Newton's proof of Kepler's second law

The general version of Kepler's second law states that the regions swept out by the radius vector of a mass point moving in a central force field in equal time intervals have equal areas. Newton also shows that the motion takes place in a plane. There is no requirement that the force fall off as $1/r^2$, just that there exist a fixed point from which the force acts.

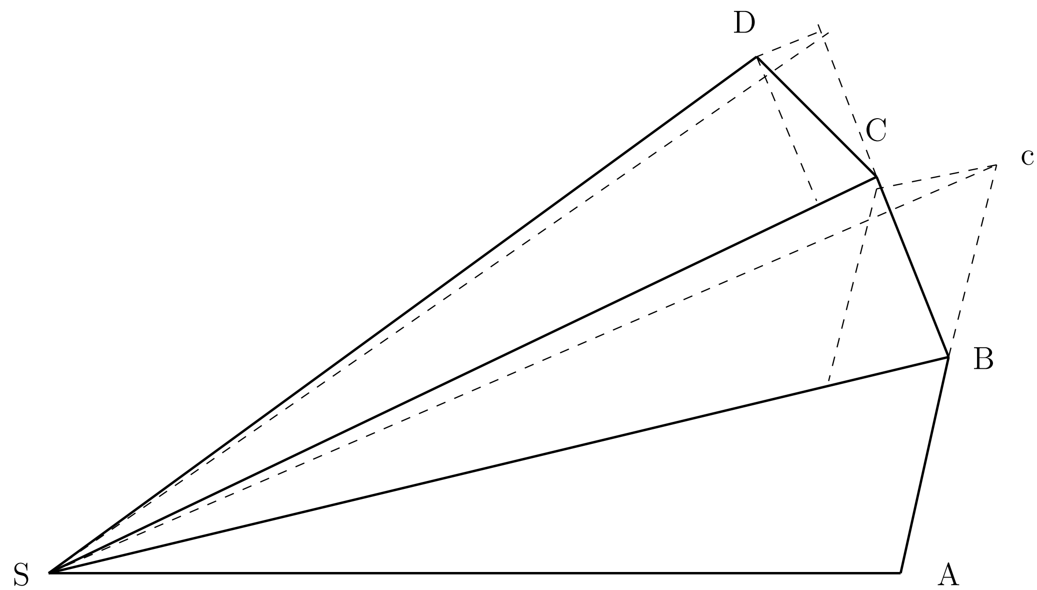

Figure 1: Newton's proof

In this figure, the center of force is at S, and the particle is initially at A, moving on some trajectory which takes it to B in some time interval Δt. If the particle at B were not subject to any forces, it would proceed in a straight line to point c in the next time interval Δt. But there's a central force acting on the particle at B, which points down the radius vector from B to S. This force results in an additional displacement of the particle from c to C. (Newton is essentially using the parallologram law of addition of displacement vectors here, and the dotted figure which looks like a parallelogram is one.)

Newton then observes that triangles SAB and SBc have the same area: they have bases of equal length and the same altitude (a perpendicular dropped from S to the line ABc). He then observes that the triangles SBc and SBC have the same area as well: they have the same base SB, and the same altitudes since the segment Cc is parallel to SB. Therefore triangles SAB and SBC have the same area, and this completes the proof.

Notice also that, by construction, the point C lies in the same plane as $S, A,$ and B. And so do all the subsequent points $D, E, \ldots$.

You should notice that Newton is being very clever here: there are only straight lines - no curves. And there are no limits. He is not computing dA/dt and showing that it's constant, the way we do today. The way he sets things up, there are no "small" quantities that have to go away as $\Delta t \to 0$. Also notice that his argument is independent of the magnitude of the displacement due to the central force. That is, we could have gotten the same result for a different point C' located anywhere at all on the line through c and parallel to SB. In fact, it seems that the force could depend on time, and could even change over time from attractive to repulsive and the result would still hold. Is Newton being too clever?

The usual calculus/physics proof of Kepler's 2^{nd} law goes as follows: we change to polar coordinates and write the acceleration vector of the particle as $\mathbf{a} = a_r \mathbf{u}_r + a_\theta \mathbf{u}_\theta$, where $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are orthonormal and point in the directions of increasing r and θ. With a bit of work (exercise in the chain rule), we find that

$$\begin{aligned} a_r &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \\ a_\theta &= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) \end{aligned}$$

By definition, in a central force field the acceleration is entirely radial, and so $a_\theta = 0$. But this leads immediately to

$$r^2\frac{d\theta}{dt} = c, \text{ a constant.}$$

We calculate the area swept out by the radius vector in a time interval Δt: Assuming that the path of the particle can be parametrized by θ, we have, first

$$A = \int\int r dr d\theta = \frac{1}{2}\int r^2 d\theta.$$

Since θ is a function of time here, this becomes

$$A = \frac{1}{2}\int_t^{t+\Delta t} r^2\frac{d\theta}{dt}dt = (c/2)\Delta t.$$

This is the same for any time interval of length Δt. Since we're only working with the θ component, the behavior of the radial force is indeed arbitrary, as Newton showed.