-1}. \)

(E4) The image of \( \exp : \mathbb{C} \rightarrow \mathbb{C} \) as a mapping equals

\[
\mathbb{C} \setminus \{0\} = \left\{ z \in \mathbb{C} \mid z \neq 0 \right\}.
\]

So, for an arbitrary \( w \in \mathbb{C}; \ w \neq 0, \) there exists \( z \in \mathbb{C} \) such that \( w = \exp z. \)

- I want you to completely digest the following proof:

**Proof.** Proving (E1) is straightforward. Indeed, \( 0 = 0 + \sqrt{-1} \cdot 0, \) thus

\[
\exp 0 = e^0 \cdot \left( \cos 0 + \sqrt{-1} \sin 0 \right)
= 1 \cdot \left( 1 + \sqrt{-1} \cdot 0 \right)
= 1.
\]

As for (E2), write complex numbers \( z \) and \( w \) as

\[
z = t + \sqrt{-1} \theta, \quad w = s + \sqrt{-1} \phi,
\]

where \( t, s, \theta, \phi \in \mathbb{R}. \) Then \( z + w = (t + s) + \sqrt{-1} \left( \theta + \phi \right), \) and hence

\[
\exp (z + w) = e^{t+s} \cdot \left( \cos \left( \theta + \phi \right) + \sqrt{-1} \sin \left( \theta + \phi \right) \right)
\]
\[
\begin{align*}
&e^t \cdot e^s \cdot \left( \left[ \cos \theta \cos \phi - \sin \theta \sin \phi \right] + \sqrt{-1} \left[ \sin \theta \cos \phi + \cos \theta \sin \phi \right] \right) \\
&= e^t \cdot e^s \cdot \left( \cos \theta + \sqrt{-1} \sin \theta \right) \left( \cos \phi + \sqrt{-1} \sin \phi \right) \\
&= e^t \left( \cos \theta + \sqrt{-1} \sin \theta \right) \cdot e^s \left( \cos \phi + \sqrt{-1} \sin \phi \right) \\
&= \left( \exp z \right) \left( \exp w \right).
\end{align*}
\]

As for (E3), we utilize (E2) which we have just proved. In (E2):

\[
\exp \left( z + w \right) = \left( \exp z \right) \left( \exp w \right),
\]

set \( w = -z \). Then

\[
\exp \left( z + (-z) \right) = \left( \exp z \right) \left( \exp (-z) \right).
\]

The left-hand side is simplified as \( \exp 0 \), which, according to (E1), equals 1. Hence

\[
1 = \left( \exp z \right) \left( \exp (-z) \right).
\]

This reads that the two complex numbers \( \exp z \) and \( \exp (-z) \) are reciprocals of each other. In particular, \( \exp z \neq 0 \). This proves (E3).

Finally, as for (E4), let \( w \in \mathbb{C}; \ w \neq 0 \) be arbitrary. Consider
\[ u = \frac{1}{|w|} w. \]

This \( u \) satisfies \(|u| = 1\). In other words, \( \text{Re} \ u \) and \( \text{Im} \ u \) satisfy

\[
\left( \text{Re} \ u \right)^2 + \left( \text{Im} \ u \right)^2 = 1.
\]

It is a common knowledge from Calculus that there exists \( \theta \in \mathbb{R} \) such that

\[
\text{Re} \ u = \cos \theta, \quad \text{and} \quad \text{Im} \ u = \sin \theta.
\]

Thus \( u = \cos \theta + \sqrt{-1} \sin \theta \). Recall that \( u \) stood for \( \frac{1}{|w|} w \), so

\[
\frac{1}{|w|} w = \cos \theta + \sqrt{-1} \sin \theta.
\]

Multiply \(|w|\) to the both sides:

\[
w = |w| \left( \cos \theta + \sqrt{-1} \sin \theta \right).
\]

Let \( \log : \left\{ r \in \mathbb{R} \mid r > 0 \right\} \longrightarrow \mathbb{R} \) be the natural logarithm. Let

\[
t = \log |w|.
\]

Then \( e^t = |w| \). Using this \( t \) and the above \( \theta \), we have

\[
\exp \left( t + \sqrt{-1} \theta \right) = e^t \cdot \left( \cos \theta + \sqrt{-1} \sin \theta \right) = |w| \left( \cos \theta + \sqrt{-1} \sin \theta \right) = w.
\]

In short, \( z = t + \sqrt{-1} \theta \) satisfies \( w = \exp z \). This proves (E4). \( \square \)

- **Note.** In the above, (E1), (E2) and (E3) are familiar looking. Within calculus,

\[
e^0 = 1,
\]
\[
\begin{cases}
    e^{t+s} = e^t \cdot e^s, \\
    e^t \neq 0, \quad \text{and} \quad e^{-t} = \frac{1}{e^t}
\end{cases}
\]
are the common knowledge. (E1), (E2) and (E3) assert that the complex variables counterpart of these are still valid.

On the other hand, (E4) is somewhat counter-intuitive (unless you are already familiar with the subject). You are familiar with the fact that for an arbitrary \( r \in \mathbb{R} \), with \( r > 0 \), there exists \( t \in \mathbb{R} \) such that \( r = e^t \). This \( t \) is unique. Meanwhile, (E4) reads as follows: For an arbitrary \( w \in \mathbb{C} \), with \( w \neq 0 \), there exists \( z \in \mathbb{C} \) such that \( w = e^z \). I want to stress that this \( z \) is not unique. As for how close or afar it is from being unique, see Corollary 3 below.

**Corollary 1.**

1. \( \exp \left( z + \sqrt{-1} \cdot 2\pi \right) = \exp z \).
2. \( \exp \left( z + \sqrt{-1} \pi \right) = -\exp z \).
3. \( \exp \left( z + \sqrt{-1} \cdot \frac{\pi}{2} \right) = \sqrt{-1} \cdot \exp z \).
4. More generally, for arbitrary integers \( k \) and \( r \) with \( r \geq 2 \),

\[
\exp \left( z + \sqrt{-1} \cdot \frac{2k \pi}{r} \right) = \zeta_r^k \cdot \exp z.
\]

**Corollary 2 (Periodicity).** Let \( m \) be an arbitrary integer. Then

\[
\exp \left( z + \sqrt{-1} \left( 2\pi m \right) \right) = \exp z.
\]

**Corollary 3.** Let \( z, w \in \mathbb{C} \) be arbitrary such that \( \exp z = \exp w \).
Then there exists an integer $m$ such that

$$w = z + \sqrt{-1} \left(2\pi m\right).$$

**Proof.** It suffices to prove the following: For $z \in \mathbb{C}$, if $\exp z = 1$, then $z$ is an integer multiple of $\sqrt{-1} \cdot 2\pi$. So, suppose $\exp z = 1$. Write $z$ as $t + \sqrt{-1} \theta$; $t, \theta \in \mathbb{R}$. Then

$$e^t \left(\cos \theta + \sqrt{-1} \sin \theta\right) = 1.$$

Take the imaginary part of the both sides and conclude that $\theta$ is an integer multiple of $\pi$. If $\theta$ is an odd integer multiple of $\pi$, then the real part of the left-hand side of the above identity is negative, hence it cannot equal 1. Hence $\theta$ is an even integer multiple of $\pi$, or the same to say, $\theta$ is an integer multiple of $2\pi$. It also follows from the above identity that $e^t = 1$. Hence $t = 0$. In short, $z = t + \sqrt{-1} \theta$ is an integer multiple of $\sqrt{-1} \cdot 2\pi$. □

