Bott periodicity and calculus of Euler classes on spheres

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A variety of computations regarding the Euler class group \(E(A_n, A_n)\) and the Grothendieck group \(K_0(A_n)\) of the algebraic sphere \(\text{Spec}(A_n)\) is done. The Euler class of the algebraic tangent bundle on \(\text{Spec}(A_n)\) is computed. It is also investigated whether every element in the Euler class group \(E(A_n, A_n)\) is the Euler class of a projective \(A_n\) module of rank \(n\).

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1. Introduction

Work on obstruction theory for projective modules started with the work of N. Mohan Kumar and M.P. Murthy [Mk,MkM,Mu1]. It is a result of Murthy [Mu1] that for a reduced (smooth) affine algebra \(A\) with \(\text{dim} A = n\), over an algebraically closed field \(k\), the top Chern class map \(C_0 : K_0(A) \to CH_0(A)\) is surjective. This result is a consequence of the result [Mu1] that given any local complete intersection ideal \(I\) of height \(n\), there is a projective \(A\)-module \(P\) with rank \((P) = n\) that maps surjectively onto \(I\).

For real smooth affine varieties such propositions will fail. Most common examples are that of real spheres. We denote the real sphere of dimension \(n\) by \(S^n\) and \(A_n\) denotes the ring of algebraic functions on \(S^n\). We have, the Chow group of zero cycles \(CH_0(A_n) = \mathbb{Z}/2\) (see 3.1) and by the theorem of Swan [Sw2], \(K_0(A_n) = KO(S^n)\). By the periodicity theorem of Bott (see 5.10), for nonnegative integers \(n = 8r + 3, 8r + 5, 8r + 6, 8r + 7\) \((r \geq 0)\) we have \(K_0(A_n) = KO(S^n) = \mathbb{Z}\). In these cases, the top Chern class map \(C_0 = 0\) and it fails to be surjective.

On the other hand, by Bott periodicity (see 5.10), \(\widetilde{K}_0(A_{8r}) = \widetilde{KO}(S^{8r}) = \mathbb{Z}, \widetilde{K}_0(A_{8r+1}) = \widetilde{KO}(S^{8r+1}) = \mathbb{Z}/(2), \widetilde{K}_0(A_{8r+2}) = \widetilde{KO}(S^{8r+2}) = \mathbb{Z}/(2), \widetilde{K}_0(A_{8r+4}) = \widetilde{KO}(S^{8r+4}) = \mathbb{Z}\). Therefore, in these cases, the question of surjectivity of the top Chern class map \(C_0 : K_0(A_n) \to CH_0(A_n)\) fully depends on the top Chern class of the generator \(\tau_n\) of \(\widetilde{K}_0(A_n)\). In analogy to the obstruction theory in topology, it makes

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more sense to consider the Euler class group $E(A_n)$ of $A_n$ as the obstruction group, instead of the Chow group $CH_0(A_n)$.

For (smooth) affine rings $A$ with dim $A = n \geq 2$, over a field $k$, the original definition of Euler class groups $E(A)$ was given by Nori [MS,BRS2]. For a projective $A$-module $P$, with $\det P = A$ and an orientation $\chi : A \to \det P$, an Euler class $e(P, \chi) \in E(A)$ was defined. We mainly refer to [BRS2], for basics on Euler class groups and Euler classes. For such a ring $A$, $\mathcal{PO}_n(A)$ will denote the set of all isomorphism classes of pairs $(P, \chi)$, where $P$ is a projective $A$-module rank $n$, with trivial determinant, and $\chi : A \to \det P$ is an isomorphism, to be called an orientation.

So, our main question is whether the Euler class map $e : \mathcal{PO}_n(A_n) \to E(A_n)$ is surjective. In fact, $E(A_n) = \mathbb{Z}$. Since for reasons given above, the Euler class map fails to be surjective for $n = 8r + 3, 8r + 5, 8r + 6, 8r + 7 (r \geq 0)$. In fact, we also prove that this map fails to be surjective. For any even integer $n \geq 2$, we prove that any even class $N \in E(A_n) = \mathbb{Z}$ is in the image of $e$. For $n = 2, 4, 8$ we prove that $e$ is surjective. For $n = 8r, 8r + 2, 8r + 4 \geq 2$, we prove that $e$ is surjective if and only if the top Stiefel-Whitney class $w_n(\tau_n) = 1$ where $\tau_n$ is the generator of $K_0(A_n)$. It remains an open question whether $w_n(\tau_n) = 1$.

Among other results in this paper, we compute (see 3.3) the Euler class of the algebraic tangent bundle $T$ over $\text{Spec}(A_n)$. As in topology (see [MiS]), $e(T, \chi) = -2$, when $n$ is even and zero when $n$ is odd. This provides a fully algebraic proof that the algebraic tangent bundles $T$ over even dimensional spheres $\text{Spec}(A_n)$ do not have a free direct summand.

Given any real maximal ideal $m$ of $A_n$, we attach (see 4.2) a local orientation $\omega$ on $m$ in an algorithmic way and compute the class $(m, \omega) = 1$ or $(m, \omega) = -1$ in $E(A_n)$.

2. Preliminaries

Following are some of the notations we will be using in this paper.

**Notations 2.1.** First, the fields of real numbers and complex numbers will, respectively, be denoted by $\mathbb{R}$ and $\mathbb{C}$. The quaternion algebra will be denoted by $\mathbb{H}$.

1. The real sphere of dimension $n$ will be denoted by $\mathbb{S}^n$. Let

$$A_n = \frac{\mathbb{R}[X_0, X_1, \ldots, X_n]}{(\sum_{i=0}^{n} X_i^2 - 1)} = \mathbb{R}[x_0, x_1, \ldots, x_n]$$

denote the ring of algebraic functions on $\mathbb{S}^n$.

2. For any real affine variety $X = \text{Spec}(A)$, let $\mathbb{R}(X) = S^{-1}A$, where $S$ is the multiplicative set of all $f \in A$ that do not vanish at any real point of $\text{Spec}(A)$. Also, $X(\mathbb{R})$ denote the set of all real points of $X$.

3. For any noetherian commutative ring $A$ and line bundles $L$ on $\text{Spec}(A)$, the Euler class group will be denoted by $E(A, L)$ and the weak Euler class group will be denoted by $E_0(A, L)$. Usually, $E(A, A)$ will be denoted by $E(A)$ and similarly $E_0(A)$ will denote $E_0(A, A)$. We refer to [BRS2] for the definitions and the basic properties of these groups.

The following theorem would be obvious to the experts (see [BRS1]).

**Theorem 2.2.** Let $X = \text{spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$. Then, the natural map

$$E_0(\mathbb{R}(X)) \to CH_0(\mathbb{R}(X))$$

is an isomorphism and $CH_0(\mathbb{R}(X)) \approx \mathbb{Z}/(2)^r$ where $r$ is the number of compact connected components of $X(\mathbb{R})$. 


Proof. It follows directly from [BRS1, Theorem 5.5] and Theorem 2.3 below that $E_0(\mathbb{R}(X)) \sim CH_0(\mathbb{R}(X))$. Also, by [BRS1, Theorem 4.10], $CH_0(\mathbb{R}(X)) \approx \mathbb{Z}/(2)^r$. $\square$

Theorem 2.3. (See [BDM].) Let $X = \text{spec}(A)$ be smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$. The following diagram of exact sequences

$$
\begin{array}{cccccc}
0 & \rightarrow & E(\mathbb{C}(L)) & \rightarrow & E(A, L) & \rightarrow & E(\mathbb{R}(X), L) & \rightarrow & 0 \\
\downarrow & & \downarrow \varphi & & \downarrow \psi & & \downarrow & & \\
0 & \rightarrow & CH(C) & \rightarrow & CH_0(A) & \rightarrow & CH_0(\mathbb{R}(X)) & \rightarrow & 0
\end{array}
$$

commute and the first vertical map $\varphi$ is an isomorphism.

Proof. We only need to prove that $\varphi$ is injective. The proof is given in the proof of [BDM, Proposition 4.29]. $\square$

We also include the following easy lemma.

Lemma 2.4. Let $A$ be any smooth affine ring over $\mathbb{R}$ with $\dim A = n \geq 2$ and $L$ be a line bundle on $\text{Spec}(A)$. Let $P$ be a projective $A$-module of rank $n$ and $\det P = L$. Let $\chi, \eta : L \rightarrow \wedge^n P$ be two orientations. Suppose $e(P, \chi) = (I, \omega)$ where $I$ is an ideal of height $n$ and $\omega$ is a local orientation on $I$ and $\eta = u\chi$ where $u$ is a unit in $A$. Then $e(P, \eta) = (I, u\omega)$.

Proof. Write $F = L \oplus A^{n-1}$. By theorem in [BRS2], there is a surjective map $f : F \rightarrow I$ that induces $(I, \omega)$ as in the commutative diagram:

$$
\begin{array}{cccccc}
P & \rightarrow & P/IP & \rightarrow & F/IF \\
\downarrow f & & \downarrow \gamma \sim \chi & & \downarrow 1 \\
I & \rightarrow & I/I^2 & \rightarrow & F/IF.
\end{array}
$$

Here $\gamma$ is an isomorphism with determinant $\chi$, and $\delta$ is any isomorphism with $\det(\delta) = u$. So $\gamma \delta \sim u\chi = \eta$ and $e(P, \eta) = (I, u\omega)$. $\square$

3. The tangent bundle

It is well known that the tangent bundle $T_n$, over the real sphere $S^n$, of even dimension $n \geq 1$, does not have a nowhere vanishing section. The purpose of this section is to compute the Euler class of the algebraic tangent bundle explicitly.

First, note that all line bundles over $S^n$, with $n \geq 2$ are trivial, we have only one Euler class group $E(A_n, A_n)$ to be denoted by $E(A_n)$. Similarly, we have only one weak Euler class group $E_0(A_n)$. The following proposition entails some of the basic facts about Euler class groups of the spheres.

Proposition 3.1. The Euler class group of the sphere is given by $E(A_n) = \mathbb{Z}$, generated by $(m, \omega)$ where $m$ is any real maximal ideal and $\omega$ is any local orientation of $m$. Similarly, the weak Euler class group is given by

$$
E_0(A_n) \approx CH_0(A_n) = \frac{\mathbb{Z}}{(2)}.
$$
**Proof.** From Theorem 2.3, we have the commutative diagram of exact sequences:

\[
\begin{array}{cccccc}
0 & \rightarrow & E^C(A_n) & \rightarrow & E(A_n) & \rightarrow & E(\mathbb{R}(\mathbb{S}^n)) & \rightarrow & 0 \\
\downarrow \varphi & & \downarrow \Theta & & \downarrow & & \downarrow \\
0 & \rightarrow & CH(\mathbb{C}) & \rightarrow & CH_0(A_n) & \rightarrow & CH_0(\mathbb{R}(\mathbb{S}^n)) & \rightarrow & 0.
\end{array}
\]

Since complex points in \( A_n \) are complete intersection [MS, Lemma 4.2], we have \( CH(\mathbb{C}) = E^C(A_n) = 0 \) and the above diagram reduces to

\[
\begin{array}{cccccc}
0 & \rightarrow & E(A_n) & \sim & E(\mathbb{R}(\mathbb{S}^n)) & \rightarrow & 0 \\
\downarrow \Theta & & \downarrow & & \downarrow & & \\
0 & \rightarrow & CH_0(A_n) & \sim & CH_0(\mathbb{R}(\mathbb{S}^n)) & \rightarrow & 0.
\end{array}
\]

We have by Theorem 2.2, \( E_0(A_n) \sim CH_0(A_n) \). Therefore, by [BRS1, Theorems 4.13, 4.10]

\[
E(\mathbb{R}(\mathbb{S}^n)) = \mathbb{Z} \quad \text{and} \quad CH_0(A_n) \approx E_0(\mathbb{R}(\mathbb{S}^n)) = \mathbb{Z}/(2).
\]

The proof is complete. \( \square \)

The following definition will be convenient for subsequent discussions.

