Stochastic heat eq. (A random field approach).

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Def: Le Chen & Robert C. Dalang. AOP 2015.

\[
\left( \frac{1}{4t} - \frac{1}{2} \frac{d^2}{dx^2} \right) u(t,x) = \mathcal{P} \left( u(t,x) \right) \mathcal{W}(t,x) \quad t > 0, \quad x \in \mathbb{R}
\]

\[ u(0,x) = \mu(x) \tag{SHE} \]

* \( t > 0 \), diffusion parameter.
* \( \mathcal{P} \) : Lipschitz.
* \( \mathcal{W} \) : space-time white noise.
* \( \mu \) : a Borel measure.

\[ \mathcal{J}_0(tx) \triangleq \int_{\mathbb{R}} G_t(x-y) \mu(dy) = \left( G_t(t,.) \ast \mu \right)(x) \]

Note: \( \mathcal{J}_0 \) is a \( C_0 \) function of \( x \).

Let \( \mathcal{J}_0(tx) \) be the heat kernel, i.e., \( \mathcal{J}_0(tx) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \). Thus, the solution to the homogeneous eq.

\[
\left\{ \begin{array}{l}
\left( \frac{1}{4t} - \frac{1}{2} \frac{d^2}{dx^2} \right) u = 0 \\
u(0,x) = \mu(-) 
\end{array} \right.
\]

Let \( W = \{ W_t(A), A \in B_6(\mathbb{R}), t > 0 \} \) be a space-time white noise on bounded Borel sets defined on a complete prob. space \( (\mathbb{R}, F, P) \). Let

\[ F_t = \sigma \left( W_0(A), 0 \leq s \leq t, A \in B_6(\mathbb{R}) \right) \quad \forall X, \quad t > 0 \]

where \( N \) is the \( \sigma \)-field generated by all \( P \)-null sets in \( F \).
Definition of a mild solution:
A process $U = (u(t+x), t \geq 0, x \in \mathbb{R})$ is called a random field if

$$R^* = (0, +\infty)$$

(1) $U$ is adapted, i.e., $H(t,x) \in \mathcal{F}_t \otimes \mathbb{R}$, $U(t,x)$ is $\mathcal{F}_t$-measurable.
(2) $U$ is jointly measurable w.r.t. $\mathbb{P} \otimes \mathcal{B}(\mathbb{R}^* \otimes \mathbb{R}) \otimes \mathcal{F}$.
(3) For all $(t,x) \in \mathbb{R}^* \times \mathbb{R}$

$$G^2 + ||u||_2^2(t,x) \leq \int_0^t \int_{\mathbb{R}} G^2(t-s,x,y) \mathbb{E}(\mathbb{P}(u(s,y))) \, ds \, dy$$

and the function $(t,x) \mapsto U(t,x)$ is measurable.

from $\mathbb{R}^* \times \mathbb{R}$ into $L^2(\mathbb{R})$ is continuous.

(4) $U$ satisfies

$$u(t,x) = J_0(t,x) + I(t,x) \quad \text{a.s.} \quad \text{for all } t > 0, x \in \mathbb{R}$$

Main result of these two lectures:

Theorem: If the initial data $u$ is a Borel measure s.t. $J_0(t,x) < +\infty$ for all $t > 0$ and $x \in \mathbb{R}$, then (SHE) has a random field sol.

Moreover:

(1) $U$ is unique (in the sense of versions).
(2) $(t,x) \mapsto u(t,x)$ is $L^p(\mathbb{R})$-continuous for all $p \geq 2$.
(3) For all $p \geq 2$ (even integers), $t > 0$ and $x \in \mathbb{R}$

$$||u(t,x)||^2_p \leq \left\{
\begin{array}{ll}
J^2(t,x) + (J^2 \phi_R(t,x) + \varepsilon^2 H(t,x)) & p = 2 \\
2J^2(t,x) + (2J^2 \phi_R(t,x) + \varepsilon^2 \tilde{H}(t,x)) & p = 2
\end{array}
\right.$$
Theorem 3

Let $f$ satisfy $18(x) \leq \frac{L^2}{4} (c^2 + x^2)$ for $x \in \mathbb{R}$, where

$$K(t,x) = K(t,x; Lp), \quad K_{t}^p(t,x) = K(t,x; 4\pi t Lp).$$

$$H(t) = H(t; Lp), \quad H_{t}^p(t) = H(t; 4\pi t Lp).$$

and

$$K(t,x; \lambda) = 6 \frac{\lambda}{2} \left( \frac{\lambda^2}{4\pi t} + \frac{\lambda^2}{2t} \right)^{\frac{3}{4}} \Phi \left( \sqrt{\frac{x}{2\lambda}} \right)$$

$$H(t; \lambda) = 2 \exp \left( \frac{-3\lambda}{2t} \right) \Phi \left( \sqrt{\frac{x}{2\lambda}} \right) - 1$$

and $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy$.