**• Exponential function as a mapping between $\mathbb{R}^2$.**

As I said earlier, I identify $\mathbb{C}$ with $\mathbb{R}^2$. Thus I want to freely go from a complex number $z = x + \sqrt{-1} y$ ($x, y \in \mathbb{R}$) to a point $(x, y)$ in $\mathbb{R}^2$, and vice versa, back and forth. Keeping that scope intact, let’s rewrite the exponential function $f(z) = \exp z$ defined earlier as a mapping between $\mathbb{R}^2$. Namely,

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad f \left( x, y \right) = \left( e^x \cos y, e^x \sin y \right).$$

So, this $f$ takes a point in $\mathbb{R}^2$ into another point in $\mathbb{R}^2$. We call this mapping the exponential mapping. Proof of the following is an easy exercise, though important:

**Basic Fact 1.** Let $f = \exp : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the exponential mapping. Then
\[
\begin{bmatrix}
\frac{\partial}{\partial x} f_1(x, y) & \frac{\partial}{\partial y} f_1(x, y) \\
\frac{\partial}{\partial x} f_2(x, y) & \frac{\partial}{\partial y} f_2(x, y)
\end{bmatrix}
= \begin{bmatrix}
e^x \cos y & -e^x \sin y \\
e^x \sin y & e^x \cos y
\end{bmatrix}.
\]

Here, I have combined the four partial derivatives in the form of a single $2 \times 2$ matrix, on purpose. Realize that the outcome matrix is of the form
\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}.
\]

This is important: This indicates that the exponential mapping is inseparably linked with the structure of $\mathbb{C}$.

**Exercise.** Let $f$ be the exponential mapping, as above. Let $c \in \mathbb{R}$ be a (constant) real number. Let $L_c$ be the straight line parallel to the $y$-axis:
\[
\ell = \left\{ (c, y) \in \mathbb{R}^2 \mid y \in \mathbb{R} \right\}.
\]
Prove that the image of $\ell$ under $f$ is the circle centered at the origin with radius $e^c$.

- **Cauchy–Riemann’s Equations.**

More generally, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a mapping. We may write
\[
g(x, y) = \left( g_1(x, y), g_2(x, y) \right).
\]
For example, for the exponential mapping in the previous page,
\[
g_1(x, y) = e^x \cos y, \quad \text{and} \quad g_2(x, y) = e^x \sin y.
\]
Assume that the derivatives $\frac{\partial g_1}{\partial x}$, $\frac{\partial g_2}{\partial x}$, $\frac{\partial g_1}{\partial y}$, $\frac{\partial g_2}{\partial y}$ all exist.
The following pair of equations is called the **Cauchy–Riemann’s equations**:

\[
\frac{\partial g_1}{\partial x} = \frac{\partial g_2}{\partial y}, \quad \text{and} \quad \frac{\partial g_2}{\partial x} + \frac{\partial g_1}{\partial y} = 0
\]

**Definition.** A **conformal mapping** is a mapping \( g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \),

\[
g(x, y) = \left( g_1(x, y), g_2(x, y) \right),
\]

which satisfies the Cauchy–Riemann’s equations.

**An important observation.**

Suppose you write the four partial derivatives in the matrix form:

\[
J g(x, y) = \begin{bmatrix}
\frac{\partial}{\partial x} g_1(x, y) & \frac{\partial}{\partial y} g_1(x, y) \\
\frac{\partial}{\partial x} g_2(x, y) & \frac{\partial}{\partial y} g_2(x, y)
\end{bmatrix},
\]

call it the **Jacobian matrix** of \( g \). Then \( g \) satisfying the Cauchy–Riemann’s equations are equivalent to \( J g \) being of the form:

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}.
\]

Please agree on this. So, the exponential mapping is an example of a conformal mapping. The exponential mapping is indeed a representative example of a conformal mapping. Now, there is another representative series of examples of conformal mappings.

**Example.** The identity mapping
\( \iota : \mathbb{R}^2 \longrightarrow \mathbb{R}^2; \quad \iota (x, y) = (x, y), \)

is a conformal mapping. In this case, the Jacobian matrix of \( \iota \) is the identity matrix

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

Note that, once moved back to \( \mathbb{C} \) from \( \mathbb{R}^2 \), then \( \iota \) as complex variable function is nothing else but

\( f(z) = z. \)

Note that this is an example of a polynomial mapping.

**Example.** Let \( a, b \in \mathbb{R} \) be constants. Then the constant mapping

\( \text{const}_{(a,b)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2; \quad \text{const}_{(a,b)} (x, y) = (a, b), \)

is a conformal mapping. In this case, the Jacobian matrix of \( \text{const}_{(a,b)} \) is the zero matrix

\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

Note that, once moved back to \( \mathbb{C} \) from \( \mathbb{R}^2 \), then \( \text{const}_{(a,b)} \) as complex variable function is nothing else but the constant function

\( f(z) = a + \sqrt{-1} b. \)

- As trivial as it may seem, the above examples actually play some important role, once combined with other facts we are going to establish one by one.

**Definition** (Addition and multiplication of mappings \( \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \)).
Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$;

\[
\begin{align*}
g(x, y) &= \left( g_1(x, y), g_2(x, y) \right), \\
h(x, y) &= \left( h_1(x, y), h_2(x, y) \right),
\end{align*}
\]

be two mappings. Define

\[
g + h : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{and} \quad gh : \mathbb{R}^2 \rightarrow \mathbb{R}^2
\]
as

\[
(g + h)(x, y) = \left( g_1(x, y) + h_1(x, y), \; g_2(x, y) + h_2(x, y) \right),
\]

and

\[
(gh)(x, y) = \left( g_1(x, y) h_1(x, y) - g_2(x, y) h_2(x, y), \; g_1(x, y) h_2(x, y) + g_2(x, y) h_1(x, y) \right),
\]

respectively. Note that, once moved back to $\mathbb{C}$ from $\mathbb{R}^2$, and regard $g$ and $h$ as complex variable functions $g(z)$ and $h(z)$, the above is nothing but the usual addition and multiplication of $g$ and $h$:

\[
g(z) + h(z), \quad \text{and} \quad g(z) h(z),
\]
in $\mathbb{C}$.

**Important Fact.**
Suppose $g$ and $h$ are both conformal. Then $g + h$ and $gh$ are both conformal.

**Proof.** The statement for the addition $g + h$ is self-evident. Thus we prove the statement for the multiplication $gh$. First,

$$\frac{\partial}{\partial x} \left( g_1 h_1 - g_2 h_2 \right)$$

$$= \frac{\partial}{\partial x} \left( g_1 h_1 \right) - \frac{\partial}{\partial x} \left( g_2 h_2 \right)$$

$$= \left( \left( \frac{\partial g_1}{\partial x} \right) h_1 + g_1 \left( \frac{\partial h_1}{\partial x} \right) \right) - \left( \left( \frac{\partial g_2}{\partial x} \right) h_2 + g_2 \left( \frac{\partial h_2}{\partial x} \right) \right)$$

$$= \left( \left( \frac{\partial g_2}{\partial y} \right) h_1 + g_1 \left( \frac{\partial h_2}{\partial y} \right) \right) + \left( \left( \frac{\partial g_1}{\partial y} \right) h_2 + g_2 \left( \frac{\partial h_1}{\partial y} \right) \right)$$

$$= \left( \left( \frac{\partial g_1}{\partial y} \right) h_2 + g_1 \left( \frac{\partial h_2}{\partial y} \right) \right) + \left( \left( \frac{\partial g_2}{\partial y} \right) h_1 + g_2 \left( \frac{\partial h_1}{\partial y} \right) \right)$$

$$= \frac{\partial}{\partial y} \left( g_1 h_2 \right) + \frac{\partial}{\partial y} \left( g_2 h_1 \right)$$

$$= \frac{\partial}{\partial y} \left( g_1 h_2 + g_2 h_1 \right).$$

In short,

$$\frac{\partial}{\partial x} \left( g_1 h_1 - g_2 h_2 \right) = \frac{\partial}{\partial y} \left( g_1 h_2 + g_2 h_1 \right).$$