**Definition 3.2.** Let \( m_0 = (x_0 - 1, x_1, \ldots, x_n) \) be the maximal ideal in \( A_n \) that corresponds to the real point \( (1, 0, \ldots, 0) \in \mathbb{S}^n \). Write \( F = A_n^\mathbb{R} \) and let \( e_1, \ldots, e_n \) be the standard basis of \( F \). Define local orientation

\[
\omega_0 : F/m_0 F \rightarrow m_0/m_0^2 \quad \text{where for } i = 1, \ldots, n, \quad \omega(e_i) = \text{image}(x_i).
\]

By Proposition 3.1, \((m_0, \omega_0)\) will generate the Euler class group \( E(A_n) = \mathbb{Z} \). This generator \((m_0, \omega_0) = 1\) will be called the **standard generator** of \( E(A_n) \). Similarly, the class of \( m_0 = 1 \) will be called the **standard generator** of \( E_0(A_n) = \mathbb{Z}/(2) \).

Unless stated otherwise, we use these standard generators in our subsequent discussions.

We compute the Euler class of the algebraic tangent bundle over \( \text{Spec}(A_n) \) as follows.

**Theorem 3.3.** Let \( T_n \) be the projective \( A_n \)-module corresponding to the tangent bundle over \( \mathbb{S}^n \). There is an orientation \( \chi : A_n \sim \wedge^n T_n \) such that, if \( n \geq 2 \) is even, then the Euler class \( e(T_n, \chi) = -2 \in E(A_n) \) and if \( n \geq 3 \) is odd, then the Euler class \( e(T_n, \chi) = 0 \in E(A_n) \).

**Proof.** Write \( m_0 = (x_0 - 1, x_1, \ldots, x_n), m_1 = (x_0 + 1, x_1, \ldots, x_n) \in \text{Spec}(A_n) \). Then \( m_0, m_1 \) correspond, respectively, to the points \( (1, 0, \ldots, 0), (-1, 0, \ldots, 0) \) in \( \mathbb{S}^n \). We have

\[
m_0 = (x_1, \ldots, x_n) + m_0^2, \quad m_1 = (x_1, \ldots, x_n) + m_1^2,
\]

and \( m_0 \cap m_1 = (x_1, \ldots, x_n) \). Write \( F = A_n^\mathbb{R} \) and let \( e_1, \ldots, e_n \) be the standard basis. For \( j = 0, 1 \) we define local orientations

\[
\omega_j : F/m_j F \rightarrow m_j/m_j^2 \quad \text{where for } i = 1, \ldots, n, \quad \omega_j(e_i) = \text{image}(x_i).
\]
Therefore, \((m_0, \omega_0) = 1\) is the standard generator of \(E(A_n) = \mathbb{Z}\). We write \(J = m_0 \cap m_1 = (x_1, \ldots, x_n)\) and define the surjective map

\[
\alpha : F \to J \quad \text{where for } i \geq 1, \quad \alpha(e_i) = x_i.
\]

Then, \(\alpha\) induces the local orientation

\[
\omega : F/JF \to J/J^2 \quad \text{where } \omega(e_i) = \text{image}(x_i).
\]

Since \((J, \omega)\) is global, it follows

\[
(m_0, \omega_0) + (m_1, \omega_1) = (J, \omega) = 0.
\]

Hence

\[
(m_1, \omega_1) = -(m_0, \omega_0) = -1.
\]

Since \(E(A_n) = E(\mathbb{R}(S^n))\), we can apply [BDM, Lemma 4.2] and we have

\[
(m_1, \omega_1) + (m_1, -\omega_1) = 0.
\]

Therefore

\[
(m_0, \omega_0) + (m_1, -\omega_1) = 2(m_0, \omega_0) = 2.
\]

Let \(D = \text{diagonal}(-x_0, 1, \ldots, 1) : F/JF \to F/JF\), then \(D\) is an automorphism and \(\det(D) = \text{image}(-x_0)\). Now, let \(\eta = \omega D : F/JF \to J/J^2\). In fact,

\[
\eta(e_1) = \text{image}(-x_0x_1) \quad \text{and} \quad \eta(e_i) = \text{image}(x_i) \quad \forall i > 1.
\]

Note that

\[
D = \text{diagonal}(-1, 1, \ldots, 1) \mod m_0, \quad D = \text{Id} \mod m_1.
\]

Since, \(\omega_i\) are the reductions of \(\omega\) modulo \(m_i\) we have

\[
(J, \eta) = (m_0, -\omega_0) + (m_1, \omega_1) = -(m_0, \omega_0) + (m_1, -\omega_1) = -2.
\]

Now we apply [BRS2, Lemma 5.1], to \(\alpha : F \to J\), with \(a = b = \text{image}(-x_0)\). We have the following:

1. Define \(T\) by the exact sequence

\[
0 \to T \to A_n \oplus F = A^{n+1} \xrightarrow{\Phi} A_n \to 0
\]

where

\[
\Phi = -(x_0, x_1, \ldots, x_n) = (b, -\alpha).
\]

2. We have \((J, \omega)\) is obtained from \((\alpha, \chi_0 = \text{Id}_{A_n})\).

3. By [BRS2, Lemma 5.1], \(T\) has an orientation \(\chi : A_n \to \bigwedge^n T\) such that

\[
e(T, \chi) = (J, \text{image}(-x_0)^{n-1} \omega) = (m_0, (1)^{n-1} \omega_0) + (m_1, \omega_1).
\]
4. If \( n \) is **EVEN**, we have
\[
e(T, \chi) = (m_0, -\omega_0) + (m_1, \omega_1) = -2.
\]

And if, \( n \) is **ODD**, we have
\[
e(T, \chi) = (m_0, \omega_0) + (m_1, \omega_1) = 0.
\]

5. Note that \( T = \ker(\Phi) \approx \ker(-\Phi) = T_n \) is the tangent bundle.

So, the proof is complete. \( \Box \)

4. An algorithmic computation in \( E(A_n) \)

**Lemma 4.1.** Let \( A_n \) be as above and let \( m_1, M_1, m_2, M_2, \ldots, m_N, M_N \in \text{spec}(A_n) \) be a set of distinct maximal ideals that correspond to distinct real points in \( S^n \). We will assume that these points are in \( S^1 = \{ x_j = 0: \forall j \geq 2 \} \subseteq S^n \). For \( i = 1, \ldots, N \), let \( L_i = 0, x_2 = 0, \ldots, x_n = 0 \) be the line passing through the pair of points corresponding to \( m_i \) and \( M_i \). Then
\[
\bigcap_{i=1}^{N} (m_i \cap M_i) = \left( \prod_{i=1}^{N} L_i, x_2, \ldots, x_n \right).
\]

**Proof.** Let \( J \) denote the right-hand side. Claim that
\[
J \subseteq m \in \text{Spec}(A_n) \quad \Rightarrow \quad m = m_i \quad \text{or} \quad m = M_i \quad \text{for some} \quad i.
\]

To see this, note for such an \( m \), we have \( L_i \in m \) for some \( i \). Therefore,
\[
m_i \cap M_i = (L_i, x_2, \ldots, x_n) \subseteq m.
\]

Hence \( m = m_i \) or \( M_i \). Let \( m \) be such a maximal ideal and assume \( m = m_i \). We have, \( L_j \notin m_i \ \forall j \neq i \) and \( J_{m_i} = (L_i, x_2, \ldots, x_n)_{m_i} = (m_i)_{m_i} \). The proof is complete. \( \Box \)

Given various points \( m \) in \( S^n \), the following is an algorithm to compute class \((m, \omega) \in E(A_n)\).

**Theorem 4.2.** As in Definition 3.2, let \( (m_0, \omega_0) = 1 \in E(A_n) = \mathbb{Z} \) be the standard generator. Let \( p = (a, b, 0, \ldots, 0) \) be a point in \( S^n \) and let \( M = (x_0 - a, x_1 - b, x_2, \ldots, x_n) \in \text{Spec}(A_n) \) be the maximal ideal corresponding to \( p \). Assume \( m_0 \neq M \) and so \( a \neq 1 \). Let
\[
L = (1-a)x_1 + b(x_0 - 1), \quad \text{so} \quad (L, x_2, x_3, \ldots, x_n) = m_0 \cap M.
\]

As in (3.2), \( F = A_n^* \) and \( e_1, \ldots, e_n \) is the standard basis of \( F \). Define
\[
\omega_M : F/\mathbb{Z}F \rightarrow M/\mathbb{Z}M^2 \quad \text{by} \quad \omega_M(e_1) = x_1 - b, \quad \omega_M(e_i) = x_i \quad \forall i \geq 2.
\]

If \( a \neq 0 \) (i.e. \( p \) is not the north or the south pole), then \( \omega_M \) is a surjective map and
\[
(m_0, \omega_0) + (M, -\text{sign}(a)\omega_M) = 0.
\]

So, if \( a > 0 \) then \( (M, \omega_M) = 1 \) and \( a < 0 \) then \( (M, \omega_M) = -1 \).
Proof. Define the surjective map
\[ f : F \rightarrow m_0 \cap M \] by \( f(e_i) = L, \quad f(e_i) = x_i \forall i \geq 2. \)
We will see that \( f \) reduces to \( \omega_0 \) modulo \( m_0 \). With \( s = -1/2, t = 1/2 \) we have, \( 1 = s(x_0 - 1) + t(x_0 + 1) \). So
\[
(x_0 - 1) = s(x_0 - 1)^2 + t(x_0^2 - 1) = s(x_0 - 1)^2 + t \sum_{i=1}^{n} -x_i^2 \in m_0^2.
\]
Therefore \( L - (1 - a)x_1 \in m_0^2 \). Also since \( M \neq m_0 \) we have \( a \neq 1 \). In fact \( a < 1 \). Hence \( f \) reduces to
\[
\omega_0 : F/m_0 F \rightarrow m_0^2.
\]
Now define
\[
\gamma_M : F/ MF \rightarrow M/ M^2 \] by \( \gamma_M(e_i) = L, \quad \gamma_M(e_i) = x_i \forall i \geq 2. \)
Since \( \gamma_M \) is the reduction of \( f \), we have
\[
(m_0, \omega_0) + (M, \gamma_M) = 0.
\]
Let \( \omega_M : F/ MF \rightarrow M/ M^2 \) be as in the statement of the theorem. We will assume \( a \neq 0 \) or equivalently, \( -1 < b < 1 \). In this case, we prove that \( \omega_M \) is a surjective map. (Note below that for \( \omega_M \) to be surjective, we need \( a \neq 0 \).) We have
\[
L = (1 - a)x_1 + b(x_0 - 1) = (1 - a)(x_1 - b) + b(x_0 - a).
\]
We also have \( a^2 + b^2 - 1 = 0 \). Now again, with \( s = -1/2a, t = 1/2a \) we have \( 1 = s(x_0 - a) + t(x_0 + a) \) and
\[
(x_0 - a) = s(x_0 - a)^2 + t(x_0^2 - a^2) = s(x_0 - a)^2 + t(b^2 - x_1^2) - t \sum_{i=2}^{n} x_i^2.
\]
Therefore, \( \omega_M \) is surjective. Further,
\[
(b^2 - x_1^2) = (b - x_1)(b + x_1) = (b - x_1)[2b - (b - x_1)] = 2b(b - x_1) - (b - x_1)^2.
\]
So,
\[
(x_0 - a) = s(x_0 - a)^2 + t[2b(b - x_1) - (b - x_1)^2] - t \sum_{i=2}^{n} x_i^2 = b/a(b - x_1) + w
\]
for some \( w \in M^2 \). So,
\[
(x_0 - a) - b/a(b - x_1) \in M^2.
\]
Therefore, modulo \( M \), we have
\[
L = (1 - a)(x_1 - b) + b(x_0 - a) \equiv (1 - a)(x_1 - b) + b(b/a)(b - x_1)
\]
or
\[ L ≡ (x_1 - b)[1 - a - b^2/a] = (x_1 - b)(a - 1)/a. \]