(4) If $f$ satisfies $18(x) \geq \frac{L^2}{4} (c^2 + x^2)$ for $x$, then

$$\|w(t,x)\|_{x}^2 \geq \frac{L^2}{4} (c^2 + x^2) + \int_{0}^{+\infty} \left( 4\pi t \right) K(t,x) + c^2 H(t)$$

where $K(t,x) = K(t,x; Lp)$ and $H(t) = H(t; Lp)$.

(5) If $f$ satisfies $18(x) = x^2 (c^2 + x^2)$ for $x \in \mathbb{R}$, then

$$\|w(t,x)\|_{x}^2 = \int_{0}^{+\infty} \left( 4\pi t \right) K(t,x) + c^2 H(t)$$

The proof of this theorem requires several technical results.

Define:

$$L_0(t,x) = L_0(t,x; 3) = x^2 \exp \left( \frac{-c^2}{4t^2} \right).$$

$$L_n(t,x) = L_n(t,x; x) = \left( \frac{L_{n-1} \ast \cdots \ast L_{n-1}}{t^{n-1}} \right) (t,x).$$

$$L_0(t,x) = \left( \frac{L_{n-1} \ast \cdots \ast L_{n-1}}{t^{n-1}} \right) (t,x).$$
Proposition 1: \[ L_t \pm b = \frac{\lambda^2}{144\pi} \] and
\[ B_n(t) = \frac{\pi^{\frac{n}{2}} b^{n + \frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \]

We have that,
\[ L_n(t+x) = L_0(t+x) B_n(t) \text{ for all } n \geq 0 \]
and
\[ K(t,x) = \sum_{n=0}^{+\infty} L_n(t+x). \]

Furthermore,
\[ (K \ast L_0)(t,x) = K(t,x) - L_0(t+x) \]
and
\[ \sum_{n=0}^{+\infty} B_n(t)^m < \infty \text{ for all } m > 0. \]

Proof: We will prove (2) by induction. If \( n \geq 0 \), then \( B_0(t) = \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \).

Suppose (2) is true for all \( k \leq n \). Now for \( k = n+1 \),
\[ L_{n+1}(t+x) = (L_n \ast L_0)(t,x) \]
\[ = \int_0^t \int_0^x L_0(t-s, x-y) B_n(t-s) \cdot L_0(s, y) \, ds \, dy \]
\[ = \int_0^t b \frac{b}{15} B(t-s) \cdot \frac{b}{15} \int_0^x 6x^2(t-s, x-y) G_0(s, y) \, dy \]
\[ = 6x^2(t,x) \cdot \frac{b}{15} \int_0^t \frac{1}{15} \beta(x) \frac{n+1}{2} \, ds \]
\[ = 6x^2(t,x) \cdot \frac{b}{15} \int_0^t \beta(x) \frac{n+1}{2} \, ds \]
\[ = \frac{t^4}{6} \frac{P(\frac{1}{2}) P(\frac{n+1}{2})}{P(\frac{n+2}{2})} \text{ Beta integral} \]
\[= L_0(t; x) B_{n+1}(t). \text{ (after some simplification)}\]

As for (3), notice that

\[
L^2(t) = \sum_{n=1}^{\infty} \frac{x^{2(n-1)}}{\Gamma\left(\frac{2n}{2}\right)} \quad \text{and} \quad \Theta^2(t) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{\Gamma\left(\frac{2n+1}{2}\right)}
\]

which implies

\[
x^2(1 + \Theta(t)) = \sum_{n=1}^{\infty} \frac{x^{h+1}}{\Gamma\left(\frac{n+1}{2}\right)} = -\frac{1}{\pi x} + \sum_{n=0}^{\infty} \frac{x^n}{\Gamma\left(\frac{n+1}{2}\right)}
\]

\[\Rightarrow \frac{1}{\pi x} + x^2(1 + \Theta(t)) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)}
\]