Second,
\[
\frac{\partial}{\partial x} \left( g_1 h_2 + g_2 h_1 \right) \\
= \frac{\partial}{\partial x} \left( g_1 h_2 \right) + \frac{\partial}{\partial x} \left( g_2 h_1 \right)
\]
\[
= \left( \left( \frac{\partial g_1}{\partial x} \right) h_2 + g_1 \left( \frac{\partial h_2}{\partial x} \right) \right) + \left( \left( \frac{\partial g_2}{\partial x} \right) h_1 + g_2 \left( \frac{\partial h_1}{\partial x} \right) \right)
\]
\[
= \left( \left( \frac{\partial g_2}{\partial y} \right) h_2 - g_1 \left( \frac{\partial h_1}{\partial y} \right) \right) + \left( - \left( \frac{\partial g_1}{\partial y} \right) h_1 + g_2 \left( \frac{\partial h_2}{\partial y} \right) \right)
\]
\[
= \left( \left( \frac{\partial g_1}{\partial y} \right) h_1 + g_1 \left( \frac{\partial h_1}{\partial y} \right) \right) + \left( \left( \frac{\partial g_2}{\partial y} \right) h_2 + g_2 \left( \frac{\partial h_2}{\partial y} \right) \right)
\]
\[
= - \frac{\partial}{\partial y} \left( g_1 h_1 \right) + \frac{\partial}{\partial y} \left( g_2 h_2 \right)
\]
\[
= - \frac{\partial}{\partial y} \left( g_1 h_1 - g_2 h_2 \right).
\]

In short,
\[
\frac{\partial}{\partial x} \left( g_1 h_2 + g_2 h_1 \right) = - \frac{\partial}{\partial y} \left( g_1 h_1 - g_2 h_2 \right).
\]

These show that the mapping \( g h \) is conformal. \( \square \)

- This has an important consequence.

**Important Corollary** (Polynomial mappings are conformal).
Let \( \alpha_d, \alpha_{d-1}, \cdots, \alpha_1, \alpha_0 \in \mathbb{C} \) be complex number constants. Let \( f : \mathbb{C} \longrightarrow \mathbb{C} \) be defined as
\[
f(z) = \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0.
\]
Then the same \( f \), regarded as a mapping from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), is conformal.

- So, in short:

\[
\text{“ A mapping } \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \text{ which arises from a polynomial with complex number coefficients is conformal. ”}
\]

Another important fact (Composite of conformal mappings).

Let \( g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \) and \( h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \);
\[
\begin{align*}
g(x, y) &= \left( g_1(x, y), g_2(x, y) \right), \\
h(x, y) &= \left( h_1(x, y), h_2(x, y) \right),
\end{align*}
\]
be two mappings. Assume both are conformal. Then their composite
\[
(h \circ g)(x, y) = \left( h_1 \left( g_1(x, y), g_2(x, y) \right), \; h_2 \left( g_1(x, y), g_2(x, y) \right) \right)
\]
is conformal.

**Proof.** By Chain Rule (from multi-variable calculus; Math 223 in our listings),
\[ J(h \circ g) = (Jh) (Jg) \]

(where the right-hand side is the matrix multiplication). Thus the assertion is a direct consequence of the fact that matrices of the form \[
\begin{bmatrix}
  a & -b \\
  b & a
\end{bmatrix}
\]
are closed under multiplication. □

- Next, let’s see examples of mappings that are not conformal.

**Example.** Let
\[ \text{conj} : \mathbb{R}^2 \rightarrow \mathbb{R}^2; \quad \text{conj} (x, y) = (x, -y). \]

This is not a conformal mapping. In this case, the Jacobian matrix ofconj is
\[
\begin{bmatrix}
  1 & 0 \\
  0 & -1
\end{bmatrix}.
\]

Note that, once moved back to \( \mathbb{C} \) from \( \mathbb{R}^2 \), conj as complex variable function is nothing else but \( f(z) = \bar{z} \).

**Example.** Let
\[ \text{abs} : \mathbb{R}^2 \rightarrow \mathbb{R}^2; \quad \text{abs} (x, y) = \left( \sqrt{x^2 + y^2}, 0 \right). \]

This is not a conformal mapping. In this case, the Jacobian matrix ofabs is
\[
\begin{bmatrix}
  x / \sqrt{x^2 + y^2} & y / \sqrt{x^2 + y^2} \\
  0 & 0
\end{bmatrix}.
\]

Note that, once moved back to \( \mathbb{C} \) from \( \mathbb{R}^2 \), abs as complex variable function is nothing else but \( f(z) = |z| \).

- **Analytic Functions.** Derivatives.
Let \( g : \mathbb{C} \longrightarrow \mathbb{C} \) be a complex variable function. We say \( g = g(z) \) is \textit{analytic}, whenever \( g \), regarded as a mapping \( g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \), is conformal.

Let's write \( g(z) \) using two real valued functions \( g_1(x, y) \) and \( g_2(x, y) \), as

\[
g(z) = g_1(x, y) + \sqrt{-1} g_2(x, y),
\]

where \( x = \text{Re} \, z \), \( y = \text{Im} \, z \), as before. Using \( g_1 \) and \( g_2 \), we define the derivative

\[
\frac{dg}{dz} = \frac{\partial g_1}{\partial x} + \sqrt{-1} \frac{\partial g_2}{\partial x}.
\]

This is natural. Indeed, remember that we used to identify the matrix

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} \quad (a, b \in \mathbb{R})
\]

with the complex number \( a + \sqrt{-1} b \). The above definition of the derivative of \( g(z) \) conforms to it, namely, the derivative \( \frac{dg}{dz} \) is the Jacobian matrix

\[
J g(x, y) = \begin{bmatrix}
\frac{\partial}{\partial x} g_1(x, y) & \frac{\partial}{\partial y} g_1(x, y) \\
\frac{\partial}{\partial x} g_2(x, y) & \frac{\partial}{\partial y} g_2(x, y)
\end{bmatrix}
\]

converted into a complex number, under such an identification. Now, this Jacobian matrix is indeed of the form

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} \quad (a, b \in \mathbb{R})
\]

under the assumption that \( g \) is conformal, by virtue of the Cauchy–Riemann’s equations. More on this definition below.

- **Local nature of the notion of analyticity.**
By the way, \( g(z) \) being analytic is a ‘local’ concept.* One can talk about \( g(z) \) being analytic at \( z = z_0 \). This leads us to the definition of an analytic function over an ‘open set’ \( U \) of \( \mathbb{C} \). Here, in case you have not taken a class on introductory analysis (a 500-level course in our course listings) or alternatively a course on general topology (we don’t seem to have an introductory course on general topology): Think of the notion of open sets the following way:

First, agree that when I talk about open sets of \( \mathbb{C} \), I am identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \). Next, a representative example of an open set of \( \mathbb{C} = \mathbb{R}^2 \) is an open disc

\[
\left\{ (x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < r^2 \right\}
\]

where \( a, b, r \in \mathbb{R} \). The center of this disc is clearly \( (a, b) \). The radius of this disc is clearly \( r \). By the way, utilizing the complex variable notations, the same disc is written as

\[
\left\{ z \in \mathbb{C} \mid |z - \alpha| < r \right\}.
\]

Here, I wrote \( a + \sqrt{-1} b \) as \( \alpha \), which is then the center of the disc. Finally, a general open set is a union (possibly infinitely many) of open discs inside \( \mathbb{C} \).