So, \( γ_M \) and \( ω_M \) differ by an isomorphism of determinant \((a - 1)/a\). Since \( a - 1 < 0 \), we have \( γ_M = -\text{sign}(a)ω_M \). Therefore,

\[ (m_0, ω_0) + (M, -\text{sign}(a)ω_M) = 0. \]

Hence, if

\[ a > 0 \quad \Rightarrow \quad (M, ω_M) = -(M, -ω_M) = (m_0, ω_0) = 1 \]

and

\[ a < 0 \quad \Rightarrow \quad (M, ω_M) = -(m_0, ω_0) = -1. \]

So, the proof is complete. □

**Remark 4.3.** We will continue to use the notations of (3.2, 4.2). As we remarked in the proof of Theorem 4.2, if \( p \) is the north pole or the south pole and \( M \) is the corresponding maximal ideal, then \( ω_M \), as defined in 4.2, will fail to define a local orientation. If \( p = (0, ±1, 0, \ldots, 0) \) is the north or the south pole, then \( M = (x_0, x_1 ± 1, x_2, \ldots, x_n) \). For \( p = N \) the north pole or \( p = S \) the south pole, a natural local orientation is defined by:

\[ ω_p : F/MF → M/M^2 \quad \text{where} \quad ω_p(e_1) = x_0, \quad ω_p(e_i) = x_i \quad ∀i ≥ 2. \]

Then \((M, ω_p) = -1 \) if \( p \) is the north pole and \((M, ω_p) = 1 \) if \( p \) is the south pole.

**Proof.** Let \( p = N = (0, 1, 0, \ldots, 0) \) be the north pole. Then \( M = (x_0, x_1 - 1, x_2, \ldots, x_n) \). Write \( L = x_0 + x_1 - 1 \). Then \( m_0 ∩ M = (L, x_2, \ldots, x_n) \). Consider the surjective map \( f : F → m_0 ∩ M \) given by these generators. Note that \( L = x_0 = x_1 - 1 \in M^2 \). This follows because \( 1 = -(x_1 - 1)/2 + (x_1 + 1)/2 \). So, it follows that \( f \) reduces to \( ω_p \). Similarly, \( f \) reduces to \( ω_0 \) on \( m_0 \). Therefore, \((M, ω_p) = -1 \).

If \( p = S = (0, -1, 0, \ldots, 0) \) is the south pole, then \( M = (x_0, x_1 + 1, x_2, \ldots, x_n) \). We replace the equation of \( L \) by \( L = x_0 - x_1 - 1 \). Then \( L = x_0 = -(x_1 + 1) \in M^2 \). It follows, that \( f \) reduced to \( ω_p \). Similarly, \( L = x_0 - x_1 - 1 \in m_0^2 \). This shows that \( f \) reduces to \(-ω_0 \) on \( m_0 \). Therefore, \((M, ω_p) - (m_0, ω_0) = 0 \). So, \((M, ω_p) = 1 \). The proof is complete. □

**Remark 4.4.** In the statement of (4.2), we assumed that \( p = (a, b, 0, \ldots, 0) \in S^1 \subseteq S^n \). Now suppose \( p \notin S^1 \) is any point in \( S^n \). Let \( e_0, \ldots, e_n \) be the standard basis of \( IR^n \). So, \( m_0 \) is the ideal of \( e_0 \). There is an orthonormal transformation \((E_0, \ldots, E_n)^t = (e_0, \ldots, e_n)^t \) of \( IR^{n+1} \) such that \( E_0 = e_0 \) and \( p = aE_0 + bE_1 \). Write \( A = (a_{ij}; i, j = 0, \ldots, n) \). It follows, \( a_{00} = 1 \) and \( a_{ij} = a_{ji} = 0 \) for all \( j = 1, \ldots, n \). We can assume \( \det A = 1 \).

Write \((Y_0, \ldots, Y_n)^t = A(X_0, \ldots, X_n)^t \). Then \( Y_0 = X_0 \), and for \( i = 1, \ldots, n \) we have \( Y_i = \sum_{j=1}^n a_{ij}X_j \). It follows that in the \( Y \)-coordinates, \( e_0 = (1, 0, \ldots, 0) \) and \( p = (a, b, 0, \ldots, 0) \). Let \( ω' \) be the local orientation on \( m_0 \) defined by \((Y_1, \ldots, Y_n) \). Since \( \det A = 1 \), it follows that \((m_0, ω') \) is the standard generator of \( E(A_n) \) (see 3.2). Now, we can write down local orientation on \( M \) in \( Y \)-coordinates, as in (4.2) and the rest of (4.2) remains valid.
5. Bott periodicity

In this section, we will give some background on Bott periodicity, mostly from [ABS,F,Sw1]. We will recall the definition of the Clifford algebras of a quadratic forms.

**Definition 5.1.** (See [ABS].) Let $k$ be a commutative ring and $(V,q)$ be a quadratic $k$-module. Then a $k$-algebra $C(q)$ with an injective map $i : V \rightarrow C(q)$ is said to be the Clifford algebra of $q$, if $i(x)^2 = q(x)$ and if it is universal with respect to this property. Following are some of the properties of $C(q)$:

1. Note $C(q) = \frac{T(V)}{I(q)}$ where $T(V)$ is the tensor algebra of $V$ and $I(q)$ is the two-sided ideal of $T(V)$ generated by $\{x^2 - q(x) : x \in V\}$.
2. The $\mathbb{Z}_2$-grading on $T(V)$ induces a $\mathbb{Z}_2$-grading on $C(q)$ as $C(q) = C_0(q) \oplus C_1(q)$ where $C_0(q)$ denotes the even part and $C_1(q)$ denotes the odd part.
3. Also, if $(V,q')$ is another quadratic $k$-module, then
   
   $$C(q \perp q') \cong C(q) \otimes C(q')$$  

   as graded rings.

   This means, the multiplication structure is given by
   $$(u \otimes x_i)(y_j \otimes v) = (-1)^{ij}u y_j \otimes x_i v$$ for $x_i \in C_1(q'), y_j \in C_1(q).

4. If $V = \bigoplus_{i=1}^n k e_i$ is free with basis $e_i$, then
   $$C(q) = \bigoplus_{0 \leq i_1 < \cdots < i_r \leq n; r \geq 0} k e_{i_1} e_{i_2} \cdots e_{i_r}.$$  

   We will mostly be concerned with this case where $V$ is free. Further, if $q = \sum_{i=1}^n a_i X_i^2$ is a diagonal form, then
   $$\forall i, j = 1, \ldots, n; \text{ with } i \neq j, \text{ } e_i^2 = a_i \text{ and } e_i e_j = -e_j e_i.$$  

**Notations 5.2.** We will introduce some notations for our convenience.

1. Let $k$ be a commutative ring and $(V,q)$ be a quadratic $k$-module and $V = \bigoplus_{i=1}^n k e_i$ is free and $q = q(X_1, \ldots, X_n)$. As in [Sw1], we denote $R_k(q) = R(q) = \frac{k[X_1, \ldots, X_n]}{(q-1)}$. We usually drop the subscript $k$ and use the notation $R(q)$.
2. Suppose $C$ is a ring. Then:
   (a) The category of finitely generated (left) $C$-modules will be denoted by $\mathcal{M}(C)$.
   (b) If $C$ has a $\mathbb{Z}_2$-grading, the category of finitely generated (left) $\mathbb{Z}_2$-graded $C$-modules will be denoted by $\mathcal{G}(C)$.
   (c) The category of finitely generated (left) projective $C$-modules will be denoted by $\mathcal{P}(C)$.
3. Given a category $\mathcal{C}$, with exact sequences, the Grothendieck group of $\mathcal{C}$ will be denoted $K(\mathcal{C})$.
4. Given a ring $R$, we will denote $K_0(R) = K(\mathcal{P}(R))$. If the rank map $\text{rank} : K_0(R) \rightarrow \mathbb{Z}$ is defined, we denote $K_0(R) = \text{rank}^{-1}(0)$.
5. Given a connected smooth real manifold $X$, the Grothendieck group of the category of real vector bundles over $X$ will be denoted by $KO(X)$. As above, $KO(X)$ will denote the kernel of the rank map.
6. For a commutative noetherian ring $R$ of dimension $n$ and $X = \text{Spec}(R)$, the Chow group of zero cycles will be denoted by $CH_0(R)$ or $CH_0(X)$. When the top Chern class is defined, $C_0 = C^n : K_0(R) \rightarrow CH_0(R)$ will denote the homomorphism defined by the top Chern class.

5.1. Generators of $\tilde{K}_0(A_n)$

In this subsection, we describe the generators of $\tilde{K}_0(A_n)$. 

Proposition 5.3. Let \( k \) be ring with \( 1/2 \in k \) and let \( q = a_1 X_1^2 + \cdots + a_n X_n^2 \) be a diagonal form. Let \( e_1, \ldots, e_n \) denote the canonical generators of \( C(-q) \). Let \( M = M^0 \oplus M^1 \in \mathcal{G}(C(-q)) \) be a \( \mathbb{Z}_2 \)-graded \( C(-q) \)-module and
\[
N = R(q \perp 1) \otimes_k M = N^0 \oplus N^1
\]
where
\[
N^0 = R(q \perp 1) \otimes_k M^0, \quad N^1 = R(q \perp 1) \otimes_k M^1.
\]
Let \( x_i \) denote the image of \( X_i \) in \( R(q \perp 1) \). Define
\[
\varphi(x) = \sum_{i=1}^n x_i (1 \otimes e_i) : N^1 \to N^0, \quad \psi(x) = \sum_{i=1}^n x_i (1 \otimes e_i) : N^0 \to N^1.
\]
Write \( q \perp 1 = q + X_0^2 \) and let \( y = x_0 = \text{image}(X_0) \in R(q \perp 1) \). Define
\[
\rho_M = \rho = \frac{1}{2} \begin{pmatrix}
1 - y & \varphi(x) \\
-\psi(x) & 1 + y
\end{pmatrix} : N \to N.
\]
That means, for \( n_0 \in N^0, n_1 \in N^1 \) we have
\[
\rho(n_0, n_1) = ((1 - y)n_0 + \varphi(x)n_1, -\psi(x)n_0 + (1 + y)n_1)/2.
\]
Then,
\[
\varphi \psi(x) = -q(x) : N^0 \to N^0, \quad \psi \varphi(x) = -q(x) : N^1 \to N^1
\]
and \( \rho \) is an idempotent homomorphism.