Because \(1 + \Theta(t) = 2 \phi\left(\frac{1}{2}x\right)\), we see that

\[
\frac{1}{\pi x} + 2x^2 \phi\left(\frac{1}{2}x\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)}.
\]

Therefore,

\[
\sum_{n=0}^{\infty} L_n(t) x = 6g_x(t; x) \sum_{n=0}^{\infty} B_n(t) = 6g_x(t; x) \sum_{n=0}^{\infty} \frac{(b\sqrt{t})^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)}
\]

\[
= 6g_x(t; x) x \cdot 2 \left(\frac{1}{\pi (b\sqrt{t})} + 2 \phi\left(\frac{1}{2}, b\sqrt{t}\right)\right)
\]
Which is eq. to 1 on p. 3. This proves 3.

Eq. 4 is a simple recursion. because

\[ R_{Kn} = \left( \sum_{n=0}^{\infty} \ln \right) B_{Kn} = \sum_{n=0}^{\infty} \ln B_{Kn} \]
\[ = \sum_{n=0}^{\infty} \ln n + 1 = \sum_{n=1}^{\infty} \ln n = k - \ln. \]

Property in 5 is due to the ratio test:

\[ \frac{B_n(t+1)^m}{B_{n+1}(t+1)^m} = \left( \frac{t+1}{t+b} \right)^m \left( \frac{P\left( \frac{n}{t} \right)}{P\left( \frac{n}{t} + \frac{1}{2} \right)} \right)^m \approx \left( \frac{t+1}{t+b} \right)^m \left( \frac{2}{n} \right)^2 \rightarrow 0 \]

\[ a_{3n \rightarrow \infty} \]

(Wash integral)

The Wash integral is defined for the predictable integrands. It needs to be extended to adapted and jointly measurable integrands similar to the Brownian case.

* Elementary random field: \( Z(t, x) = \int_{a \leq s \leq t} 1_{[s, b]}(x) W(s, x) dx \) \( a \leq b \) and \( A \subset \mathbb{R} \) an interval, \( Y \) is \( F_{a} \)-measurable.

\[ \int Z(s, x) W(ds, dxdy) = Y(W_{b}(A) - W_{a}(A)) \]

* Simple random field: A finite sum of elementary random fields.

* Simple r. f. generate the predictable \( \mathcal{P} \) field. D.
For $p \geq 2$, and $X \in L^2(\mathbb{R} \times \mathbb{R}, L^p(\mathbb{R}))$, define
\[
\|X\|_{\mathbb{H}_p}^2 \leq \iint_{\mathbb{R} \times \mathbb{R}} \|X(s,y)\|^2 \, ds \, dy < \infty.
\]

Walsh theory: $\int \int X \, dW$ is well defined if $X \in L^p$ and
\[
\|X\|_{\mathbb{H},2} < \infty.
\]

Proposition 2 (C. & Dalang). If for some $t > 0$, $p \geq 2$, a random field $X = \{X(s,y) : s \in (0,t) \text{ and } y \in \mathbb{R}\}$ satisfies

1. $X$ is adapted.
2. $X$ is jointly measurable w.r.t. $B((0,t] \times \mathbb{R}) \times \mathcal{F}$.
3. $\|X\|_{\mathbb{H},p} < \infty$.

Then $X \mathbb{1}_{(0,t]} \in \mathcal{M}_2$, where $\mathcal{M}_p$ is the closure in $L^2(\mathbb{R} \times \mathbb{R}, L^p(\mathbb{R}))$ of simple random fields w.r.t. $\| \cdot \|_{\mathbb{H},p}$ norm.

Remark: For $2 \leq p < \infty$, $\mathcal{M}_2 \supseteq \mathcal{M}_p \supseteq \mathcal{M}_q$.

* We could make such extension from predictable integrands to adapted and jointly measurable integrands thanks to the "Brownian-type" integrator (i.e., space-time white noise).

Similar results for SDE are standard. See, e.g., (Chung & Williams '90).
Proof of Proposition 2: We will prove this prop. in three step.

Step 1: Assume (2') instead of (2)

(2') $g, y \rightarrow X(s, y)$ from $(0, t)$ into $L^p(\Omega)$ is continuous.