- **More examples of open sets of \( \mathbb{C} \).**

\[
(1) \quad \left\{ (x, y) \in \mathbb{R}^2 \mid x > a \right\} \quad (a \in \mathbb{R}),
\]

or the same

\[
\left\{ z \in \mathbb{C} \mid \text{Re } z > a \right\} \quad (a \in \mathbb{R}).
\]

\[
(1)' \quad \text{In (1), change ‘>’ with ‘<’}.
\]

*Similarly, \( g(x, y) \) being conformal, for that matter, is a ‘local’ concept. From now on we mainly talk about analytic functions, not conformal mappings, though logically they are the same concept.
\[(2) \quad \left\{ (x, y) \in \mathbb{R}^2 \mid x > b \right\} \quad (b \in \mathbb{R}), \]

or the same
\[
\left\{ z \in \mathbb{C} \mid \text{Im} \ z > b \right\} \quad (b \in \mathbb{R}).
\]

\[(2)' \quad \text{In (2), change '}>' \text{ with '}<'.} \]

\[(3) \quad \left\{ (x, y) \in \mathbb{R}^2 \mid a < x < b, \ c < y < d \right\} \quad (a, b, c, d \in \mathbb{R}), \]

or the same
\[
\left\{ z \in \mathbb{C} \mid a < \text{Re} \ z < b, \ c < \text{Im} \ z < d \right\} \quad (a, b, c, d \in \mathbb{R})
\]
\[\quad \text{(the interior of a rectangular).} \]

\[(4) \quad \text{The complement of finitely many points inside } \mathbb{R}^2. \]

\[(5) \quad \text{The complement of}
\[
\left\{ (m, n) \in \mathbb{R}^2 \mid m, n \text{ are integers } \right\},
\]
\[\quad \text{inside } \mathbb{R}^2, \text{ or the same}
\[
\left\{ m + \sqrt{-1} \ n \in \mathbb{C} \mid m, n \text{ are integers } \right\},
\]
\[\quad \text{inside } \mathbb{C}. \]

\[(6) \quad \text{The complement of} \]
\( \{ (m, 0) \in \mathbb{R}^2 \mid m \text{ is an integer} \} \),
inside \( \mathbb{R}^2 \), or the same
\( \{ m \in \mathbb{C} \mid m \text{ is an integer} \} \),
inside \( \mathbb{C} \).

(7) The complement of the half-line
\( \{ (x, 0) \in \mathbb{R}^2 \mid x \leq 0 \} \),
inside \( \mathbb{R}^2 \), or the same
\( \{ z \in \mathbb{C} \mid \text{Re } z \leq 0, \text{ Im } z = 0 \} \),
inside \( \mathbb{C} \).

(7)
In (7), replace the given half-line with any other half-line inside \( \mathbb{R}^2 = \mathbb{C} \).

• Finally, most importantly:

(i) the union of either finitely many or infinitely many open sets inside \( \mathbb{C} \) is an open set in \( \mathbb{C} \), and

(ii) the intersection of finitely many open sets inside \( \mathbb{C} \) is an open set of \( \mathbb{C} \).

On the other hand:

**Caution.** The intersection of infinitely many open sets of \( \mathbb{C} \) is not necessarily an open set of \( \mathbb{C} \).

That’s all about open sets of \( \mathbb{C} \). So, from now on, when I say something is an analytic function, it does not have to be defined over the entire \( \mathbb{C} \), but it has to be
defined over some (non-empty) open set $U$ of $\mathbb{C}$. In what follows, you will typically see

“let $g : U \rightarrow \mathbb{C}$ be an analytic function”,

or alternatively

“let $g(z)$ be an analytic function defined over $U$”,

where $U$ is a specified open set of $\mathbb{C}$.

• Concerning the above definition of $\frac{dg}{dz}$, the following hold:

**Important Facts about derivatives.**

(1) Suppose $g$ is analytic over an open set $U \subseteq \mathbb{C}$. Then $\frac{dg}{dz}$ is analytic over the same open set $U \subseteq \mathbb{C}$.

(2) $\frac{dg}{dz}(z_0) = \lim_{\eta \to 0} \frac{g(z_0 + \eta) - g(z_0)}{\eta}$.

Here, note that $\eta$ underneath the limit symbol is a complex variable. Thus, the fraction inside the limit symbol is the division of a complex number by a complex number.

This is not to be confused with $\frac{g(z_0 + \eta) - g(z)}{|\eta|}$.

(3) Conversely, suppose the limit in (2) exists at each $z_0$ on an open set $U \subset \mathbb{C}$. Then $g$ is analytic over $U$.

(4a) $\frac{dz}{dz} = 1$.  
(4b) $\frac{d\alpha}{dz} = 0 \quad (\alpha \in \mathbb{C} \text{ is a constant})$.

• The above statement (1) may look innocuous. However, this is a very strong
fact. If you say you can prove the statement (1) easily from the Cauchy–Riemann’s equations, as in

$$\frac{\partial}{\partial x} \left( \frac{\partial g_1}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial g_2}{\partial y} \right) \quad \text{(by Cauchy–Riemann)}$$

$$= \frac{\partial}{\partial y} \left( \frac{\partial g_2}{\partial x} \right),$$

and

$$\frac{\partial}{\partial x} \left( \frac{\partial g_2}{\partial x} \right) = -\frac{\partial}{\partial x} \left( \frac{\partial g_1}{\partial y} \right) \quad \text{(by Cauchy–Riemann)}$$

$$= -\frac{\partial}{\partial y} \left( \frac{\partial g_1}{\partial x} \right),$$

that’s only ‘half-right’. Namely, in calculus with real variables, if $g(x)$ has a derivative $g'(x)$, that does not guarantee that $g'(x)$ again has a derivative. The statement (1) is inferring that, if $g(z)$ is analytic over $U$, then there is a guarantee that the second partial derivatives

$$\frac{\partial^2 g_1}{\partial x^2}, \quad \frac{\partial^2 g_1}{\partial x \partial y}, \quad \frac{\partial^2 g_1}{\partial y^2},$$

$$\frac{\partial^2 g_2}{\partial x^2}, \quad \frac{\partial^2 g_2}{\partial x \partial y}, \quad \frac{\partial^2 g_2}{\partial y^2}$$

all exist, and furthermore they satisfy an appropriate set of equations which turns out to be the Cauchy–Riemann’s equations for $\frac{dg}{dz}$. This is not at all self-evident.

We will not prove this. A course on complex variables (the 600 level in our listings) typically covers it. I will give a reference (at the end of this present notes, page **).

- The exponential function (again).
Below has already been (essentially) proved.

**Fact.** Let

\[ f = \exp : \mathbb{C} \longrightarrow \mathbb{C}; \quad f(z) = \exp z = e^z \]

be the exponential function. Then

1. \( f(z) \) is analytic over \( \mathbb{C} \).
2. \[ \frac{d}{dz} \exp z = \exp z, \quad \text{or the same} \quad \frac{d}{dz} e^z = e^z. \]

- **Break-up rule, Leibniz rule.**

Let \( U \subseteq \mathbb{C} \) be an open set. Let \( g \) and \( h \) be both analytic functions over \( U \). Then

1. \( g + h \) is an analytic function over \( U \). Moreover, the ‘break-up rule’ holds:

\[ \frac{d}{dz} (g + h) = \frac{dg}{dz} + \frac{dh}{dz}. \]

2. \( g h \) is an analytic function over \( U \). Moreover, ‘Leibniz rule’ holds:

\[ \frac{d}{dz} (g h) = \frac{dg}{dz} \cdot h + g \cdot \frac{dh}{dz}. \]

