Sample Final of Math 121, Fall, 2014

Print your name on every page. There are 6 pages with 15 problems. Two detachable blank pages are provided at the back of this test for use as a scratch paper only, and any work left on this scratch paper will NOT be graded.

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(18 = 6 × 3 points) Part I. Multiple-choice problems. Circle the correct answer. No partial credit possible.

4. Let \( f \) and \( g \) be functions with continuous derivatives \( f' \) and \( g' \), respectively, well defined on the real line, such that \( \int_1^3 f(x) \, dx = 4 \) and \( \int_1^5 f(x) \, dx = 6 \). Furthermore we have the following data.

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & f(x) & g(x) & f'(x) & g'(x) \\
\hline
1 & -3 & -1 & 2 & -3 \\
3 & 5 & 1 & -1 & -2 \\
5 & -5 & 0 & -2 & 3 \\
\hline
\end{array}
\]

(i) \( (g \circ f)'(3) = \)
(a) -3, (b) 2, (c) -4, (d) 4, (e) none of the above

(ii) If \( h(x) = f(x)^2 g(x^2) \), then \( h'(1) = \)
(a) -15, (b) -48, (c) -42, (d) 36, (e) none of the above

(iii) \( \lim_{x \to 5} \frac{f(x)g(x)}{x^2 - 25} = \)
(a) 1, (b) -1.5, (c) \( \infty \), (d) -0.6, (e) none of the above

(iv) If \( H(x) = \int_0^{\ln x} f(t)^2 \, dt \), then \( H'(e^3) = \)
(a) \( \frac{25}{3} \), (b) \( 8e^3 \), (c) 25, (d) \( 25e^{-3} \), (e) none of the above

(v) \( \int_0^{\ln^3 3} e^t f(e^t + 2) \, dt = \)
(a) 4, (b) 0, (c) 2, (d) \( \infty \), (e) none of the above

(vi) \( \int_1^3 xf'(x) \, dx = \)
(a) 14, (b) 24, (c) 12, (d) 11, (e) none of the above
(45 = 15 × 3 points) **Part II.** Short Questions. Fill each blank with the **correct and complete** answer. **No partial credit** possible.

2. The slope of the tangent line to the curve \( x^2 + x \ln y + xy^2 = 5 + \ln 2 \) at the point \((1, 2)\) is

3. What is the exact volume of the solid obtained by revolving, about the line \(y = -1\), the region

\[ R = \{(x, y) : 0 \leq x \leq 1 \text{ and } x \leq y \leq e^{3x}\} \]?

4. If 600 cm\(^2\) of material is available to make an open-top box with a square base, what is the largest possible volume of the box?

5. Let \( F(x) = \int_0^{x^2} e^{-f(t)} dt \) for a differentiable function \( f \) on \((-\infty, \infty)\). Then \( F'(x) = \)

6. In three hours, the velocity of a car at each half hour was recorded as follows:

<table>
<thead>
<tr>
<th>Time (hours)</th>
<th>0</th>
<th>.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity (mi/h)</td>
<td>25</td>
<td>45</td>
<td>50</td>
<td>55</td>
<td>40</td>
<td>45</td>
<td>37</td>
</tr>
</tbody>
</table>

What is the numerical estimate of the **average** velocity (in miles/hour) of the car over these three hours, obtained via the Simpson’s approximation \( S_6 \)?
7. A cable that weighs 10 lb per linear foot is used to lift 900 lb of coal up a mine-shaft. How much work, measured in ft-lb, is done in lifting the coal from 200 feet below the ground to 30 feet below the ground?

8. A tank of the shape of a circular cone with its vertex pointing downward (and its top horizontal) is completely filled with water. Assume that the radius of its circular top is 1 m and its height (i.e. the distance from the vertex to the top) is 1 m.
   (i) What is the work, measured in newton-meter (i.e. joule or kg·m²/s²), needed to pump all of the water out of this tank over its top? (Note that the density of water is 1000 kg/m³ and the gravitational acceleration is 9.8 m/s².)

   (ii) Let \( h(t) \) be the water level (i.e. the distance from the vertex to the water surface) in the tank \( t \) seconds after we start to pump the water out of this tank at the constant rate of 0.1 m³/s. How fast, in m/s, is the water level dropping at the moment when the water level is 0.4 m? (Give the correct rate of change in its absolute value, ignoring the ±-sign.)

