Answer the following questions. Show detailed work. Answers without sufficient work will receive no credit.

1. (10 points each) Evaluate the following limits. Justify your answer by applying the limit laws, a computation, or a theorem from class. Guessing the limit from a graph or a table will receive no credit.

a. \( \lim_{x \to 2} \frac{3x^2 - 10x + 8}{x^4 - 16} = \lim_{x \to 2} \frac{(3x - 4)(x - 2)}{(x^2 - 4)(x^2 + 4)} \)
   \[ = \lim_{x \to 2} \frac{(3x - 4)(x - 2)}{(x - 2)(x + 2)(x^2 + 4)} \]
   \[ \text{prime quadratic, doesn't factor} \]
   \[ = \frac{2}{(4)(8)} = \frac{1}{16} \]

b. \( \lim_{x \to 0} \frac{\cos(\pi x^2)}{x^2} \)
   \( \text{Since } \lim_{x \to 0} \cos(\pi x^2) = \cos 0 = 1, \text{ and } \]
   \( \lim_{x \to 0} x^2 = 0, \text{ then: } \lim_{x \to 0} \frac{\cos(\pi x^2)}{x^2} = \text{DNE} \)

c. \( \lim_{x \to \infty} \left( \sqrt{5x^2 + 7x + 1} - \sqrt{5x^2 + 4x + 8} \right) \cdot \frac{\sqrt{5x^2 + 7x + 1} + \sqrt{5x^2 + 4x + 8}}{\sqrt{5x^2 + 7x + 1} + \sqrt{5x^2 + 4x + 8}} \)
   \[ = \lim_{x \to \infty} \frac{(5x^2 + 7x + 1) - (5x^2 + 4x + 8)}{\sqrt{5x^2 + 7x + 1} + \sqrt{5x^2 + 4x + 8}} \]
   \[ = \lim_{x \to \infty} \frac{3x - 7}{\sqrt{5x^2 + 7x + 1} + \sqrt{5x^2 + 4x + 8}} \]
   \[ \text{Slashed method} \]
   \[ \lim_{x \to \infty} \frac{3x}{\sqrt{5x^2 + 5x^2}} = \lim_{x \to \infty} \frac{3x}{2\sqrt{5}x} = \frac{3}{2\sqrt{5}} \]

   d. \( \lim_{x \to \pi^-} \frac{\tan(\frac{\pi x}{4})}{e^{100}} \)
   \[ \lim_{x \to \pi^-} \tan(\frac{\pi x}{4}) = \tan(\frac{\pi}{4}) = 1 \]
   \[ \lim_{x \to \pi^-} \frac{e^{\tan(\frac{\pi x}{4})}}{e^{100}} = \frac{e^1}{e^{100}} = \frac{1}{e^{99}} \]
2. (30 points) The function \( f = f(x) \) is defined as follows:

\[
 f(x) = \begin{cases} 
 1 + \sqrt{1-x} & ; x \leq 1 \\
 1 - \frac{1}{x+8} & ; x > 1 
\end{cases}
\]

a. Evaluate \( f \circ f(-8) \), and \( f(1 + x^2) \) where \( x \) is any nonzero real number.

\[
 f(-8) = 1 + \sqrt{1-(-8)} = 1 + \sqrt{9} = 1 + 3 = 4.
\]

\[
 \Rightarrow f \circ f(-8) = f(f(-8)) = f(4) = 1 - \frac{1}{4+8} = 1 - \frac{1}{12} = \frac{11}{12}
\]

Since \( x \neq 0 \), then \( x^2 > 0 \) \( \Rightarrow \) \( 1 + x^2 > 1 \) \( \Rightarrow \) \( f(1 + x^2) = \frac{1}{1 + x^2 + 8} + 1 = 1 - \frac{1}{x^2 + 9} \)

b. Find all the possible values of \( x \) at which the function \( f \) is continuous. Justify your answer.

For \( x < 1 \):

\( f(x) = 1 + \sqrt{1-x} \) is cont. on \((-\infty, 1)\) since const. functions, square roots are cont. and sum of cont. functions is cont.

For \( x > 1 \):

\( f(x) = 1 - \frac{1}{x+8} \); since \( x + 8 \neq 0 \) on \((1, \infty)\) then \( \frac{1}{x+8} \) is cont., hence \( f(x) \) is also cont.

At \( x = 1 \):

\[
 f(1) = \lim_{x \to 1} 1 + \sqrt{1-x} = 1 \neq \lim_{x \to 1} 1 - \frac{1}{x+8} = \frac{8}{9} \Rightarrow f \text{ is not cont. at } x = 1
\]

c. Using the limit definition of the derivative, evaluate \( f'(2) \).

\[
 f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}
\]

\[
 = \lim_{h \to 0} \frac{1 - \frac{1}{2+h+8} - (1 - \frac{1}{2+8})}{h}
\]

\[
 = \lim_{h \to 0} \frac{h}{h} \left[ x - \frac{1}{10+h} - x + \frac{1}{10} \right] = \lim_{h \to 0} \frac{h}{10(10+h)} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{h}{10} = \frac{1}{10^2}
\]

3. (15 points) Use the intermediate value theorem to prove that the equation: \( e^x = 2 - x \) has a real solution in the interval \((0, 1)\).

Let \( f(x) = e^x(2-x) = e^x - 2 + x \); \( x \) in \([0, 1]\).

Then \( f(x) = 0 \) is equivalent to solving \( e^x = 2 - x \).

Now, we have:

1) \( f(x) \) is cont. on \([0, 1]\), since \( e^x, 2-x \) are both cont. on \( \mathbb{R} \).

2) \( f(0) = e^0 - 2 + 0 = 1 - 2 + 0 = -1 < 0 \), \( f(1) = e^1 - 2 + 1 = e - 1 > 0 \)

\( \Rightarrow \) There exists a \( c \) in \((0, 1)\) with \( f(c) = 0 \).
4. (5 points each) **Multiple choice:** In each of the following, circle only the one choice that would make the corresponding statement true. You do not have to show work for this question.

(i) The function \( y = G(x) \) is not continuous at \( x = a \). Then

(a) \( \lim_{x \to a} G(x) \) does not exist   (b) \( G(x) \) is not differentiable at \( x = a \)
(c) \( G(x) \) is an even function   (d) none of the above

(ii) If the function \( g(x) = e^x + c |\sin x| \) is differentiable on \((-\infty, \infty)\), then \( c \)

(a) must be positive   (b) must be negative   (c) must be zero   (d) could be any real number

(iii) If \( \lim_{x \to a} h(x) = \infty \) and \( \lim_{x \to a} k(x) = \infty \), then \( \lim_{x \to a} (h(x) - k(x)) \)

(a) does not exist   (b) is 0   (c) is infinity   (d) depends on the the functions \( h(x), k(x) \)

(iv) If \( \lim_{x \to a} h(x) = \infty \) and \( \lim_{x \to a} k(x) = \infty \), then \( \lim_{x \to a} \frac{h(x)}{k(x)} \)

(a) does not exist   (b) is 0   (c) is infinity   (d) depends on the the functions \( h(x), k(x) \)

(v) The function \( F(x) = \begin{cases} 1 - x^2, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases} \)

(a) differentiable everywhere   (b) continuous but not differentiable everywhere   
(c) discontinuous   (d) not enough information to decide