Fix $\epsilon > 0$ and $\frac{\epsilon}{2} < \frac{\epsilon}{2}$. Choose $a > 0$ s.t.

$$\int_{\Omega} \|X(s^0, y)\|_p^2 \, ds \, dy < \epsilon$$

Define

$$X_{n,m}(t) = \sum_{j=0}^{n} \sum_{i=0}^{m} X(t_j, x_i) \frac{1}{(t_j + j \cdot \frac{\epsilon}{n})} \frac{1}{(x_i, x_{i+1})}$$

where

$$t_j = \frac{j \cdot \epsilon}{n} \quad j = 0, \ldots, n$$

$$x_i = \frac{i \cdot \epsilon}{n} - a \quad i = 0, \ldots, 2m$$

$X_{n,m}$ is predictable by construction.
$L^p(\mathbb{R})$ continuity implies that when \( \|x\|_{\infty} \) are large enough,

$$\|x(z_1, y_2) - x(z_1, y_2)\|_p < \frac{\varepsilon}{\alpha}$$

where \((z_1, y_1), (z_1, y_2)\) are in each small rectangle in Fig.

Finally,

$$\|X z_{\text{ext}} - X_{\text{ext}}\|_p \leq \varepsilon + \sum \frac{\varepsilon^2}{\alpha^2} \int d \text{dy}$$

$$= \varepsilon + \sum \frac{2\varepsilon^2 - \varepsilon}{\alpha^2} \leq \varepsilon^2 + C \varepsilon^2 \to 0 \; \text{for} \; \varepsilon$$

So, \( X z_{\text{ext}} \in M_p \subseteq M_2 \).

Step 2: Assume (2) + \( X \) being bounded. Choose mollifier properly. (A good exercise to complete this part).

Step 3: Assume (2). Truncation arguments.

The next proposition gives a convenient form of the BDG inequality (Barkholder-Davis-Gundy).

**Proposition 3:** Let \( G(t, x) \) be a deterministic function. If \( Z \) is adapted and jointly measurable and

$$E \left( \int_0^1 G^2(t, x - y) Z^2(s, y) \, ds \, dy \right) < \infty \quad \forall t \to x \in \mathbb{R},$$

then \( G(t, x - y) Z(s, y), s \in (0, t), y \in \mathbb{R} \) \( \subseteq M_2 \).
and so the stochastic integral

$$(G \otimes Z \dot{W})_{t \times t} = \int_0^t \int_{\mathbb{R}} G(t-s, x-y) Z(s, y) W(dy) ds$$

is well-defined. Moreover,

$$\| (G \otimes Z \dot{W})_{t \times t} \|^p \leq Z_p^2 \| G(t-t, x-x) Z(, 0) \|^p$$

$$= Z_p^2 \int_0^t \int_{\mathbb{R}} G(t-s, x-y) Z(s, y) \| Z(s, y) \|^p ds dy$$

**Remark:** This Proposition under stronger condition (i.e., predictability) appeared already in Foundan & Khoshnevisan EJP 09 and Conus & Khoshnevisan PTRF '12.

$$Z_p = 2^p$$

**Proof:**

$$\| (G \otimes Z \dot{W})_{t \times t} \|^p = \mathbb{E} \left( \int_0^t \int_{\mathbb{R}} G(t-s, x-y) Z(s, y) W(dy) ds \right)^p$$

$$(BDA) \leq Z_p^p \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} (G(t-s, x-y) Z(s, y))^2 ds dy \frac{p}{2} \right]^{\frac{p}{2}}$$

So

$$\| (G \otimes Z \dot{W})_{t \times t} \|^2 \leq Z_p^2 \left\| \int_0^t \int_{\mathbb{R}} G^2(t-s, x-y) \frac{Z^2(s, y)}{2} ds dy \right\|^p$$

Minkowski's inequality

$$\leq Z_p^2 \int_0^t \int_{\mathbb{R}} G^2(t-s, x-y) \left( Z(s, y) \right)^2 ds dy$$

$$= Z_p^2 \int_0^t \int_{\mathbb{R}} G^2(t-s, x-y) \left( Z(s, y) \right)^2 ds dy$$

$\square$
The next proposition tells us that the $L^p(R)$-continuity is stable under the stochastic integral. As we have seen in the steps of the proof of proposition 2, the $L^p(R)$-continuity is a sufficient condition for the joint measurability.