**Proof.** Assume both \( g \) and \( h \) are analytic over \( U \). The analyticity of \( g + h \) and
$g$ and $h$ over $U$ have already been proved. Let’s write $g(z)$ and $h(z)$ using real valued functions $g_1(x, y), g_2(x, y), h_1(x, y)$ and $h_2(x, y)$, as

$$g(z) = g_1(x, y) + \sqrt{-1} g_2(x, y),$$

$$h(z) = h_1(x, y) + \sqrt{-1} h_2(x, y),$$

where $x = \text{Re } z$, $y = \text{Im } z$. Then the assertions immediately follow from

$$(1) \quad \frac{\partial}{\partial x} \left( \begin{bmatrix} g_1(x, y) - g_2(x, y) \\ g_2(x, y) - g_1(x, y) \end{bmatrix} + \begin{bmatrix} h_1(x, y) - h_2(x, y) \\ h_2(x, y) - h_1(x, y) \end{bmatrix} \right)$$

$$= \frac{\partial}{\partial x} \left( \begin{bmatrix} g_1(x, y) - g_2(x, y) \\ g_2(x, y) - g_1(x, y) \end{bmatrix} \right) + \frac{\partial}{\partial x} \left( \begin{bmatrix} h_1(x, y) - h_2(x, y) \\ h_2(x, y) - h_1(x, y) \end{bmatrix} \right),$$

and

$$(2) \quad \frac{\partial}{\partial x} \left( \begin{bmatrix} g_1(x, y) - g_2(x, y) \\ g_2(x, y) - g_1(x, y) \end{bmatrix} \right) \cdot \left( \begin{bmatrix} h_1(x, y) - h_2(x, y) \\ h_2(x, y) - h_1(x, y) \end{bmatrix} \right)$$

$$= \left( \frac{\partial}{\partial x} \begin{bmatrix} g_1(x, y) - g_2(x, y) \\ g_2(x, y) - g_1(x, y) \end{bmatrix} \right) \cdot \begin{bmatrix} h_1(x, y) - h_2(x, y) \\ h_2(x, y) - h_1(x, y) \end{bmatrix}$$

$$+ \begin{bmatrix} g_1(x, y) - g_2(x, y) \\ g_2(x, y) - g_1(x, y) \end{bmatrix} \cdot \left( \frac{\partial}{\partial x} \begin{bmatrix} h_1(x, y) - h_2(x, y) \\ h_2(x, y) - h_1(x, y) \end{bmatrix} \right). \quad \Box$$

Important Corollary (Polynomial mappings).
Let \( \alpha_d, \alpha_{d-1}, \cdots, \alpha_1, \alpha_0 \in \mathbb{C} \) be complex number constants. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be defined as

\[
f(z) = \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0.
\]

Then \( f(z) \) is an analytic function over the entire \( \mathbb{C} \), and moreover,

\[
\frac{df}{dz} = d\alpha_d z^{d-1} + (d-1)\alpha_{d-1}z^{d-2} + \cdots + 2\alpha_2 z + \alpha_1.
\]

• In other words, you can differentiate a polynomial function on \( z \) (with coefficients in \( \mathbb{C} \)) ‘as usual’, meaning you can pretend that the coefficients are real and just apply the familiar differentiation rule of polynomials with real variables.

**Quotient rule – I.**

Let \( U \subseteq \mathbb{C} \) be an open set. Let \( g \) be an analytic function over \( U \). Then the following hold:

1. Define a subset \( V \) of \( U \) as follows:
   \[
   V = \left\{ z \in U \mid g(z) \neq 0 \right\}.
   \]

Then \( V \) is also an open subset of \( \mathbb{C} \).

2. \( \frac{1}{g} \) is an analytic function over \( V \).

3. (Quotient rule – I): \( \frac{d}{dz} \left( \frac{1}{g} \right) = -\frac{1}{g^2} \cdot \frac{dg}{dz} \).

**Proof.** We will not prove the assertion (1). See Remark below.

As for (2), recall that for a complex number \( \alpha; \alpha \neq 0, \)
Thus

\[ \frac{1}{\alpha} = \frac{1}{|\alpha|^2} \bar{\alpha}. \]

Let's write \( g(z) \) using real valued functions \( g_1(x, y) \) and \( g_2(x, y) \), as

\[ g(z) = g_1(x, y) + \sqrt{-1} g_2(x, y), \]

where \( x = \text{Re } z \), \( y = \text{Im } z \). Then

\[ \frac{1}{g} = \frac{g_1}{g_1^2 + g_2^2} - \sqrt{-1} \frac{g_2}{g_1^2 + g_2^2}. \]

First,

\[ \frac{\partial}{\partial x} \frac{g_1}{g_1^2 + g_2^2} = \frac{\partial g_1}{\partial x} \cdot \left( g_1^2 + g_2^2 \right) - g_1 \frac{\partial}{\partial x} \left( g_1^2 + g_2^2 \right) \]

\[ \left( g_1^2 + g_2^2 \right)^2 \]

\[ = \left( g_1^2 + g_2^2 \right) \cdot \frac{\partial g_1}{\partial x} - 2 g_1^2 \frac{\partial g_1}{\partial x} - 2 g_1 g_2 \frac{\partial g_2}{\partial x} \]

\[ \left( g_1^2 + g_2^2 \right)^2 \]

\[ = \left( - g_1^2 + g_2^2 \right) \cdot \frac{\partial g_1}{\partial x} - 2 g_1 g_2 \frac{\partial g_2}{\partial x} \]

\[ \left( g_1^2 + g_2^2 \right)^2 \].

In short,
\[ (\ast) \quad \frac{\partial}{\partial x} \frac{g_1}{g_1^2 + g_2^2} = \frac{\left( -g_1^2 + g_2^2 \right) \cdot \frac{\partial g_1}{\partial x} - 2 g_1 g_2 \frac{\partial g_2}{\partial x}}{(g_1^2 + g_2^2)^2} \]

In (\ast), switch the role of \( x \) and \( y \):

\[ (** \ast) \quad \frac{\partial}{\partial y} \frac{g_1}{g_1^2 + g_2^2} = \frac{\left( -g_1^2 + g_2^2 \right) \cdot \frac{\partial g_1}{\partial y} - 2 g_1 g_2 \frac{\partial g_2}{\partial y}}{(g_1^2 + g_2^2)^2} \]

In (\ast), switch \( g_1 \) and \( g_2 \), and then negate:

\[ (# \ast) \quad \frac{\partial}{\partial x} \frac{-g_2}{g_1^2 + g_2^2} = \frac{2 g_1 g_2 \frac{\partial g_1}{\partial x} + \left( -g_1^2 + g_2^2 \right) \cdot \frac{\partial g_2}{\partial x}}{(g_1^2 + g_2^2)^2} \]

In (#), switch the role of \( x \) and \( y \):

\[ (## \ast) \quad \frac{\partial}{\partial y} \frac{-g_2}{g_1^2 + g_2^2} = \frac{2 g_1 g_2 \frac{\partial g_1}{\partial y} + \left( -g_1^2 + g_2^2 \right) \cdot \frac{\partial g_2}{\partial y}}{(g_1^2 + g_2^2)^2} \]

If you take into account the Cauchy–Riemann’s equations for \( g \):

\[ \frac{\partial g_1}{\partial x} = \frac{\partial g_2}{\partial y}, \quad \text{and} \quad \frac{\partial g_2}{\partial x} + \frac{\partial g_1}{\partial y} = 0 \]

then clearly “(\ast) = (##\ast)” and “(**) + (#) = 0”. These are nothing else but the Cauchy–Riemann’s equations for

\[ \frac{1}{g} = \frac{g_1}{g_1^2 + g_2^2} - \sqrt{-1} \frac{g_2}{g_1^2 + g_2^2}. \]

This proves that \( \frac{g_1}{g_1^2 + g_2^2} - \sqrt{-1} \frac{g_2}{g_1^2 + g_2^2} \) is conformal, or, equivalently, that \( \frac{1}{g} \) is analytic.