Proof. By direct multiplication, it follows \( \varphi \psi = -q(x), \psi \varphi = -q(x) \). Again, we have
\[
\rho^2 = \frac{1}{4} \begin{pmatrix}
(1 - y)^2 - \varphi(x)^2 & 2\varphi(x) \\
-2\psi(x) & -\psi(x)\varphi(x) + (1 + y)^2
\end{pmatrix}
\]
which is
\[
\frac{1}{4} \begin{pmatrix}
(1 - y)^2 + q(x) & 2\varphi(x) \\
-2\psi(x) & q(x) + (1 + y)^2
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
2(1 - y) & 2\varphi(x) \\
-2\psi(x) & 2(1 + y)
\end{pmatrix} = \rho.
\]
This completes the proof. \( \square \)

Definition 5.4. We use the notation as in Proposition 5.3. Define a functor
\[
\alpha : \mathcal{G}(C(-q)) \to \mathcal{P}(R(q \perp 1)) \quad \text{by} \quad \alpha(M) = \ker(\rho_M).
\]
Since, \( k \to R(q \perp 1) \) is flat, it follows easily that \( \alpha \) is an exact functor. Therefore, \( \alpha \) induces a homomorphism
\[
\Theta_q : K(\mathcal{G}(C(-q))) \to \mathcal{K}_0(R(q \perp 1))
\]
where \( \forall M \in \mathcal{G}(C(-q)) \)
\[
\Theta_q([M]) = [\alpha(M)] - \text{rank}(\alpha(M)).
\]
Before we proceed, we will describe \( \alpha(M) \) in (5.4) by patching two trivial bundles on the two (algebraic) hemispheres along the (algebraic) equator, as follows.

**Proposition 5.5.** We will use all the notations of (5.3, 5.4). We have \( q = q(X_1, \ldots, X_n), q \perp 1 = q + X_0^2 \) and \( y = x_0 = \text{image}(X_0) \). Let \( M = M^0 \oplus M^1 \in G(C(-q)), N = N^0 \oplus N^1, \varphi, \psi \) be as in (5.3). Write

\[
F^0 = N^0_{1+y} = R(q + X_0^2)_{1+y} \otimes M^0, \quad F^1 = N^1_{1-y} = R(q + X_0^2)_{1-y} \otimes M^1.
\]

Then \( \alpha(M) \) is obtained by patching \( F^0 \) and \( F^1 \) via \( \psi_{1-y^2} \). In particular, if \( k \) is a field, \( \text{rank}(\alpha(M)) = \dim_k M/2 = \dim_k M_0 \).

**Proof.** Define

\[
\sigma : F^0_{1-y} \to F^1_{1+y} \quad \text{by} \quad \sigma(n_0) = \frac{-\psi(n_0)}{1+y}
\]

and

\[
\eta : F^1_{1+y} \to F^0_{1-y} \quad \text{by} \quad \sigma(n_1) = \frac{\varphi(n_1)}{1-y^2}
\]

Then for \( n_0 \in F^0_{1-y} \), we have

\[
\eta \sigma(n_0) = \frac{-\varphi \psi(n_0)}{1-y^2} = \frac{q(x_1, \ldots, x_n)(n_0)}{1-y^2} = n_0.
\]

So, \( \eta \sigma = 1 \) and similarly, \( \sigma \eta = 1 \). Consider fiber product

\[
\begin{array}{ccc}
R(q \perp 1) & \longrightarrow & R(q \perp 1)_{1-y} \\
\downarrow & & \downarrow \\
R(q \perp 1)_{1+y} & \longrightarrow & R(q \perp 1)_{1-y^2}
\end{array}
\]

and define \( P(\sigma) \) by the patching diagram

\[
\begin{array}{ccc}
P(\sigma) & \longrightarrow & F^1 \\
\downarrow & & \downarrow \\
F^0 & \longrightarrow & F^0_{1-y} \quad \longrightarrow \quad F^1_{1+y}.
\end{array}
\]

Define

\[
f_0 : F^0 = N^0_{1+y} \to \alpha(M)_{1+y} \quad \text{by} \quad f_0(n_0) = \left( n_0, \frac{\psi(n_0)}{1+y} \right)
\]

and

\[
f_1 : F^1 = N^1_{1-y} \to \alpha(M)_{1-y} \quad \text{by} \quad f_1(n_1) = \left( \frac{-\varphi(n_1)}{1-y}, n_1 \right).
\]
We check that \( f_0, f_1 \) are well-defined isomorphisms. Recall that
\[
\rho_M = \rho = \frac{1}{2} \begin{pmatrix} 1 - y & \varphi(x) \\ -\psi(x) & 1 + y \end{pmatrix}.
\]

Using the identities \( q(x) + y^2 = 1, \varphi\psi = -q, \psi\varphi = -q \), direct computation shows
\[
f_0(n_0) = \left( n_0, \frac{\psi(n_0)}{1 + y} \right), \quad f_1(n_1) = -\left( \frac{\psi(n_1)}{1 - y}, n_1 \right) \in \ker(\rho) = \alpha(M).
\]

So, \( f_0, f_1 \) are well defined. Clearly, \( f_0, f_1 \) are injective and their surjectivity can also be checked directly. Now consider the patching diagram:

\[
\begin{array}{c}
P(\sigma) \\
\downarrow f \\
\downarrow \alpha(M) \\
\downarrow f_0 \\
\downarrow \alpha(M)_{1+y} \\
\downarrow \sigma \\
F_0 \\
\downarrow \alpha(M)_{1+y} \\
\downarrow \alpha(M)_{1-y^2} \\
\downarrow \sigma \\
\downarrow \alpha(M)_{1-y^2} \\
\downarrow \alpha(M) \\
\end{array}
\]

We check \( f_1\sigma = f_0 \). For \( n_0 \in F_0^{1-y} \), we have
\[
f_1\sigma(n_0) = -\left( \frac{-\varphi(n_0)}{1 - y}, \frac{-\psi(n_0)}{1 + y} \right)
= -\left( \frac{-q(x)}{1 - y^2}, \frac{-\psi(n_0)}{1 + y} \right) = \left( n_0, \frac{\psi(n_0)}{1 + y} \right) = f_0(n_0),
\]
since \( q(x) = 1 - y^2 \). In this patching diagram above, \( f \) is obtained by properties of fiber product diagrams. Now, since \( f_0, f_1 \) are isomorphisms, \( f : P(\sigma) \to \alpha(M) \) is also an isomorphism. Let \( P(\psi_{1-y^2}) \) denote the projective module obtained by patching \( F_0 \) and \( F_1 \) via \( \psi_{1-y^2} \). Since \( P(\sigma) \approx P(\psi_{1-y^2}) \), the proposition is established. \( \square \)

### 5.2. Further background on Bott periodicity

For the benefit of the readership, in this subsection, we give some further background on Bott periodicity from \([ABS,Sw1]\). We establish that \( \Theta_q \) defined in (5.4) is a surjective homomorphism, when \( q = \sum_{i=1}^n X_i^2 \in \mathbb{H}[X_1, \ldots, X_n] \). In this case, \( R(q \perp 1) = A_n \). We have the following proposition.

**Proposition 5.6.** (See \([ABS]\).) We continue to use notations as in (5.3, 5.4). The composition
\[
K(G(C(\perp q \perp -1))) \longrightarrow K(G(C(\perp q))) \stackrel{\Theta_q}{\longrightarrow} K_0(R(q \perp 1))
\]
is zero. Further, as in [ABS,Sw1], define \( ABS(q) \) by the exact sequence
\[
K(G(C(−q ⊥ 1))) → K(G(C(−q))) → ABS(q) → 0.
\]
So, there is a homomorphism \( α_q : ABS(q) → \overline{KO}(R(q ⊥ 1)) \) such that the diagram

\[
\begin{array}{ccc}
K(G(C(−q ⊥ 1))) & → & K(G(C(−q))) & → & ABS(q) & → & 0 \\
\uparrow{α_q} & & \downarrow{α_q} & & & & \downarrow{0} \\
\overline{KO}(R(q ⊥ 1)) & & & & & & \\
\end{array}
\]

commute.

**Proof.** We reinterpret the proof of Swan [Sw1, 7.7], and sketch a direct proof. Let \( e_1, \ldots, e_n \) denote the canonical generators of \( C(−q) \) and \( f \) be the other generator of \( C(−q ⊥ 1) \). Let \( M = M^0 ⊕ M^1 ∈ G(C(−q − Z^2)) \). Write \( N = M ⊗ R(q ⊥ X^2_0) \) and define \( f^* : M → M \) such that \( f^*_|M^0 = f|_{M^0}, f^*_|M^1 = − f|_{M^1} \) and similarly define \( e^*_i \).

With the notations as in (5.3, 5.4), we have \( ρ = \frac{1−γ^2}{2} \), where
\[
γ = \begin{pmatrix} y & −φ \\ ψ & −y \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & −y \end{pmatrix} + \sum_{i=1}^{n} x_i e^*_i.
\]
So, \( α(M) = \{ w ∈ N : w = γ(w) \} \). Also note \( γ^2 = 1 \). Define
\[
L^0 = ker(f^* − 1) = \{ (m_0, fm_0) : m_0 ∈ M^0 \} = \{ (− fm_1, m_1) : m_1 ∈ M^1 \}
\]
and
\[
L^1 = ker(f^* + 1) = \{ (fm_1, m_1) : m_1 ∈ M^1 \} = \{ (m_0, − fm_0) : m_0 ∈ M^0 \}.
\]
So, \( M = L^0 ⊕ L^1 \) and \( N = Q^0 ⊕ Q^1 \) with \( Q^0 = L^0 ⊗ R(q ⊥ X^2_0), Q^1 = L^1 ⊗ R(q ⊥ X^2_0) \). We check that \( diagonal(1, −1)L^0 ⊆ L^1 \) and \( e^*_i L^0 ⊆ L^1 \). So, \( γ(Q^0) ⊆ Q^1 \) and similarly, \( γ(Q^1) ⊆ Q^0 \).

We have \( Q^0 ∩ α(M) ⊆ Q^0 ∩ Q^1 = 0 \), and for \( n = n^0 + n^1 \) with \( n^i ∈ Q^i \), we have \( n = (n^0 − γ(n^1)) + (n^1 + γ(n^1)) ∈ Q^0 + α(M) \). So, \( N = Q^0 ⊕ α(M) \) and \( α(M) ≅ N/Q^0 = Q^1 \) is free. The proof is complete. □

The following theorem relates topological and algebraic \( K \)-groups.

**Theorem 5.7.** (See [ABS,F,Sw1].) Let \( q = X^2_1 + \cdots + X^2_n ∈ \mathbb{R}[X] \). Then the following is a commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
ABS(q) & → & \overline{KO}(S^n) \\
α_q \downarrow & & \downarrow \sim \\
\overline{KO}(A_n) & → & \overline{KO}(S^n).
\end{array}
\]

In particular, the homomorphism
\[
θ_q : K(G(C(−q))) → \overline{KO}(R(q ⊥ 1)) \text{ is surjective.}
\]
Proof. Note that $R(q \perp 1) = \frac{R[X_0, X_1, \ldots, X_n]}{(X_0^2 + q - 1)} = A_n$. The diagonal isomorphism is established in [ABS]. The horizontal (equivalently the vertical) homomorphism is an isomorphism due to the theorem of Swan [Sw2, Theorem 3]. So, the proof is complete. □

5.3. Patching matrices

Proposition 5.5 exhibits the importance of a suitable description of the homomorphism $\psi_{1-y^2}$, as a matrix. We are interested in the cases of real spheres $\text{Spec}(A_n)$ with $K_0(A_n) \approx KO(S^n)$ nontrivial. So, $q = \sum_{i=1}^{n} X_i^2 \in \mathbb{R}[X_1, \ldots, X_n]$ and $n = 8r, 8r + 1, 8r + 2, 8r + 4$. We only need to consider the irreducible $\mathbb{Z}_2$-graded modules $M$ over $C_n = C(-q)$.

We will include the following from [ABS] regarding Bott periodicity.