**Proposition 4:** If $P_{22}$ and $Y$ is adapted, jointly measurable s.t.

$$\int_0^t \mathcal{G}(t-s, x-y) \, dY(s, y) \in L^p \quad \forall t \in \mathbb{R}, x \in \mathbb{R}$$

then $\mathcal{G}(t-0, x-0) Y(t, 0) \in M_2$ and the stochastic integral

$$W(t,x) = \int_0^t \int_0^x \mathcal{G}(t-s, x-y) Y(s, y) \, dW(ds, dy)$$

satisfies the property that if $Y$ has locally bounded $p$th moment, i.e., $\forall K \subset \mathbb{R}^+ \times \mathbb{R}$ compact,

$$\sup_{t \in K} \| Y(t, x) \|_p < \infty$$

which is the case if $Y$ is $L^p(R)$-continuous, then $W$ is $L^p(R)$-continuous on $\mathbb{R}^+ \times \mathbb{R}$.

**Remark:** The proof is quite technical even though the result looks quite standard. (See [C.D. Malam Prop 3.4.43])

* This proposition gives us a way to check the joint measurability in the Picard iteration latter.
Proposition 5: If the initial measure (nonnegative) $\beta$ s.t.

$$\beta_0(x) < +\infty \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R},$$

then

$$\left( \int_0^\infty G_\beta(t, x) h_0(\cdot) ds \right) \leq 2 E \int_0^2 (2t, x) \int_0^t h_0(t, \cdot) ds$$

In particular, for some $C_t > 0$,

$$\left( \int_0^2 L_0 \right)(x) \leq \left( \int_0^2 L \right)(x) \leq C_t \int_0^2 (2t, x) < +\infty$$

Proof: Notice that

$$\int_0^2 (t, x) = \int_{\mathbb{R}^2} G_\beta(1 + x, y) G_\alpha(t, x - y) \mathbb{M}(dy_1) \mathbb{M}(dy_2).$$

$$= \int_{\mathbb{R}^2} G_\beta(t, x - y) G_\alpha(t, y) \mathbb{M}(dy_1) \mathbb{M}(dy_2).$$

[Lemma A.4 & C. Dalang]: $G_\beta(t, x) G_\alpha(t, y) = G_\beta(\frac{t + x + y}{t}, \frac{t + x + y}{t}) \times G_\alpha(\frac{t + x + y}{t}, -y).$

$$\left( \int_0^2 G_\beta(t, x) h_0(\cdot) ds \right)(x) = \int_0^t ds \int_{\mathbb{R}^2} G_\beta(2s, x - z) h_0(t, z) \mathbb{M}(dz).$$

$$\times G_\alpha(\frac{2s + x + y}{2s}, \frac{2s + x + y}{2s}) G_\beta(\frac{2s + x + y}{2s}, y - y_2) \mathbb{M}(dy_1) \mathbb{M}(dy_2)$$

$$\leq G_\beta(t, x - y) G_\alpha(t, x - y_2) \frac{1}{0} \frac{1}{0} \frac{1}{0}$$

[CD Dalang Lemma A.5]
\[
\leq 2 \begin{bmatrix} t \end{bmatrix} J_0 (2t \pm x) \cdot \int_0^t \frac{h(s)}{s} ds \quad \square
\]

Now we are ready to prove theorem on p. 2.

**Proof of Theorem:** Fix p \( \geq 2 \). (Picard Iteration).

**Step 1:** Define \( U_0 (t, x) = \int_0^t f(u(t, s)) g_r(t - s, x - y) w(ds)dy \).

We need only to verify the integrability condition, which is true due to the linear growth cond. on \( f \) and Proposition 5. Hence, Prop. 4 implies that

\[ I(t) = \int_0^t f(u(s, y)) g_r(t - s, x - y) w(ds)dy \]

is well defined and \( (t, x) \mapsto I_t(t, x) \) is \( L^p(\Omega) \) continuous.

Define \( U_1 (t, x) = J_0 (t, x) + I(t, x) \). Then by the BDG ineq. (Prop. 3),

\[
\| U_1 (t, x) \|^2 \leq b_p J_0^2 (t, x) + \left( C^2 + b_p J_0^2 + \frac{2}{\gamma} \right) L_{0_p} (t, x)
\]

\[
\leq b_p J_0^2 (t, x) + \left( C^2 + b_p J_0^2 \right) \Theta p \quad \text{for } K = 1
\]

where \( b_p = \left\{ \begin{array}{ll} \frac{1}{2} & \text{if } p = 2 \\ p & \text{if } p \geq 2 \end{array} \right. \). Therefore, we have proved properties (i) - (iv) in Step 2.
Step 2: Assume that for all $k \leq n$,

$$I_k(t,x) = \int_0^t P(u_{k-1}(s,y)) G_k(t,x,y) W(ds,dy)$$

is well defined so that,

(i) $u_k := I_0 + I_k$ is adapted,
(ii) $t \mapsto u_k(t,x)$ is $L^p(\Omega)$-continuous,
(iii) $\|u_k(t,x)\|^2_p \leq 5p I_0(t,x) + \left( \frac{c^2}{4} + 5p J_0 \right) \mathbb{E} \rho(t,x)$.