Finally, as for (3), \( \frac{d}{dz} \left( \frac{1}{g} \right) \) clearly equals “(\ast) + \sqrt{-1} (##)”, namely,
\[
\frac{\left(-g_1^2 + g_2^2\right) \cdot \frac{\partial g_1}{\partial x} - 2g_1 g_2 \frac{\partial g_2}{\partial x}}{\left(g_1^2 + g_2^2\right)^2}
\]
\[
+ \sqrt{-1} \frac{2g_1 g_2 \frac{\partial g_1}{\partial x} + \left(-g_1^2 + g_2^2\right) \cdot \frac{\partial g_2}{\partial x}}{\left(g_1^2 + g_2^2\right)^2}
\]
\[
= \frac{\left(-g_1^2 + g_2^2 + \sqrt{-1} \cdot 2g_1 g_2\right) \cdot \frac{\partial g_1}{\partial x}}{\left(g_1^2 + g_2^2\right)^2}
\]
\[
+ \frac{\left(-2g_1 g_2 + \sqrt{-1} \cdot \left(-g_1^2 + g_2^2\right)\right) \cdot \frac{\partial g_2}{\partial x}}{\left(g_1^2 + g_2^2\right)^2}
\]
\[
= \frac{-g_1^2 + g_2^2 + \sqrt{-1} \cdot 2g_1 g_2}{\left(g_1^2 + g_2^2\right)^2} \cdot \left(\frac{\partial g_1}{\partial x} + \sqrt{-1} \frac{\partial g_2}{\partial x}\right)
\]
\[
= \frac{-\left(g_1 - \sqrt{-1} g_2\right)^2}{\left(g_1^2 + g_2^2\right)^2} \cdot \left(\frac{\partial g_1}{\partial x} + \sqrt{-1} \frac{\partial g_2}{\partial x}\right)
\]
\[
= -\frac{\overline{g}^2}{|g|^4} \cdot \frac{d g}{d z} = -1 \cdot \frac{1}{g^2} \cdot \frac{d g}{d z},
\]
as required. Now our proof of assertion (3) is complete. \(\square\)

**Definition (Zero locus).** Let \(g(z)\) be an analytic function over an open set \(U \subseteq \mathbb{C}\). The set of points
\[ Z(g) = \left\{ z \in U \mid g(z) = 0 \right\} \]

is called the zero locus of \( g \), or the zeroes of \( g \). The following is a consequence of Factorization Theorem:

• **Important Facts about the zeroes of a polynomial.**

  (1) For a polynomial function \( f(z) \), its zero locus \( Z(f) \) is finite.

  (2) Conversely, for any finite set of points \( S \subseteq \mathbb{C} \), there exists a polynomial function \( f(z) \) such that \( Z(f) \) equals \( S \).

• You may see the above (1) and (2) as trivial, and may not see their importance. The truth is, those encapsulate the most important underlying idea in algebraic geometry. You will see this more clearly as the semester progresses.

**Remark.** On the other hand, for a general analytic function \( g \) defined over an open set \( U \subseteq \mathbb{C} \) its zero locus may or may not be finite.

As I noted earlier, if \( U \subseteq \mathbb{C} \) is an open set, then the complement of a finite set inside \( U \) is still an open set. But you might worry that potentially one cannot discard the possibility that the complement of \( Z(g) \) inside \( U \) for an analytic function \( g(z) \) defined over \( U \) might not be an open set, if \( Z(g) \) is an infinite set. That is a legitimate concern. However, the assertion (1) of ‘Quotient Rule’ four pages ago rules out such a possibility. In fact, the following result holds, whose proof I will not cover in class. I refer to any complex analysis textbook.

**Isolatedness Theorem (of a zero of an analytic function).**

Let \( U \subseteq \mathbb{C} \) be an open set. Let \( g(z) \) be an analytic function defined over \( U \). Suppose \( z_0 \in Z(g) \). Then there exists a positive real number \( \varepsilon \) such that the open
disc centered at \( z_0 \) having radius \( \varepsilon \) does not contain any zero of \( g \) other than \( z_0 \) itself:

\[
\left\{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \right\} \cap Z(g) = \left\{ z_0 \right\}.
\]

- So, for example, let

\[
S = \left\{ \frac{1}{n} \in \mathbb{C} \mid n \text{ is an integer, } n \neq 0 \right\}
\]

\[
= \left\{ \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \pm \frac{1}{5}, \pm \frac{1}{6}, \cdots \right\}.
\]

Then you cannot create an example of an analytic function \( g \) defined over some open set \( U \subseteq \mathbb{C} \) that contains 0; \( U \ni 0 \), and such that \( Z(f) \) equals \( S \cap U \).

You can still create an example of an analytic function defined over the complement of 0 inside \( \mathbb{C} \) and whose zero locus equals the above \( S \). Here is what comes on top of my head:

\[
g(z) = \sin \frac{\pi}{z}.
\]

This \( g(z) \) cannot be extended as a function over \( \mathbb{C} \) which is analytic at \( z = 0 \). The same function \( g(z) \) as above is said to have an “essential singularity” at \( z = 0 \). As for the definition of \( \sin z \), see page 76.

- Similarly, you cannot create an example of an analytic function \( g \) defined over some open set \( U \) such that \( U \cap \mathbb{R} \neq \emptyset \) and such that \( Z(g) \) equals \( \mathbb{Q} \cap U \).

- **Prime (’ ) notation for the derivative.**

In what follows we use the prime notation for the derivative, just like what we
do in the case of real variable derivatives. So, the derivative \( \frac{dg}{dz} \) can at times be denoted as \( g' \), or \( g'(z) \), when no confusion is likely. Thus

\[
    g'(z) = \frac{dg}{dz}.
\]

The advantage of this notation is clear when you substitute the variable \( z \) with another quantity (such as a constant complex number or a quantity involving other variables) after taking the derivative. Namely, we may write

\[
    g'(0) \quad \text{for} \quad \left. \frac{dg}{dz} \right|_{z=0}, \quad g'(\alpha^2 + \beta^2) \quad \text{for} \quad \left. \frac{dg}{dz} \right|_{z=\alpha^2+\beta^2}, \text{ etc.}
\]

Also, the Leibniz rule is written simply as

\[
    (gh)' = g'h + gh'.
\]

• **Higher order derivatives (Notation).**

Furthermore, in what follows we use the notation \( g^{(m)}(z) \) to mean the \( m \)-th order derivative. Namely:

\[
    g^{(1)}(z) = \frac{d}{dz}g(z),
\]

\[
    g^{(2)}(z) = \frac{d}{dz} \left( \frac{d}{dz}g(z) \right),
\]

\[
    g^{(3)}(z) = \frac{d}{dz} \left( \frac{d}{dz} \left( \frac{d}{dz}g(z) \right) \right),
\]

\[
    g^{(4)}(z) = \frac{d}{dz} \left( \frac{d}{dz} \left( \frac{d}{dz} \left( \frac{d}{dz}g(z) \right) \right) \right), \quad \text{etc.}
\]

**Quotient rule – II.**

Let \( U \subseteq \mathbb{C} \) be an open set. Let \( g \) and \( h \) be analytic functions over \( U \).
Then the following hold:

(1) \( \frac{h}{g} \) is an analytic function over the complement of \( Z(g) \) inside \( U \).

(2) (Quotient rule – II): \( \frac{d}{dz} \left( \frac{h}{g} \right) = \frac{h' \cdot g - h \cdot g'}{g^2} \).

**Proof.** This is a mere combination of ‘Leibniz rule’ and ‘Quotient rule – I’. \( \square \)

**Example.**

(1) \( \frac{d}{dz} z^5 = 5z^4 \).