Theorem 5.8. Let $q_n = q = \sum_{i=1}^{n} X_i^2 \in \mathbb{R}[X_1, \ldots, X_n]$ and $C_n = C(-q)$. Write $a_n = (\dim_{\mathbb{R}} I_n)/2 = \dim_{\mathbb{R}} I_n^0$ where $I_n = I_n^0 \oplus I_n^1$ is an irreducible $\mathbb{Z}_2$-graded $C_n$-module. The following chart summarizes some information regarding $C_n = C(-q)$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$C_n$</th>
<th>$K(G(C_n))$</th>
<th>$ABS(q_n)$</th>
<th>$a_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{C}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{H}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{H} \oplus \mathbb{H}$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{M}_2(\mathbb{H})$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{M}_4(\mathbb{C})$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{M}_8(\mathbb{R})$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{M}_8(\mathbb{R}) \oplus \mathbb{M}_8(\mathbb{R})$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>$\mathbb{M}_{16}(\mathbb{R})$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>8</td>
</tr>
</tbody>
</table>

Further

$C_{n+8} \approx C_8 \otimes C_n \approx \mathbb{M}_{16}(\mathbb{R}) \otimes C_n$

and

$K(G(C_{n+8})) \approx K(G(C_n)), \quad ABS(q_{n+4}) \approx ABS(q_8), \quad a_{n+8} = 16a_n$.

The following corollary will be of some interest to us.

Corollary 5.9. With notations as above (5.8), for nonnegative integers $r$, we have

$C_{8r} \approx \mathbb{M}_{16^r}(\mathbb{R}), \quad C_{8r+1} \approx \mathbb{M}_{16^r}(\mathbb{C}), \quad C_{8r+2} \approx \mathbb{M}_{16^r}(\mathbb{H}), \quad C_{8r+4} \approx \mathbb{M}_{2 \cdot 16^r}(\mathbb{H})$

and

$a_{8r} = 16^r/2, \quad a_{8r+1} = 16^r, \quad a_{8r+2} = 2 \cdot 16^r, \quad a_{8r+4} = 4 \cdot 16^r$.

Proof. First part follows from (5.8). For the later part, let $I_n$ be an irreducible $C_n$-module. Note that for $n = 8r, 8r + 1, 8r + 2, 8r + 4$, Clifford algebras $C_n$ are matrix algebras. Form general theory, $I_n$ is isomorphic to the module of column vectors. So, $\dim_{\mathbb{R}} I_n$ is easily computable. One can also establish, by induction, that there are $\mathbb{Z}_2$-graded $C_n$-modules $I_n$ with $\dim I_n = \dim I_n$. So, $I_n$ is irreducible and $a_n = (\dim I_n)/2 = (\dim I)/2$. This completes the proof. □
Theorem 5.10. Following chart describes the $\widetilde{KO}(S^n)$ groups.

<table>
<thead>
<tr>
<th>$n$</th>
<th>8r</th>
<th>8r + 1</th>
<th>8r + 2</th>
<th>8r + 3</th>
<th>8r + 4</th>
<th>8r + 5</th>
<th>8r + 6</th>
<th>8r + 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\widetilde{KO}(S^n)$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Proof. It follows from Theorem 5.7 and Corollary 5.9. For a complete proof the reader is referred to the book [H] or [ABS].

Now we state our result on the matrix representation of $\psi$. In fact, we will do it more formally at the polynomial ring level.

Proposition 5.11. Let $n = 8r, 8r + 1, 8r + 2, 8r + 4$ be a nonnegative integer and $q(X) = \sum_{i=1}^{n} X_i^2 \in \mathbb{R}[X_1, \ldots, X_n]$. As before, $C_n = C(-q)$ and $e_1, \ldots, e_n$ are the canonical generators of $C_n$.

Let $M = M^0 \oplus M^1$ be a $\mathbb{Z}_2$-graded irreducible $C_n$-module. Write $m = a_0 = \dim_{\mathbb{R}} M^0$. Define

$$\psi = \psi_n = \sum_{i=1}^{n} X_i (1 \otimes e_i) : \mathbb{R}[X_1, \ldots, X_n] \otimes M^0 \to \mathbb{R}[X_1, \ldots, X_n] \otimes M^1$$

and

$$\phi = \phi_n = \sum_{i=1}^{n} X_i (1 \otimes e_i) : \mathbb{R}[X_1, \ldots, X_n] \otimes M^1 \to \mathbb{R}[X_1, \ldots, X_n] \otimes M^0.$$

Then, there are choices of bases $u_1, \ldots, u_m$ of $M^0$ and $v_1, \ldots, v_m$ of $M^1$ such that the matrix $\Gamma$ of $\psi$ and the matrix $\Delta$ of $\phi$ have the following properties:

1. Each row and column of $\Gamma$, $\Delta$ has exactly $n$ nonzero entries and for $i = 1, \ldots, n$ exactly one entry in each row and column is $\pm X_i$.
2. As a consequence, $\Delta = -\Gamma^t$ and they are orthogonal matrices.

Proof of $(1) \Rightarrow (2)$. Suppose we have bases of $M^0, M^1$ as above that satisfy (1). Write $u = (u_1, \ldots, u_m)^t, v = (v_1, \ldots, v_m)^t$. Then we have $-q(x)(u) = \phi(u) = \Gamma \Delta(v)$. So, $\Gamma \Delta = -q$. Let $\Gamma_i^t$ denote the $i$th-row of $\Gamma$ and $\Delta_i^t$ denote the $i$th-column of $\Delta$. So, $\Gamma_i^t \Delta_i^t = -\sum_{i=1}^{n} X_i^2$. Comparing two sides, we have $\Gamma_i^t = -(\Delta_i^t)^t$. So, $\Delta = -\Gamma^t$. Since $\Gamma \Delta = -q$, we have $\Gamma, \Delta$ are orthogonal matrices. Proof of (1) comes later.

Before we get into the proof of 5.11, we wish to deal with the initial cases of $n = 2, 4, 8$ with some extra details.

Lemma 5.12. Let $q = X_1^2 + X_2^2 \in \mathbb{R}$ and $C_2 = C(-q)$. Then $C_2 = \mathbb{H}$, where the canonical basis of $\mathbb{R}^2 \subseteq C_2$ is $e_1 = i, e_2 = j$ and the matrices of $\psi_2$ and $\phi_2$ have the property (1) of Proposition 5.11.

Proof. A matrix representation of $\psi$ is given by

$$
\begin{pmatrix}
\psi(1) \\
\psi(k)
\end{pmatrix} =
\begin{pmatrix}
X_1 & X_2 \\
X_2 & -X_1
\end{pmatrix}
\begin{pmatrix}
i \\
j
\end{pmatrix}.
$$

Similarly, we can get a matrix representation of $\phi$. The proof is complete.

Now we consider the case $n = 4$. We will include additional information that will be useful later.
Lemma 5.13. Let \( q = \sum_{i=1}^{4} X_i^2 \in \mathbb{R}[X_1, X_2, X_3, X_4] \) and \( C_4 = C(-q) \). Then:

1. We have \( C_4 = \mathbb{M}_2(\mathbb{H}) \) where the canonical basis of \( \mathbb{R}^4 \subseteq C_4 \) is given as follows:
   \[
   e_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad e_2 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix},
   \]

   and

   \[
   e_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.
   \]

2. Following [F], write \( w_4 = 1 + e_1e_2e_3e_4 \). We have the following identities,
   \[
   e_1e_2e_3e_4w_4 = w_4, \quad -e_1e_2w_4 = e_2e_4w_4, \quad e_1e_3w_4 = e_2e_4w_4, \quad -e_1e_4w_4 = e_2e_3w_4.
   \]

3. Let \( M = C_4w_4 \). Then, \( M \) is irreducible.

4. Then \( \Psi_4, \Phi_4 \) have the desired property (1) of (5.11).

Proof. Proof of (1) follows by direct checking. Identities in (2) are obvious. The statement (3) is a theorem of Fossum [F]. To see a proof, let \( M = Cw_4 = M^0 \oplus M^1 \) be the \( \mathbb{H}_2 \)-graded decomposition of \( M \). We have:

\[
M^0 = \mathbb{R}w_4 + \sum_{i < j} \mathbb{R}e_1e_jw_4 + \mathbb{R}e_1e_2e_3e_4w_4; \quad M^1 = \sum_{i < j < k} \mathbb{R}e_1e_je_kw_4.
\]

Using the identities above, it is easy to check that a basis of \( M^0 \) is given by

\[
u_1 = w_4, \quad u_2 = -e_1e_2w_4, \quad u_3 = -e_1e_3w_4, \quad u_4 = -e_1e_4w_4
\]

and a basis of \( M^1 \) is given by

\[
v_1 = e_1w_4 = e_1u_1, \quad v_2 = e_2w_4, \quad v_3 = e_3w_4, \quad v_4 = e_4w_4 = e_1e_2e_3w_4.
\]

Since dimension of an irreducible module over \( C_4 = \mathbb{M}_2(\mathbb{H}) \) is eight, \( M \) is irreducible. This establishes (3).

Now, we write down the matrix of \( \Psi_4, \Phi_4 \) with respect to the above bases:

\[
\begin{pmatrix}
\psi(u_1) \\
\psi(u_2) \\
\psi(u_3) \\
\psi(u_4)
\end{pmatrix} = \begin{pmatrix}
X_1 & X_2 & X_3 & X_4 \\
-X_2 & X_1 & X_4 & -X_3 \\
-X_3 & -X_4 & X_1 & X_2 \\
-X_4 & X_3 & -X_2 & X_1
\end{pmatrix} \begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix}.
\]

Also

\[
\begin{pmatrix}
\phi(v_1) \\
\phi(v_2) \\
\phi(v_3) \\
\phi(v_4)
\end{pmatrix} = \begin{pmatrix}
-X_1 & X_2 & X_3 & X_4 \\
-X_2 & -X_1 & X_4 & -X_3 \\
-X_3 & -X_4 & -X_1 & X_2 \\
-X_4 & X_3 & -X_2 & -X_1
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{pmatrix}.
\]

The proof is complete. □

Now we will consider the case of \( n = 8 \).
Lemma 5.14. Let \( q = \sum_{i=1}^{8} X_i^2 \in \mathbb{R}[X_1, \ldots, X_8] \) and \( C_8 = C(-q) \). Let \( E_1, E_2, \ldots, E_8 \) be the canonical generators of \( C_8 \). Following Fossum [F], let

\[
w_8 = (1 + E_1 E_2 E_5 E_6 + E_1 E_3 E_5 E_7 + E_1 E_4 E_5 E_8)(1 + E_1 E_2 E_3 E_4)(1 + E_5 E_6 E_7 E_8).
\]

Then \( M = C_8 w_8 \) is irreducible and \( \Psi_8, \Phi_8 \) have the desired property (1) of (5.11).

Proof. It is a theorem of Fossum [F], that \( M = C_8 w_8 \) is irreducible. A basis of \( M \) is also given in [F]. We will describe this basis of \( M \) and provide a proof of irreducibility. By (5.1), there is an isomorphism \( C_8 \cong C_4 \otimes C_4 \). As in (5.13), we denote the canonical generators of \( C_4 \) by \( e_1, e_2, e_3, e_4 \) and \( w_4 = 1 + e_1 e_2 e_3 e_4 \).