9. Consider the parametric curve \( \gamma \) defined by \((x, y) = (\frac{1}{3}t^3 - t + 3, t^2 + 1)\) with \( t \geq 0 \). The length of the part of \( \gamma \) that joins the points \((3, 1)\) and \((9, 10)\) is

10. Assume that \( \int_{-3}^{2} f(x) \, dx = -1 \), \( \int_{-1}^{5} f(x) \, dx = 8 \), and \( \int_{-3}^{5} f(x) \, dx = 6 \), for some continuous function \( f \) on \( \mathbb{R} \).
   (i) \( \int_{-1}^{2} f(x) \, dx = \)

   (ii) What value must be taken by \( f \) at some point in \([-1, 2]\)?
11. What is the exact volume of the solid obtained by revolving, about the $y$-axis, the region

$$R = \{(x, y) : 1 \leq x \leq \pi \text{ and } 0 \leq y \leq e^{x^2}\}?$$

12. The line $y = -x + 4$ and the parabola $y = x^2 - 2$ intersect at two points $(-3, 7)$ and $(2, 2)$, and bound (or enclose) a unique bounded region $R$. Let $S$ be the solid that has the region $R$ (in the $xy$-plane) as its base such that each of its cross-sections perpendicular to the $x$-axis is a half-disk. (Note that this solid is not a solid of revolution.)

(i) The area $A$ of the region $R$ is

(ii) Express the volume $V$ of $S$ as a single concrete definite integral without actually computing the value.

(iii) Find the length $\ell$ of the whole boundary of the region $R$, as a single concrete definite integral without actually computing the value.

Part III. True-false Problems. Circle the correct answer, T (standing for ‘True’) or F (standing for ‘False’). No partial credit possible.
(8 = 4 × 2 points) 12. Determine whether each of the following statements is true or false.

(1) T F ····· $\int_1^9 f(x) \, dx = \int_1^3 f(x) \, dx - \int_3^9 f(x) \, dx$ for any continuous function $f$ on $(-\infty, \infty)$.

(2) T F ····· $\int_2^1 f(x) \, dx < 0$ for any positive (i.e. $f(x) > 0$ for all $x$) continuous function $f$ on $(-\infty, \infty)$.

(3) T F ····· $\lim_{h \to 0} \frac{1}{b} \int_a^{a+h} f(t^2) \, dt = 2af(a^2)$ for any continuous function $f$ on $(-\infty, \infty)$ and any $a \in (-\infty, \infty)$.

(4) T F ····· $\frac{1}{6} \int_3^9 f(t) \, dt \leq |f(3)| + |f(9)|$ for any continuous function $f$ on $(-\infty, \infty)$.

(14 = 7 × 2 points) 13. The graph of a continuous function $f$ on the closed interval $[-6, 7]$ is shown in the following figure, where the arc is a semicircle. Let $g(x) = \int_{-2}^{x} f(t) \, dt$ for $-6 \leq x \leq 7$.

(1) T F ····· $g(-6) < 0$.

(2) T F ····· $g(1) = g(3) = 6 - \pi$.

(3) T F ····· $g$ is not differentiable at $x = -4, 0, 1, 3, 5$.

(4) T F ····· $g''(4) = 4$.

(5) T F ····· $g$ has a local minimum at $x = \frac{7}{2}$.

(6) T F ····· $g$ is concave up on the interval $(-4, 0)$.

(7) T F ····· $g$ has an absolute maximum at $x = 2$ in the interval $[-6, 7]$.
Part IV. Standard Essay Problems. **Show your work** to support your answers unless otherwise instructed. Solutions obtained only from calculators will not get any credit.

(15 points) 15. Compute **EXACTLY ONE** of the following integrals and **CROSS OUT** the other one that is not to be graded. (In the real final, you may be given **only one** problem to solve.)

(i) \[ \int_{-2}^{3} \frac{1}{(x-1)^8} \, dx. \]

(ii) \[ \int \frac{x^2 - 5x + 3}{x^3 - 9x} \, dx. \] (Note that \( x^3 - 9x = x(x-3)(x+3) \).)
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