**Linear operators and linear differential equations**

**Definition:** An operator is a function whose domain is a set of functions (not a set of real or complex numbers).

**Examples:** In these examples, all our functions are assumed to be differentiable functions of \( t \). The functions are written as \( f, g, h, \) etc., and not \( f(t), g(t), h(t) \), since the last three are numbers, and not functions.

- Multiplication of a function by the number 2 is an operator: it could be written \( M_2[x] = 2x \). This means that for all \( t \), the function \( M_2[x](t) = 2x(t) \).

- Differentiation with respect to \( t \) is an operator, denoted by \( D \): by definition, for any differentiable function \( f \),
  \[
  D[x] = x', \text{ or for all } t, D[x](t) = x'(t).
  \]
  Taking two derivatives:
  \[
  D^2[x] = D[D[x]] \text{ or for all } t, D^2[x](t) = x''(t).
  \]

- Integration from 0 to \( t \):
  \[
  \text{Int}_0^t[x](t) = \int_0^t x(s) \, ds.
  \]

- Some other operators:
  \[
  \text{Sq}[x] = x^2 : \text{that is, } \text{Sq}[x](t) = x^2(t) = x(t)x(t).
  \]
  \[
  \text{Sqrt}[x] = \sqrt{x}.
  \]

- There are also the well-known binary operators:
  \[
  [f + g] = f + g; \text{ and } [fg] = fg \text{ (meaning } (fg)(t) = f(t)g(t) \text{ for all } t).\]

- And last, but not least, the identity operator \( I \):
  \[
  I[x] = x.
  \]
  The identity operator applied to \( x \) just gives back \( x \). It "acts like" multiplication by 1. For completeness, we define \( D^0 = I \).

We can use addition and multiplication, together with other operators, to make more complicated operators:

\[
D^2[x] + \text{Sq}[x] + M_3[x] = g, \text{ meaning that, for all } t,
\]
\[
x''(t) + x^2(t) + 3x(t) = g(t).
\]

This is a second order ODE, of course, which is where we’re heading.

\[ \square \]

**Definition:** An operator \( L \) is said to be linear if, for any constants \( c_1, c_2 \) and any (smooth) functions \( x_1, x_2 \),
\[
L[c_1x_1 + c_2x_2] = c_1x_1 + c_2x_2.
\]

**Example:** The second order differential equation \( x'' + px' + qx = f \) can be written as \( L[x] = f \), where
\[
L = D^2 + pD + qI.
\]

So \( L[x] = x'' + px' + qx \), and \( L \) is linear.

**Proof:** For any \( c_1, c_2, x_1, x_2 \), we compute
\[
L[c_1x_1 + c_2x_2] = (c_1x_1 + c_2x_2)'' + p(c_1x_1 + c_2x_2)' + q(c_1x_1 + c_2x_2)
= c_1x_1'' + c_2x_2'' + c_1px_1' + c_2px_2' + c_1qx_1 + c_2qx_2
= c_1(x_1'' + px_1' + qx_1) + c_2(x_2'' + px_2' + qx_2)
= c_1L[x_1] + c_2L[x_2].
\]

For this reason, \( L \) is said to be a linear second order differential operator and the DE \( L[x] = f \) is called a second order linear differential equation.

\[ \square \]

**Definition:** A linear differential operator (LDO) of degree \( n \) is a polynomial in \( D \) of degree \( n \) whose coefficients are continuous functions of \( t \):
\[
L = a_n(t)D^n + a_{n-1}(t)D^{n-1} + \cdots + a_1(t)D + a_0(t)I.
\]

That is
\[
L[x] = a_nD^n[x] + a_{n-1}D^{n-1}[x] + \cdots + a_1D[x] + a_0x
= a_nx^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1x' + a_0x.
\]

Note that we generally write just \( a_n \) rather than \( a_n(t) \), which is supposed to be understood.

It is tedious but straightforward to check that such an \( L \) is linear in the sense of the definition.

\[ \square \]

**Definition:** A linear differential equation of order \( n \) (LDE) is an equation of the form \( L[x] = f \), where \( L \) is an \( n^{th} \) order LDO.

\[ \square \]

**Definition:** The linear differential equation is said to be homogeneous if the right hand side is zero: \( L[x] = 0 \).

Solutions to linear homogeneous differential equations obey the superposition principle: if \( x_1 \) and \( x_2 \) are two solutions, then so is the linear combination \( c_1x_1 + c_2x_2 \) for any constants \( c_1 \) and \( c_2 \).
Proof: A function $y(t)$ is a solution to the homogeneous equation if $L[y] = 0$. By linearity, $L[c_1x_1 + c_2x_2] = c_1L[x_1] + c_2L[x_2]$ (and using the fact that $x_1$ and $x_2$ are solutions), this $= c_1 \cdot 0 + c_2 \cdot 0 = 0$. So $c_1x_1 + c_2x_2$ is also a solution.

Examples: (a) The equation $x'' + \omega^2 x = 0$ ($\omega$ is a constant) is linear. You can easily check that

$$x_1(t) = \cos(\omega t), \text{ and } x_2 = \sin(\omega t)$$

are both solutions to this LDE. It follows from the superposition principle that $x(t) = A \cos(\omega t) + B \sin(\omega t)$ is also a solution for any numbers $A, B$.

(b) A first order linear homogeneous DE, as you know, has the form $x' + px = 0$. If $\mu = e^{\int p}$ is the integrating factor, then

$$x_1(t) = e^{-\int_0^t p(s) ds}$$

is one solution, and so is $c_1x_1(t)$ for any constant $c_1$.

(c) There are also linear partial differential equations, in which the differential operators take the form

$$\frac{\partial}{\partial x} \frac{\partial}{\partial t}$$

and so on. So a linear partial differential equation for an unknown function $u(t, x)$ will be a polynomial in the partial derivatives with respect to $t$ and $x$. For instance,

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0$$

is a homogeneous linear PDE of second order known as the heat equation or, in a slightly different context, the diffusion equation.

Exercise: Prove that solutions to the heat equation satisfy the superposition principle.

Note: the partial differential operators above are often written as $\partial_t, \partial_{xx}$, etc.