Lab 2 - The logistic map and chaos

Introduction: discrete dynamical systems

An ODE is often called a continuous (or smooth) dynamical system. Once the IVP has been solved, you can start at any point x_0 at $t = 0$ and move smoothly along the unique solution to the IVP.

There are also *discrete* dynamical systems. In one dimension, they are simply given by functions (or *maps* as we like to call them) $f : \mathbb{R} \to \mathbb{R}$. Starting from any point x, the discrete dynamical system produces the *sequence* of points $\{x, f(x), f^2(x), f^3(x), \ldots\}$, where by $f^2(x)$, we mean the second iterate of f at x, $f(f(x))$, and similarly for $f^k(x)$. In general, such a sequence is infinite (i.e., takes on infinitely many different values). Such a sequence is called the *orbit* of x under the map f.

A discrete dynamical system can also be thought of as a first order difference equation, in the sense that the orbits are given by the equation

$$x_{n+1} = f(x_n); \text{ for } n = 1, 2, \ldots, \text{ with } x_1 \text{ given.}$$

The mathematical problem here is that these difference equations can seldom be solved explicitly, even for simple maps like the logistic map below. We therefore have to study them by examining the structure of their orbits:

For certain values of x it could happen that

- $f(x) = x$. In this case, x is said to be an *equilibrium point* of the dynamical system. A point x such that $f(x) = x$ is also called a *fixed point* of f.
- $f(x) \neq x$, but $f^2(x) = x$. Then x is said to lie on an orbit of *period 2*.
- In general, if $f^n(x) = x$, but $f^k(x) \neq x$ for $k < n$, then x lies on an orbit of period n.

Often, a discrete dynamical system will have the form $y = f(x, a)$, where a is some adjustable parameter, and the behavior of the dynamics may depend critically on this parameter. In this lab, we'll look at such a system.

The logistic (or quadratic) map

The *logistic map* is so-named because it comes from the right hand side of the logistic differential equation $y' = ax(1 - x)$. So it's given by the map

$$y = f(x) = ax(1 - x), \tag{1}$$

where, for this lab, the parameter a satisfies $0 \le a \le 4$. This is also known as the *quadratic map.* The reason for requiring $0 \le a \le 4$ is that, for these values of k, the function f maps the unit interval into itself: if $0 \le x \le 1$, then $0 \le f(x) \le 1$. This ensures that iterations of the function stay in the unit interval, where they may be conveniently observed.

The logistic map is often used as a (seriously oversimplified) discrete model of population growth, where the difference equation is given by

$$y_{n+1} = ay_n(1 - y_n).$$

If you recall, the solutions to the logistic DE are uncomplicated, and completely determined by the fact that $x = 1$ is a stable equilibrium point, while $x = 0$ is unstable. This is true for any positive value of the parameter k.

For the logistic map, the fixed points (equilibria) are the values of x such that

$$\begin{aligned} x &= f(x), \text{ or} \\ x &= ax(1 - x), \text{ or, if } x \neq 0, \\ 1 &= a(1 - x) \text{ or} \\ x &= 1 - 1/a, \end{aligned}$$

(together with $x = 0$, which is always an equilibrium solution).

□ Definition: A fixed point x_0 of f is said to be *stable* if, for x sufficiently close to x_0, $f(x)$ is closer still. That is, $|f(x) - x_0| < |x - x_0|$ when x is close to x_0. It is *unstable* if the inequality is reversed.

In words, the point x_0 is stable if, for any point x sufficiently close to x_0, the sequence of iterates $\{x, f(x), f(f(x)), f(f(f(x))), \ldots\}$ *converges* to x_0. The fixes point is unstable if the sequence of iterates (i.e., the orbit) recedes from x_0. Stable (unstable) fixed points or orbits are also known as *attracting (repelling)* fixed points or orbits.

EXAMPLE: Take $a = 2$ for the logistic map. The fixed point is $x_0 = 1 - 1/2 = 1/2$. Take some nearby point, such as $x = .6$. Then we find, using (1), that

$$\begin{aligned} f(.6) &= 2(.6)(1 - .6) = .48 \\ f(.48) &= 2(.48)(1 - .48) = .4992 \\ f(.4992) &= 2(.4992)(1 - .4992) = .49999872, \text{ and so on.} \end{aligned}$$

So here the sequence of iterates (i.e., the orbit of .6) is $\{.6, .48, .4992, .49999872, \ldots\}$, and it certainly appears that this is converging to $x_0 = 1/2$. But of course, this is not a proof.

THEOREM: SUPPOSE $f(x_0) = x_0$. IF $f'(x)$ IS CONTINUOUS AND $|f'(x_0)| < 1$ (RESP. $|f'(x_0)| > 1$), THEN x_0 IS STABLE (RESP. UNSTABLE).

PROOF: We do the "stable" part. We want to show that, if x is close to x_0 and $|f'(x_0)| < 1$, that $|f(x) - x_0| < |x - x_0|$. This follows right away from the mean value theorem, which says that, if f is differentiable on $[a, b]$, then there exists a $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Now, take the absolute value of both sides, and put $b = x, a = x_0$ to get

$$|f(x) - f(x_0)| = |f(x) - x_0| \text{ (fixed point) } = |f'(c)|\,|x - x_0|.$$

Now $|f(x_0)| = \alpha < 1$, and f' is continuous. This means there is some interval of the form $(x_0 - \epsilon, x_0 + \epsilon)$ in which $|f'| < 1$.

And in this interval, we must have $|f(x) - x_0| < |x - x_0|$ $\square$

Notice that the theorem says nothing about the case $|f'(x_0)| = 1$, which must be dealt with separately if it occurs.

To return to our example, when $a = 2$, $f(x) = 2x(1 - x)$, $f'(x) = 2 - 4x$, and at $x_0 = 1/2$, $|f'(x_0)| = 0 < 1$. So this equilibrium is stable, as we guessed. What about the other equilibrium point $x = 0$? Is it stable or not?

♣ Exercise: Show that, in the theorem above, if $|f'(x_0| > 1$, the equilibrium point is unstable.

♣ Exercise: For the quadratic map $Q(x) = x^2 + c$:

1. Find the values of the two fixed points (these will depend on c). What is the largest value of c for which there is a real fixed point?
2. Find the c-interval in which this fixed point is stable, and the interval in which it is unstable. The value of c separating these two regions is called a bifurcation point.

Bifurcations

In the 1970s, a biologist, Robert May, was using the discrete logistic map as a population model, and obtained some unexpected results. Unlike the case of the logistic differential equation, whose behavior is relatively insensitive to the particular value of a, he found that the behavior of the discrete map changed dramatically with a.

For the logistic map, we have an equilibrium point at $x_0 = 1 - 1/a$. The derivative of f at that point is

$$\begin{aligned} f'(x_0) &= a(1 - 2x_0) \\ &= a(1 - 2(1 - 1/a) \\ &= -a + 2 \end{aligned}$$

So $|f'(x_0)| < 1$ if $|2 - a| < 1$. Thus this equilibrium point is stable if $1 < a < 3$. And it is unstable for $a > 3$, and for $a < 1$. So the "behavior" of the function changes at $a = 1$ and $a = 3$ which are called a *bifurcation points*. The precise nature of these, and other changes, as a marches from 0 to 4, is the subject of this lab.