Similarly to Step 1, we are going to apply Prop. 4 to $y = P(u_k)$.

- Adaptedness: Yes
- Jointly measurable: As implied by the LDU-continuity
- Integrable condition: $\lim \mathbb{E} \rho(t, \cdot) \leq K$

Hence,

$$I_{n+1}(t,x) = \int_0^t G_n(t,s,x,y) P(u_n(s,y)) W(ds,dy)$$

is well-defined and $L^p(\Omega)$-continuous by Prop. 4.

Define $u_{n+1}(t,x) = I_0(t,x) + I_{n+1}(t,x)$.

Then by the BDG ineq. (Prop. 3),

$$\|u_{n+1}(t,x)\|^2_p \leq 5p I_0(t,x) + \left( \frac{c^2}{4} + 5p J_0 \right) \mathbb{E} \rho(t,x)$$

by induction assumption.

$$\leq 5p I_0(t,x) + \left( \frac{c^2}{4} + 5p J_0 \right) \mathbb{E} \rho(t,x).$$
Therefore (ii) - (iv) are true for \( k = m + 1 \) and so true for all \( k \).

**Step 3.** We claim that \( \{ u_{n} (t, x) \}_{n \in \mathbb{N}} \) is Cauchy and let \( u (t, x) \) be its limit.

Define \( F_{n} (t, x) = \| u_{n} (t, x) - u_{n+1} (t, x) \|^{2} \). By BDG, (Prop. 3),

\[
F_{n} (t, x) \leq (F_{n-1} \ast L_{0,p}) (t, x). \quad \forall \nu \sim \lambda = \ldots \,
\]

For \( n = 0 \), \( F_{0} (t, x) \leq (\overline{c}^{2} + \overline{j}^{2}) \ast L_{0,p} (t, x) \). Hence,

\[
F_{n} (t, x) \leq (F_{n-1} \ast L_{0,p}) (t, x) \leq \cdots \leq (\overline{c}^{2} + \overline{j}^{2}) \ast L_{0,p} (t, x)
\]

\[
= (\overline{c}^{2} + \overline{j}^{2}) \ast L_{0,p} (t, x) \cdot B_{n} (t) \quad \text{as} \quad n \to \infty.
\]

Hence,

\[
\sum_{n=0}^{\infty} F_{n} (t, x)^{1/n} \leq \left( (\overline{c}^{2} + \overline{j}^{2}) \ast L_{0,p} (t, x) \right)^{1/n} \sum_{n=0}^{\infty} |B_{n} (t)|^{1/n} < \infty
\]

It suffices to take \( m = 2 \).

We further claim that \( u (t, x) \) is \( L^{0, \infty} - \text{continuous} \). (Exercise) (Prop. 4)

**Step 4: (Verification)** We will show that \( u (t, x) \) in Step 3 is a random field sol. (See the definition on p. 2)

(Exercise)
Step 5: (Uniqueness). Let \( u \) and \( v \) be two solutions to (5.10) with the same initial data. Denote \( W(t,x) = u(t,x) - v(t,x) \).

We have shown in Step 3 that \( W(t,x) \) is \( L^2(\mathbb{R}) \)-continuous. By Itô's isometry,

\[
E(W^2(t,x)) \leq (E(W^2) \ast L_0^*(t,x)), \quad L_0^*(t,x) = L_0(t,x; L_0)
\]

Hence,

\[
(E(W^2) \ast L_0^*) (t,x) \leq (E(W^2) \ast L_0^* \ast K^*) (t,x)
\]

\[
\leq (E(W^2) \ast K^*) (t,x).
\]

So,

\[
(E(W^2) \ast L_0^*) (t,x) = 0.
\]

Therefore, \( E(W^2(t,x)) = 0 \) for all \( t > 0, x \in \mathbb{R} \), which implies that \( W(t,x) = 0 \) a.s. for all \( t > 0, x \in \mathbb{R} \).

This completes the whole proof of the Theorem. \( \square \)