We will identify \( C_8 = C_4 \otimes C_4 \). Under this identification, the canonical generators of \( C_8 \) are given by \( E_i = e_i \otimes 1 \) for \( i = 1, 2, 3, 4 \) and \( E_i = 1 \otimes e_{i-4} \) for \( i = 5, 6, 7, 8 \). Also, \( w_8 \) is identified as

\[
w_8 = (w_4 \otimes w_4 + e_1 e_2 w_4 \otimes e_1 e_2 w_4 + e_1 e_3 w_4 \otimes e_1 e_3 w_4 + e_1 e_4 w_4 \otimes e_1 e_4 w_4).
\]

We denote \( w = w_8 \). Let \( M = C_8 w_8 = M^0 \oplus M^1 \) be the \( \mathbb{Z}_2 \)-graded decomposition of \( M \). We denote the basis [F] of \( M^0 \) by \( u_i \) as in the table:

<table>
<thead>
<tr>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
<th>( u_7 )</th>
<th>( u_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( = w )</td>
<td>( E_1 E_2 w )</td>
<td>( E_1 E_3 w )</td>
<td>( E_2 E_3 w )</td>
<td>( E_1 E_5 w )</td>
<td>( E_2 E_5 w )</td>
<td>( E_3 E_5 w )</td>
<td>( E_1 E_2 E_3 E_5 w )</td>
</tr>
</tbody>
</table>

and similarly, \( v_i \) will denote the basis [F] of \( M^1 \) as follows:

<table>
<thead>
<tr>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
<th>( v_5 )</th>
<th>( v_6 )</th>
<th>( v_7 )</th>
<th>( v_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( = E_1 w )</td>
<td>( E_2 w )</td>
<td>( E_3 w )</td>
<td>( E_1 E_2 E_3 w )</td>
<td>( E_5 w )</td>
<td>( E_1 E_2 E_5 w )</td>
<td>( E_1 E_3 E_5 w )</td>
<td>( E_2 E_3 E_5 w )</td>
</tr>
</tbody>
</table>

The following multiplication table will be useful for our purpose:

<table>
<thead>
<tr>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
<th>( u_7 )</th>
<th>( u_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_1 )</td>
<td>( v_1 )</td>
<td>( -v_2 )</td>
<td>( -v_3 )</td>
<td>( v_4 )</td>
<td>( -v_5 )</td>
<td>( v_6 )</td>
<td>( v_7 )</td>
</tr>
<tr>
<td>( E_2 )</td>
<td>( v_2 )</td>
<td>( v_1 )</td>
<td>( -v_4 )</td>
<td>( -v_3 )</td>
<td>( -v_6 )</td>
<td>( -v_5 )</td>
<td>( v_8 )</td>
</tr>
<tr>
<td>( E_3 )</td>
<td>( v_3 )</td>
<td>( v_4 )</td>
<td>( v_1 )</td>
<td>( v_2 )</td>
<td>( -v_7 )</td>
<td>( -v_8 )</td>
<td>( -v_5 )</td>
</tr>
<tr>
<td>( E_4 )</td>
<td>( v_4 )</td>
<td>( -v_3 )</td>
<td>( v_2 )</td>
<td>( -v_1 )</td>
<td>( v_8 )</td>
<td>( -v_7 )</td>
<td>( v_6 )</td>
</tr>
<tr>
<td>( E_5 )</td>
<td>( v_5 )</td>
<td>( v_6 )</td>
<td>( v_7 )</td>
<td>( v_8 )</td>
<td>( v_1 )</td>
<td>( v_2 )</td>
<td>( v_3 )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>( v_6 )</td>
<td>( -v_5 )</td>
<td>( -v_8 )</td>
<td>( v_7 )</td>
<td>( v_2 )</td>
<td>( -v_1 )</td>
<td>( -v_4 )</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>( v_7 )</td>
<td>( v_8 )</td>
<td>( -v_5 )</td>
<td>( -v_6 )</td>
<td>( v_3 )</td>
<td>( -v_4 )</td>
<td>( v_1 )</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>( -v_8 )</td>
<td>( v_7 )</td>
<td>( -v_6 )</td>
<td>( v_5 )</td>
<td>( v_4 )</td>
<td>( -v_3 )</td>
<td>( v_2 )</td>
</tr>
</tbody>
</table>

This table is constructed by using the identities in (5.13). For the benefit of the reader, we give proof of one of them. We prove \( E_6 u_1 = E_6 w = v_6 \). First, we have \( E_6 w = -E_5 (E_5 E_6 w) = -E_5 (1 \otimes e_1 e_2) w \).

We compute

\[
E_5 E_6 w = (1 \otimes e_1 e_2) w = (w_4 \otimes e_1 e_2 w_4 + e_1 e_2 w_4 \otimes e_1 e_2 w_4 + e_1 e_2 e_3 w_4 \otimes e_1 e_2 e_3 w_4 + e_1 e_2 e_4 w_4 \otimes e_1 e_2 e_4 w_4)
\]

\[
= (-e_1 e_2 \otimes 1)[e_1 e_2 w_4 \otimes e_1 e_2 w_4 + w_4 \otimes w_4 + e_2 e_3 w_4 \otimes e_2 e_3 w_4 + e_2 e_4 w_4 \otimes e_2 e_4 w_4]
\]

\[
= -E_1 E_2 [e_1 e_2 w_4 \otimes e_1 e_2 w_4 + w_4 \otimes w_4 + (-e_1 e_4 w_4 \otimes -e_1 e_4 w_4) + e_1 e_3 w_4 \otimes e_1 e_3 w_4].
\]
Therefore, \( E_5E_6w = (1 \otimes e_1e_2)w = -E_1E_2w \). So,

\[
E_6w = -E_5(E_5E_6w) = -E_5(-E_1E_2w) = E_1E_2E_5w = v_6.
\]

This establishes \( E_6u_1 = v_6 \).

A similar multiplication table \( (E_i v_j) \) can be constructed using the fact \( E_i v_j = u_k \leftrightarrow -v_j = E_iu_k \). This shows that the vector space \( V \) generated by \( \{u_i, v_j : \ i, j = 1, \ldots, 8\} \) is a \( C_8 \)-left module. So \( V = M \). Since, an irreducible \( C_8 \)-module has dimension sixteen, \( M \) is irreducible.

Now we compute the matrix \( \Gamma \) of \( \Phi \) with respect to these bases. We have, \( \Psi(u_i) = \sum_{j=1}^{8} X_j E_j u_i \). The \( (i, j) \)th entry of the matrix \( \Gamma \) of \( \Psi \) is \( \pm X_k \) if and only if \( E_k u_i = \pm v_j \). For a fixed \( i \) there is exactly one \( j \) such that \( E_k u_i = \pm v_j \) and similarly for a fixed \( j \) there is exactly one \( i \) such that \( E_k u_i = \pm v_j \). So, for \( k = 1, \ldots, 8; \pm X_k \) appears exactly once in each row and column. In fact, The matrix of \( \Psi \) is

\[
\Gamma = \begin{pmatrix}
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & -X_8 \\
-X_2 & X_3 & -X_4 & X_5 & -X_6 & X_7 & X_8 & X_7 \\
-X_3 & X_4 & X_5 & -X_6 & -X_7 & X_8 & -X_5 & -X_6 \\
-X_4 & X_5 & X_6 & X_7 & -X_8 & -X_5 & X_6 & X_5 \\
-X_5 & -X_6 & X_7 & X_8 & -X_1 & -X_2 & -X_3 & X_4 \\
-X_6 & -X_7 & X_8 & X_5 & -X_2 & X_1 & -X_4 & -X_3 \\
-X_7 & X_8 & X_5 & -X_6 & -X_3 & X_4 & X_1 & X_2 \\
-X_8 & -X_7 & X_6 & X_5 & -X_4 & -X_3 & X_2 & -X_1 \\
\end{pmatrix}.
\]

Similar argument can be given for \( \Phi \). So, \( \Psi, \Phi \) have the property (1) of (5.11).

We remark that the property (2) of (5.11) was established as a consequence of property (1). Alternatively, we can use the fact \( E_i v_j = u_k \leftrightarrow -v_j = E_iu_k \) to establish property (2). This completes the proof of (5.14). \( \square \)

Now we are ready to give a complete proof of Proposition 5.11.

**Proof of 5.11.** We already proved (1) \( \Rightarrow \) (2). So, we only need to prove (1). The case \( n = 1 \), is obvious and Lemmas 5.12, 5.13, 5.14, respectively, establish the proposition in the cases \( n = 2, 4, 8 \).

We will use induction, i.e. we assume that (1) of the proposition is valid for some \( m = 8r \), \( 8r + 1, 8r + 2, 8r + 4 \) and prove that the same is valid for \( n = m + 8 \).

First, we set up some notations. For a matrix \( A \), the \( i \)th-row will be denoted by \( rA_i \) and the \( i \)th-column will be denoted by \( cA_i \). We have \( m + 8 \) variables \( X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+8} \). For \( i = 1, \ldots, 8 \), we will write \( Y_i = X_{m+i} \). In our cases, there is only one irreducible \( \mathbb{Z}_2 \)-graded \( C_m \)-module, which will be denoted by \( M(m) = M(m)^0 \oplus M(m)^1 \). We have \( C_{m+8} = C_m \otimes C_8 \). Comparing dimensions (see 5.9), we have \( M(m+8) = M(m) \otimes M(8) \).

Write \( N = \dim_{\mathbb{R}} M(m)^0 \). We assume that there are bases \( u_1, \ldots, u_N \) of \( M(m)^0 \) and \( v_1, \ldots, v_N \) of \( M(m)^1 \) and bases \( \mu_1, \ldots, \mu_8 \) of \( M(8)^0 \) and \( v_1, \ldots, v_8 \) of \( M(8)^1 \) such that

\[
\begin{pmatrix}
\Psi_m(u_1) \\
\vdots \\
\Psi_m(u_N)
\end{pmatrix} = A(X)
\begin{pmatrix}
v_1 \\
\vdots \\
v_N
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\Psi_8(\mu_1) \\
\vdots \\
\Psi_8(\mu_8)
\end{pmatrix} = B(Y)
\begin{pmatrix}
v_1 \\
\vdots \\
v_8
\end{pmatrix}
\]

where \( A(X) = (a_{ij}(X_1, \ldots, X_m)) \) and \( B(Y) = (b_{ij}(Y_1, \ldots, Y_8)) \) have the properties \( \Gamma \) of the proposition. Also

\[
M(m+8)^0 = M(m)^0 \otimes M(8)^0 \oplus M(m)^1 \otimes M(8)^1 \quad \text{with basis } u_i \otimes \mu_j, \ v_i \otimes v_j
\]

and

\[
M(m+8)^1 = M(m)^1 \otimes M(8)^0 \oplus M(m)^0 \otimes M(8)^1 \quad \text{with basis } v_i \otimes \mu_j, \ u_i \otimes v_j.
\]
The canonical generators of $C_m$ will be denoted by $e_1, \ldots, e_m$ and the canonical generators of $C_8$ will be denoted by $e'_1, \ldots, e'_8$. With $E_1 = e_1 \otimes 1, \ldots, E_m = e_m \otimes 1; E'_1 = e'_1 \otimes e'_1, \ldots, E'_8 = E_{m+8} = 1 \otimes e'_8$, we have

$$
\Phi_{m+8} = \sum_{i=1}^{N} X_i E_i + \sum_{i=1}^{8} Y_i E'_i.
$$

We have

$$
\Phi_{m+8}(u_1 \otimes \mu_1) = \sum_{i=1}^{N} X_i E_i (u_1 \otimes \mu_1) + \sum_{i=1}^{8} Y_i E'_i (u_1 \otimes \mu_1)
= \sum_{i=1}^{N} a_1i(X) v_i \otimes \mu_1 + \sum_{i=1}^{8} b_{i}(Y) u_1 \otimes v_i.
$$

We will use the notations $u = (u_1, \ldots, u_N)^t, v = (v_1, \ldots, v_N)^t, \mu = (\mu_1, \ldots, \mu_8)^t, v = (v_1, \ldots, v_8)^t$. With these notation,

$$
\Phi_{m+8}(u_1 \otimes \mu_1) = r A(X)_1 v \otimes \mu_1 + r B(Y)_1 u_1 \otimes v.
$$