(2) \( \frac{d}{dz} \left( \sqrt{-1}z^3 - z^2 + \left(1 - \sqrt{-2}\right)z + \sqrt{3} \right) = \sqrt{-1} \cdot 3z^2 - 2z + \left(1 - \sqrt{-2}\right) \).

(3) \( \frac{d}{dz} \left( \left(z^2 + \sqrt{-1}\right) \cdot \left(\exp z\right) \right) = 2z \cdot \left(\exp z\right) + \left(z^2 + \sqrt{-1}\right) \cdot \left(\exp z\right) = \left(z^2 + 2z + \sqrt{-1}\right) \cdot \left(\exp z\right) \).

(4) \( \frac{d}{dz} \frac{1}{z^3 + z^2 + z + 1} = \frac{-1}{(z^3 + z^2 + z + 1)^2} \cdot \left(3z^2 + 2z + 1\right) \).

(5) \( \frac{d}{dz} \frac{z + \sqrt{-1}}{3z^2 + 1} = \frac{1 \cdot \left(3z^2 + 1\right) - \left(z + \sqrt{-1}\right) \left(6z\right)}{\left(3z^2 + 1\right)^2} \)

\[ = \frac{-3z^2 - \sqrt{-1} \cdot 6z + 1}{\left(3z^2 + 1\right)^2} \].

- **Chain Rule.** Let \( g(z) \) and \( h(w) \) be two analytic functions, defined over an
open set $U \subseteq \mathbb{C}$, and an open set $V \subseteq \mathbb{C}$, respectively. Suppose $g(U) \subseteq V$ (namely, the image of $U$ under $g$ is contained in $V$), so the composite of $g$ and $h$:

$$h \circ g : U \xrightarrow{g} V \xrightarrow{h} \mathbb{C}.$$ 

makes sense. Then $h \circ g : U \rightarrow \mathbb{C}$ is analytic, and moreover

$$\frac{d}{dz} (h \circ g)(z) = \left( \frac{dh}{dw} \right)_{w=g(z)} \cdot \left( \frac{dg}{dz} \right).$$

Or, in short,

$$\left( h \circ g \right)'(z) = h'(g(z)) \cdot g'(z).$$

**Proof.** This assertion has essentially been proved already when we have proved the fact that the composite of two conformal mappings is again conformal (in page 56–57). □

**Example.** Each of the following functions is analytic:

$$\exp(z^2), \quad \exp(\exp z) \quad \text{and} \quad \exp(z + z^{-1}).$$

Their derivatives:

$$\frac{d}{dz} \exp(z^2) = 2z \cdot \exp(z^2),$$

$$\frac{d}{dz} \exp(\exp z) = \left( \exp z \right) \cdot \exp(\exp z),$$

$$\frac{d}{dz} \exp(z + z^{-1}) = \left( 1 - z^{-2} \right) \cdot \exp(z + z^{-1}).$$

**Terminology (Rational function).**
Suppose $f(z)$ and $g(z)$ are both polynomials. Then the function $\frac{f(z)}{g(z)}$ is analytic over the complement of $Z(g)$ inside $\mathbb{C}$. It is called a **rational function** in $z$. Polynomials are clearly rational functions. Adding and multiplying two rational functions yield rational functions. Dividing one rational function by another yields a rational function. Examples of rational functions which are not polynomials:

\[
\frac{1}{z}, \quad \frac{\sqrt{-1} z}{z^2 + 3z}, \quad \frac{4z^3 + 4z}{z^4 - \sqrt{-1}z^3 - z^2 + \sqrt{-1}z + 1}, \quad \text{etc.}
\]

**Exercise.** First verify that each of the following functions on $z$ are analytic over an appropriate open set of $\mathbb{C}$. Then calculate the derivative.

\[
\begin{align*}
(1) & \quad z^4 - 2\sqrt{-1}z^2 + 1. \\
(2) & \quad z^3 \exp z. \\
(3) & \quad \exp z + \sqrt{-1} \exp (2z). \\
(4) & \quad \frac{z + \sqrt{-2}}{2\sqrt{-1}z^2 - \sqrt{2}}. \\
(5) & \quad \frac{\exp z - \exp (-z)}{\exp z + \exp (-z)}. \\
(6) & \quad \exp \frac{z + \sqrt{-1}}{z - \sqrt{-1}}.
\end{align*}
\]

[Answers]: Each of these functions (1) through (6) is a composite of rational functions and the exponential function, and hence is analytic. Their derivatives:

\[
\begin{align*}
(1) & \quad 4z^3 - 4\sqrt{-1}z. \\
(2) & \quad \left(z^3 + 3z^2\right) \exp z. \\
(3) & \quad \exp z + 2\sqrt{-1} \exp (2z). \\
(4) & \quad \frac{-2\sqrt{-1}z^2 + 4\sqrt{2}z - \sqrt{2}}{\left(2\sqrt{-1}z^2 - \sqrt{2}\right)^2}. \\
(5) & \quad \frac{4}{\left(\exp z - \exp (-z)\right)^2}. \\
(6) & \quad \frac{-2\sqrt{-1}}{\left(z - \sqrt{-1}\right)^2} \cdot \exp \frac{z + \sqrt{-1}}{z - \sqrt{-1}}.
\end{align*}
\]

**Taylor’s Theorem** (for Analytic Functions).
Let $g(z)$ be an analytic function defined over an open set $U$ of $\mathbb{C}$. Let $z_0 \in U$. Let $R$ be a positive real number satisfying the condition

$$D = \left\{ z \in \mathbb{C} \mid |z - z_0| < R \right\} \subseteq U.$$ 

Then there exists an infinite sequence $\{\alpha_n\}_{n=0,1,2,3,...}$ of complex numbers such that for an arbitrary $z \in D$ the value $g(z)$ is written as a convergent series

$$g(z) = \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n.$$ 

Moreover,

$$\alpha_n = \frac{g^{(n)}(z_0)}{n!}.$$ 

The above infinite series is called the Taylor series of $g(z)$ with center $z_0$.

- **Taylor series for the exponential function.**

  $$\exp z = e^z = \frac{1}{0!} + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots.$$ 

**Definition (Trigonometric Functions).** Define

$$\cos z = \frac{e^{i\sqrt{-1}z} + e^{-i\sqrt{-1}z}}{2},$$

$$\sin z = \frac{e^{i\sqrt{-1}z} - e^{-i\sqrt{-1}z}}{2i\sqrt{-1}}.$$ 

**Definition (Hyperbolic Trigonometric Functions).** Define
\[ \begin{align*}
\cosh z &= \frac{e^z + e^{-z}}{2}, \\
\sinh z &= \frac{e^z - e^{-z}}{2}.
\end{align*} \]

- **Taylor series for sin, cos, sinh, cosh.**

\[ \begin{align*}
\cos z &= \frac{1}{0!} - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 - \frac{1}{6!} z^6 + \frac{1}{8!} z^8 - \cdots, \\
\sin z &= \frac{1}{1!} z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \frac{1}{9!} z^9 - \cdots, \\
\cosh z &= \frac{1}{0!} + \frac{1}{2!} z^2 + \frac{1}{4!} z^4 + \frac{1}{6!} z^6 + \frac{1}{8!} z^8 + \cdots, \\
\sinh z &= \frac{1}{1!} z + \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \frac{1}{7!} z^7 + \frac{1}{9!} z^9 + \cdots.
\end{align*} \]

- The following are self-evident:

\[ e^{\sqrt{-1} z} = \left( \cos z \right) + \sqrt{-1} \left( \sin z \right) \quad \text{(Euler's formula)}, \]

\[ \cosh z = \cos \left( \sqrt{-1} z \right), \quad \sinh z = \frac{\sin \left( \sqrt{-1} z \right)}{\sqrt{-1}}. \]

- The following are also self-evident:

**Formula.**

\[ \frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z, \]

\[ \frac{d}{dz} \sinh z = \cosh z, \quad \frac{d}{dz} \cosh z = \sinh z. \]

- **Radius of convergence.**

Let \( \{ \alpha_n \}_{n=0,1,2,3,\ldots} \) be an infinite sequence of complex numbers. Consider
\[(\ast) \quad \sum_{n=0}^{\infty} \alpha_n \cdot (z - z_0)^n.\]

Then either one of the following holds:

(i) \((\ast)\) is convergent whenever \(|z - z_0| < R\), and moreover is divergent whenever \(|z - z_0| > R\), for some real number \(R > 0\),

(ii) \((\ast)\) is convergent for an arbitrary \(z \in \mathbb{C}\), or

(iii) \((\ast)\) is convergent for \(z = 0\) and is divergent otherwise.