For $i = 1, \ldots, N$, likewise, we get

$$
\Phi_{m+8}(u_i \otimes \mu_1) = r A(X)_i v \otimes \mu_1 + r B(Y)_1 u_i \otimes v.
$$

Therefore, 

$$
\Phi_{m+8}(u \otimes \mu_1) = 
\begin{pmatrix}
    r A(X)_1 & 0 & \ldots & 0 \\
    r A_2(X) & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    r A_N(X) & 0 & \ldots & 0 
\end{pmatrix}
\begin{pmatrix}
    v \otimes \mu_1 \\
    \vdots \\
    v \otimes \mu_8 \\
    u_1 \otimes v \\
    \vdots \\
    u_N \otimes v 
\end{pmatrix}.
$$

Given a row vector $a(Y)$ of length 8, let $\mathcal{R}(a(Y)) \in M_{N \times 8N}$ denotes the matrix as on the right-hand side of the above matrix. With such notations,

$$
\Phi_{m+8} (u \otimes \mu_1) = (A(X) 0 \ldots 0 | \mathcal{R}(r B_1(Y)))
\begin{pmatrix}
    v \otimes \mu_1 \\
    \vdots \\
    v \otimes \mu_8 \\
    u_1 \otimes v \\
    \vdots \\
    u_N \otimes v
\end{pmatrix}.
$$

Using similar calculations for $\Phi_{m+8}(u \otimes \mu_i)$ we have

$$
\begin{pmatrix}
    \Phi_{m+8}(u \otimes \mu_1) \\
    \Phi_{m+8}(u \otimes \mu_2) \\
    \vdots \\
    \Phi_{m+8}(u \otimes \mu_8)
\end{pmatrix} =
\begin{pmatrix}
    A(X) & 0 & \ldots & 0 & \mathcal{R}(r B_1(Y)) \\
    0 & A(X) & \ldots & 0 & \mathcal{R}(r B_2(Y)) \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \ldots & A(X) & \mathcal{R}(r B_8(Y))
\end{pmatrix}
\begin{pmatrix}
    v \otimes \mu_1 \\
    \vdots \\
    v \otimes \mu_8 \\
    u_1 \otimes v \\
    \vdots \\
    u_N \otimes v
\end{pmatrix}.
$$
This gives the upper $8N$ rows of the matrix of $\Psi_{m+8}$ which form a $8N \times (8N + 8N)$ matrix. Now we proceed to compute

\[
\begin{pmatrix}
\Psi_{m+8}(v_1 \otimes v) \\
\Psi_{m+8}(v_2 \otimes v) \\
\vdots \\
\Psi_{m+8}(v_N \otimes v)
\end{pmatrix}.
\]

Again, the matrices of $\Phi_m, \Phi_8$ are respectively $-A(X)^t, -B(Y)^t$. We have,

\[
\Psi_{m+8}(v_1 \otimes v_1) = \sum_{i=1}^{m} X_i (e_1 \otimes 1)(v_1 \otimes v_1) + \sum_{i=1}^{8} Y_i (1 \otimes e_i')(v_1 \otimes v_1) = \Phi_m(v_1) \otimes v_1 - v_1 \otimes \Phi_8(v_1) = -\sum_{j=1}^{N} a_{j1}(X)u_j \otimes v_1 + \sum_{j=1}^{8} b_{j1}(Y)v_1 \otimes \mu_j = -rA_1(X)^tu \otimes v_1 + rB_1(Y)^tv_1 \otimes \mu.
\]

Similarly, for $i = 1, \ldots, 8$, we have

\[
\Psi_{m+8}(v_1 \otimes v_i) = \Phi_m(v_1) \otimes v_i - v_1 \otimes \Phi_8(v_i) = -rA_1(X)^tu \otimes v_i + rB_1(Y)^tv_1 \otimes \mu;
\]

and for $k = 1, \ldots, N$, we have

\[
\Psi_{m+8}(v_k \otimes v_i) = \Phi_m(v_k) \otimes v_i - v_k \otimes \Phi_8(v_i) = -rA_k(X)^tu \otimes v_i + rB_k(Y)^tv_k \otimes \mu.
\]

So, the left half of the matrix of $\Psi_{m+8}(v_1 \otimes v)$ is given by

\[
( cB_1^t \ 0 \ \cdots \ 0 \ | \ cB_2^t \ 0 \ \cdots \ 0 \ | \ \cdots \ | \ cB_8^t \ 0 \ \cdots \ 0 ) \in \mathbb{M}_{8 \times 8N}.
\]

Similarly, the left half of the matrix of $\Psi_{m+8}(v_2 \otimes v)$ is given by

\[
( 0 \ cB_1^t \ \cdots \ 0 \ | \ 0 \ cB_2^t \ \cdots \ 0 \ | \ \cdots \ | \ 0 \ cB_8^t \ \cdots \ 0 ) \in \mathbb{M}_{8 \times 8N}.
\]

So, the left-lower block of the matrix $\Psi_{m+8}$ is given by

\[
\begin{pmatrix}
\begin{pmatrix}
cB_1(Y)^t \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix} & 0 \cdots 0 & \cdots & cB_8(Y)^t \\
\begin{pmatrix}
0 & cB_1(Y)^t & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots \\
0 & \vdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & cB_1(Y)^t & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\end{pmatrix}^t
\]

\[
= (\mathcal{R}(rB_1(Y)))^t = (Upper-Right \ Block)^t \in \mathbb{M}_{8N \times 8N}.
\]

Now we compute the right half of the same matrix of

\[
\begin{pmatrix}
\Psi_{m+8}(v_1 \otimes v) \\
\Psi_{m+8}(v_2 \otimes v) \\
\vdots \\
\Psi_{m+8}(v_N \otimes v)
\end{pmatrix}.
\]
Let $I_8$ denote the identity matrix of order 8. Then, the right half of this matrix is given by

\[
\begin{pmatrix}
a_{11}(X)I_8 & a_{21}(X)I_8 & \cdots & a_{N1}(X)I_8 \\
a_{12}(X)I_8 & a_{22}(X)I_8 & \cdots & a_{N2}(X)I_8 \\
\vdots & \vdots & \ddots & \vdots \\
a_{1N}(X)I_8 & a_{2N}(X)I_8 & \cdots & a_{NN}(X)I_8
\end{pmatrix}.
\]

Now, the upper left and lower right blocks of the matrix of $\psi_{m+8}$ involve only the variables $X_1, \ldots, X_m$ and the upper right and lower left blocks involve only the variables $Y_1, \ldots, Y_8$. Recall that $A(X)$ and $B(Y)$ have the properties of $\Gamma$ of the proposition. Examining all four blocks of the matrix of $\psi_{m+8}$, it follows that the matrix of $\psi_{m+8}$ also has the property of $\Gamma$ of the proposition. By symmetry, the matrix of $\Phi_{m+8}$ also has the property of $\Delta$ of the proposition. This completes the proof of Proposition 5.11. □

6. Complete intersections

In this final section, we consider the question whether a local complete intersection ideal $I$ of $A_n$, with $\text{height}(I) = n$, is the image of a projective $A_n$-module $P$ of rank $n$. For an affirmative answer to this question for all such ideals $I$ it is necessary that the top Chern class map $C_0 : K_0(A_n) \to CH_0(A_n)$ is surjective. Since $CH_0(A_n) = \mathbb{Z}_2$ and for $n = 8r + 3, 8r + 5, 8r + 6, 8r + 7$, by (5.10), $K_0(A_n) = \mathbb{Z}$, the top Chern class map $C_0$ fails to be surjective. So, in these cases, the question has a negative answer.

We consider the stronger question whether each element of the Euler class group $E(A_n) = \mathbb{Z}$ is Euler class of a projective $A_n$-module $P$ of rank $n$. We start with the following two theorems about even classes and odd classes.

**Theorem 6.1.** Let $A_n$ be the ring of algebraic functions on $\mathbb{S}^p$, as in nation (2.1), and $n \geq 2$ be even. Let $N = 2r$ be an even integer. Then there is a stably free $A_n$-module $P$ of rank $n$ and an orientation $\chi : A_n \sim \text{det}(P)$ such that $e(p, \chi ) = N$.

**Proof.** By Lemma 2.4, we can assume $N \geq 0$. Let $m_1, \ldots, m_N$ be even number of distinct real maximal ideals and assume that the corresponding real points are on $\mathbb{S}^1 = (x_2 = 0, \ldots, x_n = 0)$. As in Lemma 4.1,

\[
\bigcap_{i=1}^{N} m_i = (L, x_2, \ldots, x_n) \quad \text{where} \quad L = \prod_{i=1}^{N/2} L_i \quad \text{with} \quad L_i \text{ linear}.
\]

Write $F = A_n^n$ and $J = \bigcap_{i=1}^{N} m_i = (L, x_2, \ldots, x_n)$. The standard basis of $F$ will be denoted by $e_1, \ldots, e_n$.

Define

\[
f : F \to J \quad \text{by} \quad f(e_1) = L, \quad f(e_i) = x_i \quad \forall i \geq 2.
\]

Let

\[
\omega : F/ JF \to J/ J^2 \quad \text{and} \quad \text{for} \quad i = 0, 1 \quad \omega_i : F/ m_iF \to m_i/ m_i^2,
\]

be induced by $f$. Therefore

\[
(J, \omega) = \sum_{i=1}^{N} (m_i, \omega_i) = 0 \in E(A_n).
\]
We can assume
\[(m_i, \omega_i) = 1 \quad \forall i = 1, \ldots, s, \quad (m_i, \omega_i) = -1 \quad \forall i = s + 1, \ldots, N.\]

Let \(u \in A\) be such that \(u - 1 \in m_i\) for \(i = 1, \ldots, s\) and \(u + 1 \in m_i\) for \(i = s + 1, \ldots, N\). Note \(u^2 - 1 \in J\).

Define \(P\) by the exact sequence
\[
0 \to P \to A_n \to F \xrightarrow{(u,-f)} A_n \to 0.
\]

By [BRS2, Lemma 5.1], \(P\) has an orientation \(\chi\) such that
\[
e(P, \chi) = u^{-(n-1)}(f, \omega) = \sum_{i=1}^{s} (m_i, \omega_i) + \sum_{i=s+1}^{N} -(m_i, \omega_i) = N.
\]

The proof is complete. \(\Box\)

**Theorem 6.2.** Let \(A_n\) be the ring of algebraic functions on \(\mathbb{S}^n\). Assume \(n = \dim A_n\) is even. Then, there exists a projective \(A_n\)-module \(P\) with \(\text{rank}(P) = n\) and an orientation \(\chi : A_n \xrightarrow{\sim} \det P\) with \(e(P, \chi) = N\) for some odd integer \(N\) if and only if the same is possible for all odd integers \(N\).

**Proof.** Suppose \(N\) odd and \(e(P, \chi) = N\). By Lemma 2.4, we can assume \(N > 0\). Now assume \(M\) be any other odd integer. Again, we can assume \(M > 0\). Let \(m_1, \ldots, m_M\) be distinct real maximal ideals and \((m_i, \omega_i) = 1 \in \text{EL}(A_n) = \mathbb{Z}\). Write \(F = A_n^0\) and \(I = \bigcap_{i=1}^{M} m_i\). Let \(\omega : F/IF \to 1/I^2\) be obtained from \(\omega_1, \ldots, \omega_M\). Then \(M = (I, \omega_1)\). Note that the weak Euler class group
\[
E_0(A_n) \approx CH_0(A_n) = \mathbb{Z}/(2).
\]

Therefore, the weak Euler class \(e_0(P) = \text{image}(N) = 1 = \text{image}(M) = (I)\). So, by proposition [BRS2, Proposition 6.4], there is a projective \(A_n\)-module \(Q\) of rank \(n\) and a surjective map \(f : Q \to I\), and also \([P] = [Q] \in K_0(A_n)\). Fix an orientation \(\chi_0 : A_n \xrightarrow{\sim} \det Q\). Using an isomorphism \(\gamma : F/IF \sim Q/IQ\), with \(\det \gamma = \chi_0\), \(f\) induces orientations \(\eta : F/IF \to 1/I^2\) and \(\eta_i : F/m_iF \to m_i/m_i^2\) for \(i = 1, \ldots, M\). Then, by definition,
\[
e(Q, \chi_0) = (I, \eta) = \sum_{i=1}^{M} (m_i, \eta_i).
\]

We can assume that
\[(m_i, \eta_i) = 1 \quad \forall i \leq s, \quad \text{and} \quad (m_i, \eta_i) = -1 \quad \forall i > s.
\]

Pick \(u \in A_n\) such that \(u - 1 \in m_i\) for \(i \leq s\) and \(u + 1 \in m_i\) for \(i > s\). Let \(Q'\) be defined by
\[
0 \to Q' \to A_n \to Q \xrightarrow{(u,-f)} A_n \to 0.
\]

Then, by [BRS2, Lemma 5.1], \(Q'\) has an orientation \(\chi'\) such that
\[
e(Q', \chi') = (I, \bar{u}^{-(n-1)} \eta) = \sum_{i=1}^{s} (m_i, \bar{u}^{-(n-1)} \eta_i) + \sum_{i=s+1}^{N} (m_i, \bar{u}^{-(n-1)} \eta_i).
\]

Here \(n\) is even. So, \(e(Q', \chi') = \sum_{i=1}^{s} (m_i, \eta_i) + \sum_{i=s+1}^{N} (m_i, -\eta_i) = M\). This completes the proof. \(\Box\)
Before we proceed, we need the following proposition that relates top Chern classes with Stiefel–Whitney classes.

**Proposition 6.3.** Let $A_n$ be the ring of algebraic functions on the real sphere $\mathbb{S}^n$ and $X = \text{Spec}(A_n)$. Then, the following diagram

$$
\begin{array}{ccc}
K_0(X) & \longrightarrow & KO(\mathbb{S}^n) \\
\downarrow C_0 & & \downarrow w_n \\
CH_0(X) & \longrightarrow & H^n(\mathbb{S}^n, \mathbb{Z}/2) \\
\end{array}
$$

commutes, where $C_0$ denotes the top Chern class map and $w_n$ denotes the top Stiefel–Whitney class.

**Proof.** Note, $\text{CH}_0(\mathbb{R}(X)) = \mathbb{Z}/2(2)$ and $H^n(\mathbb{S}^n, \mathbb{Z}/2)) = \mathbb{Z}/2(2)$. Any element in $\widetilde{K}_0(X)$ can be written as $[P] - [A^n_0]$, where $P$ is a projective $A_n$-module of rank $n$. By Bertini’s theorem (see [BRS1, 2.11]), we can find a surjective map $f : P \rightarrow I$ where $I = m_1 \cap m_2 \cap \cdots \cap m_N$ is intersection of $N$ distinct maximal ideals. Assume $m_1, \ldots, m_r$ are real maximal ideals and $m_{r+1}, \ldots, m_N$ are the complex maximal ideals. For $i = 1, \ldots, r$, let $y_i \in \mathbb{S}^n$ be the point corresponding to $m_i$. So, $C_0(P) = \tilde{r} \in \mathbb{Z}/2$, where $\tilde{r}$ is the image of $r$ in $\mathbb{Z}/2$.

Let $\tilde{P}$ denote the bundle on $\mathbb{S}^n$ induced by $P$. Then $f$ induces a section $s$ on the bundle $\tilde{P}$, transversally intersecting the zero section, exactly on the points $y_1, \ldots, y_r$. So, $w_n(\tilde{P}) = \tilde{r}$. The proof is complete. □

**Remark.** In a subsequent paper [MaSh], a more general version of Proposition 6.3 was proved later.

Now, we have the following corollary to Theorem 6.2.

**Corollary 6.4.** Let $A_n$ be the ring of algebraic functions on $\mathbb{S}^n$. Assume $n = \dim A_n \geq 2$ is even. Then, the following are equivalent:

1. $e(P, \chi) = 1$ for some projective $A_n$-module $P$ with rank $P = n$ and orientation $\chi : A_n \rightarrow \text{det} P$.
2. For some odd integer $N$, $e(P, \chi) = N$ for some projective $A_n$-module $P$ with rank $P = n$ and orientation $\chi : A_n \rightarrow \text{det} P$.
3. For any odd integers $N$, $e(P, \chi) = N$ for some projective $A_n$-module $P$ with rank $P = n$ and orientation $\chi : A_n \rightarrow \text{det} P$.
4. The top Chern class $\text{C}_0(P) = 1$ for some projective $A_n$-module $P$ with rank $P = n$.
5. The Stiefel–Whitney class $w_n(V) = 1$ for some vector bundle $V$ with rank $V = n$.

Let $n = 8r, 8r + 2, 8r + 4$ and let $P_n$ be a projective $A_n$-module of rank $n$ such that $[P_n] - n = \tau_n$ is the generator of $\widetilde{K}_0(A_n)$. Then above conditions are equivalent to $w_n(P_n) = 1$ (which is equivalent to $\text{C}_0(P_n) = 1$).

**Proof.** By (6.2), (1) $\iff$ (2) $\iff$ (3). It is obvious that (2) $\iff$ (4). Also by (6.3), we have (4) $\iff$ (5), because we can assume [Sw2] that $V$ is algebraic.

For the later part, we only need to prove that (5) $\Rightarrow$ $w_n(P_n) = 1$. To prove this assume that $w_n(P_n) = 0$ and let $V$ be any vector bundle of rank $n$ over $\mathbb{S}^n$. We have $[V] - \text{rank}(V) = \text{kt}_n$. So, the total Stiefel–Whitney class $w(V) = w(\text{kt}_n) = w(\text{t}_n)^k = 1$. So, $w_n(V) = 0$. The proof is complete. □

**Remark 6.5.** We have the following summary. Let $P$ denote a projective $A_n$-module of rank $n \geq 2$ and $\chi : A_n \rightarrow \text{det} P$ be an orientation. Then
1. For \( n = 8r + 3, 8r + 5, 8r + 7 \) we have \( \tilde{K}_0(A_n) = 0 \). So, the top Chern class \( C_0(P) = 0 \). By [BDM], \( P \approx Q \oplus A_n \). Therefore \( e(P, \chi) = 0 \).

2. For \( n = 8r + 6 \), we have \( \tilde{K}_0(A_n) = 0 \). So, \( C_0(P) = 0 \), and hence \( e(P, \chi) \) is always even. Further, by (6.1), for any even integer \( N \) there is a projective \( A_n \)-module \( Q \) with \( \text{rank}(Q) = n \) and an orientation \( \eta : A_n \to \det Q \), such that \( e(Q, \eta) = N \).

3. For \( n = 8r + 1 \), we have \( \tilde{K}_0(A_n) = \mathbb{Z}/2 \). If \( e(P, \chi) \) is even then \( C_0(P) = 0 \). So, \( P \approx Q \oplus A_n \) and \( e(P, \chi) = 0 \) for all orientations \( \chi \). So, only even value \( e(P, \chi) \) can assume is zero.

4. Now consider the remaining cases, \( n = 8r, 8r + 2, 8r + 4 \). We have \( \tilde{K}_0(A_{8r}) = \mathbb{Z}, \tilde{K}_0(A_{8r+2}) = \mathbb{Z}/2, \tilde{K}_0(A_{8r+4}) = \mathbb{Z} \). As in the case of \( n = 8r + 6 \), for any even integer \( N \), for some \((Q, \eta)\) the Euler class \( e(Q, \eta) = N \). The case of odd integers \( N \) was discussed in (6.4).

This leads us to the following question.

**Question 6.6.** Suppose \( n = 8r, 8r + 1, 8r + 2, 8r + 4 \) and \( \tau_n \) is the generator of \( \tilde{K}_0(A_n) \). Then, whether \( w_n(\tau_n) = 1 \) (which is equivalent to \( C_0(\tau_n) = 1 \))?

Apparently, answer to this question is not known. For \( n = 1 \), the question has affirmative answer. We will be able to answer this question for \( n = 2, 4, 8 \) using the description (5.11) of the patching matrix \( \psi_n \).

**Theorem 6.7.** Let \( n = 2, 4, 8 \) and \( A_n \) denote the ring of algebraic functions on \( \mathbb{S}^n \). Let \( \tau_n \) be the generator of \( \tilde{K}_0(A_n) \). Then, the top Chern class \( C_0(\tau_n) = 1 \).

**Proof.** Here \( n = 2, 4, 8 \) and \( q = q_n = \sum_{i=1}^n X_i^2 \). If \( M = M^0 \oplus M^1 \) is an irreducible \( \mathbb{Z}_2 \)-graded \( C_n \)-module, then \( \tau_n = [P] - \text{rank}(P) \), where \( P = \alpha(M) \) as defined in Proposition 5.5. We have \( R(q + X_0^2) = A_n \) and \( P \) is obtained by patching together \( F^0 = (A_n)_{1+x_0} \otimes M^0 \) and \( F^1 = (A_n)_{1-x_0} \otimes M^1 \) via \( \psi = \psi_n \). In the cases of \( n = 2, 4, 8 \), by (5.9), \( \text{rank}(P) = \dim M_0 = n \). By (5.11), with respect some bases of \( M^0, M^1 \), the matrix of \( \psi \) has the first column \((x_1, \ldots, x_n)^t \).

We write \( y = x_0 \) and \( F = A_n^n \). Let \( e_1, \ldots, e_n \) denote the standard basis of \( F \). We identify \( F^0 = F_{1+y}, F^1 = F_{1-y} \) and consider \( \psi \) as a matrix with first column \((x_1, \ldots, x_n)^t \).

Let \( I = (y - 1, x_1, \ldots, x_n) \) be the ideal of the north pole of \( \mathbb{S}^n \). Then, \( I_{1+y} = (x_1, \ldots, x_n) \). Define surjective maps \( f_0 : F^0 \to I_{1+y} \) where \( f_0(e_i) = x_i \) for \( i = 1, \ldots, n \); and \( f_1 : F^1 \to I_{1-y} \) where \( f_1(e_i) = 1 \) and \( f_1(e_i) = x_i \) for \( i = 2, \ldots, n \). We have the following patching diagram:

The map \( f \) is induced by the properties of fiber product diagrams. Since \( f_0, f_1 \) are surjective, so is \( f \). Therefore, the top Chern class \( C_0(P) = \text{cycle}(A_n/I) = 1 \). Since \( \tau_n = [P] - n \), we have \( C_0(\tau_n) = 1 \). This completes the proof. \( \square \)
Corollary 6.8. Let $n = 2, 4, 8$. Then, given any integer $N$ there is a projective $A_n$-module $Q$ of rank $n$ and orientation $\chi : A_n \to \det Q$, such that the Euler class $e(Q, \chi) = N$.

Also, suppose $I$ is a locally complete intersection ideal of height $n$ and $\omega : (A_n/I)^n \to I/I^2$ is a surjective homomorphism. Then, there is a projective $A_n$-module $P$ of rank $n$, and orientation $\chi : A_n \to \det P$ and a surjective homomorphism $f : P \to I$ such that $(I, \omega)$ is induced by $(P, \chi)$.

Proof. First part follows immediately from (6.1, 6.4, 6.7). The later part follows from [BRS2, Corollary 4.3]. □

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References