**Definition (of radius of convergence).**

- If (i) occurs, then we say the radius of convergence for \((\ast)\) is \(R\).

- If (ii) occurs, then we say the radius of convergence for \((\ast)\) is infinity. Write \(R = +\infty\).

- If (iii) occurs, then we say the radius of convergence for \((\ast)\) is 0. Write \(R = 0\).

**Fact.** Let \(g(z)\) be an analytic function defined over an open set \(U \subseteq \mathbb{C}\). Let \(z_0 \in U\). Then the radius of convergence \(R\) for the Taylor series of \(g(z)\) with center \(z_0\) is positive, and it is bigger than or equal to any positive real number \(R'\) such that

\[
\left\{ z \in \mathbb{C} \mid |z - z_0| < R' \right\} \subseteq U.
\]

**Terminology (Entire Function).**

A function analytic over the entire \(\mathbb{C}\) is called an entire function.
Corollary. Let $g(z)$ be an entire function. Then the radius of convergence $R$ for the Taylor series of $g(z)$ with any center is $+\infty$.

Example. (1) Any polynomial function is an entire function.
(2) The exponential function $\exp z$ is an entire function.
(3) Each of $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, is an entire function.

- The following is self-evident:

Facts. (1) Adding and multiplying two entire functions yield entire functions.
(2) Let $g(z)$ be an entire function, which does not have any zeroes: $Z(g) = \emptyset$. Then $\frac{1}{g(z)}$ is also an entire function.
(3) Suppose
   (i) $g(z)$ is an entire function,
   (ii) $h(w)$ is an analytic function defined over an open set $U \subseteq \mathbb{C}$, and moreover
   (iii) $g(\mathbb{C}) \subseteq U$.

Thus the composite of $g$ and $h$:

$$h \circ g : \mathbb{C} \xrightarrow{g} U \xrightarrow{h} \mathbb{C}.$$  

makes sense. Then $h \circ g : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function.

(4) In particular, the composite of two entire functions is an entire function.
(5) The derivative of an entire function is an entire function.

Definition (Connected open subset of $\mathbb{C}$).

An open set $U \subseteq \mathbb{C}$ is said to be disconnected, if $U$ is written as a disjoint
union of two open subsets $U_1$ and $U_2$ of $\mathbb{C}$:

$$U = U_1 \cup U_2, \quad \text{and} \quad U_1 \cap U_2 = \emptyset.$$  

An open set $U \subseteq \mathbb{C}$ is said to be connected, if it is not disconnected.

**Examples.** $\mathbb{C}$ itself is a connected open subset of $\mathbb{C}$. Also, all examples in page 59–61 (all of (1) through (7); (1)', (2)' and (7)') are connected open subsets of $\mathbb{C}$.

On the other hand, choose any two open subsets of $\mathbb{C}$ that are disjoint. Then (regardless of whether each of them is connected or not) their union is an example of a disconnected open set of $\mathbb{C}$. So, for example, the union of two discs which do not overlap is an example of a disconnected open set of $\mathbb{C}$.

**• Coincidence Theorem and Analytic Continuation.**

Let me first state the ‘coincidence theorem’. This is essentially a consequence of the isolatedness of zeroes of analytic functions (page 71).

**Theorem (of coincidence).**

Let $f(z)$ and $g(z)$ be two analytic functions both defined over a connected open set $U \subseteq \mathbb{C}$. Let \( \{ \alpha_n \}_{n=0,1,2,3,\ldots} \) be an infinite sequence of complex numbers. Suppose

\begin{align*}
(i) \quad & \alpha_n \in U \quad (n = 0, 1, 2, 3, \ldots), \\
(ii) \quad & f(\alpha_n) = g(\alpha_n) \quad (n = 0, 1, 2, 3, \ldots), \quad \text{and moreover} \\
(iii) \quad & \lim_{n \to \infty} \alpha_n = z_0 \quad \text{exists, and } z_0 \in U.
\end{align*}

Then $f(z) = g(z)$ holds for an arbitrary $z \in U$.

**Corollary (Coincidence Theorem – II).**

Let $f(z)$ and $g(z)$ be two analytic functions both defined over an open set
$U \subseteq \mathbb{C}$. Suppose there exists another open set $V \subseteq \mathbb{C}$ which is smaller than $U$: $V \subseteq U$. Suppose $f(z) = g(z)$ holds for an arbitrary $z \in V$. Then $f(z) = g(z)$ holds for an arbitrary $z \in U$.

- **Analytic continuation (Idea).**

  The above leads to the notion of analytic continuation. Namely, suppose you have an analytic function $g(z)$ defined over a connected open set, say $D_0$. Let’s suppose $D_0$ is a disc. Choose $z_1 \in D_0$ and write $g(z)$ as a Taylor series with center $z = z_1$. Let $R$ be the radius of convergence for that Taylor series. So the Taylor series is convergent over the disc $D_1$ with center $z_1$ having radius $R$. Sometimes (not always) that disc $D_1$ is not entirely contained in $D_0$, namely, a portion of $D_1$ stretches outside of $D_0$. Then by coincidence theorem there exists a unique analytic function over the union of $D_0$ and $D_1$ which, over $D_0$, equals the original $g(z)$. In other words, this way you have ‘extended’ $g(z)$, which was originally an analytic function over $D_0$, to an analytic function defined over an enlarged open set $D_0 \cup D_1 \subseteq \mathbb{C}$.

  Now, you call that enlarged open set as $U_1$ and choose a new point $z_2$ in $U_1$ which is not in $D_0$. Sometimes (not always) a portion of the disc with center $z_2$ and with radius being equal to the radius of convergence for the Taylor series of $g(z)$, call it $D_2$, stretches outside of $U_1$. You call that enlarged open set as $U_2$ and choose a new point $z_2$ in $U_2$ which is not in $U_1$, and and so on so forth.

  Now, this way you obtain a sequence of discs $D_0$, $D_1$, $D_2$, $D_3$, · · ·. You might say that you keep doing it and eventually you get the largest connected open set $U \subseteq \mathbb{C}$ over which $g(z)$ is defined and is analytic. However, the reality is subtler than this. Namely, you cannot entirely dismiss the following possibility:

  Sometimes (not always), for some $n$, the intersection of $D_n$ with $U_n = D_0 \cup D_1 \cup D_2 \cup \cdots \cup D_{n-1}$ is disconnected. If that happens, then there is no guarantee that the value for $g(z)$
Example (Riemann’s zeta function).

\[ \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \]

is convergent for \( z \) in

\[ \left\{ z \in \mathbb{C} \mid \Re z > 1 \right\}. \]

Here,

\[ \frac{1}{n^z} = \exp \left( - \left( \log n \right) \cdot z \right). \]

- There exists an analytic function, once again denoted by \( \zeta(z) \), defined over the complement of \( z = 1 \) inside \( \mathbb{C} \), that coincides with \( \zeta(z) \) when \( \Re z > 1 \). This function is called \( \text{the Riemann’s zeta function} \).

--- To be continued ---