Math 558 S09

Review problems for first exam

These are not "practice problems." The problems on the exam may look very different.
But they do cover all of the material you are responsible for on the exam.
On the exam you will need to show your work. So you should practice doing that.
Problems with an asterisk are on material taught the week of February 17.

I. Calculations

1. Suppose one root of $x^2 + ax + 15$ is 3.
   (a) Find the other root
   (b) Find the coefficient a.
2. Multiply $3(\cos 1.3\pi + i \sin 1.3\pi) \cdot 15(\cos 2.1\pi + i \sin 2.1\pi)$. Write your answer in polar form only.
2. Find:
   (a) $(1 - i)^{25}$
   (b) $[2(\cos \frac{5\pi}{9} + i \sin \frac{5\pi}{9})]^6$.

Write your answers in both in Cartesian and in polar form.

3. Find all elements of $\sqrt[5]{\zeta}$ where $\zeta = 243(\cos \frac{25\pi}{16} + i \sin \frac{25\pi}{16})$. Write your answers in polar form.
4. if $o(\zeta) = 99$, what's ζ^{200}?
5. What's $2^{47} + 3^{47}$ in $\mathbb{Z}_{47}$?
6. Calculate:
   (a) 3^{30} in $\mathbb{Z}_{29}$
   (b) 3^{30} in $\mathbb{Z}_{31}$
7. Let $\zeta = \cos \frac{5\pi}{9} + i \sin \frac{5\pi}{9}$
   (a) What's $o(\zeta)$?
   (b) What's $o(\zeta^3)$?
   (c) What's $o(\zeta^5)$?
8. Working in $\mathbb{Z}_5$
   (a) What's $o(3)$?
   (b) What's $o(3^{18})$?
9. Working in $\mathbb{Z}_6$
   (a) What's $o(5)$?
   (b) What's $o(5^{25})$?
10. Find all elements of $\mathbb{Z}_{36}$ which have multiplicative inverses.
11. Find all elements of $\mathbb{Z}_{36}$ which are roots of unity.
12. What's the multiplicative inverse of 5 in $\mathbb{Z}_{12}$? What's the additive inverse of 5 in $\mathbb{Z}_{12}$?
13. Using the Euclidean algorithm, find (324, 24).
14. Not using the Euclidean algorithm, find (a, b) where $a = 2^2 3^3 5^5$ and $b = 2^5 3^2 5^2$.
15. In $\mathbb{Z}_4$, which elements a satisfy the equation $a^3 + a = 2$?

16. In $\mathbb{Z}_5$, which elements a satisfy the equation $a^{125} + a^{120} - a^3 - 1 = 0$?

17. Find the quotient and remainder when $x^5 + x^3$ is divided by $x^3 + 2$ in (a) $\mathbb{R}$, (b) $\mathbb{Z}_5$, (c) $\mathbb{Z}_3$

Definitions[1]

1. Define *root of unity* [i.e., "ζ is a root of unity iff..."].
2. Define $o(r)$ [i.e., $o(r) = n$ iff...]
3. Define *primitive* n^{th} *root of unity.*
4. Define *greatest common divisor of* m, n.
5. Define m, n *are relatively prime.*

True/false[2]

1.Let r be a root of unity, either complex or modular.

(a) If $o(r) = 34$ then $r^{17} = 1$.

(b) If $o(r) = 17$ then $r^{34} = 1$.

(c) If $o(r) = 34$ then $r^{17} = -1$

(d) If $o(r) = 17$ then $r^{36} = r^4$

(e) If $o(r) = 17$ then $r^{36} = r^2$

2. Let r be a root of unity, either complex or modular.

(a) If $o(r) = 8$ then $r^3 = r^7$.

(b) If $o(r) = 2$ then $r^3 = r^7$.

(c) If $o(r) = 5$ then $r^3 = r^7$.

3. There are $A, B \in \mathbb{Z}$ so $9A + 12B = 3$.

4. (a) If 5 divides $12n$ then 5 divides n.

(b) If 5 divides n and 12 divides n, then 60 divides n.

(c) If 15 divides $36n$ then 15 divides n.

(d) If 5 divides n and 10 divides n then 50 divides n.

5. (a) Every field is a ring.

(b) Every ring is a field.

6. (a) $\mathbb{Z}_{10}$ is a field.

(b) $\mathbb{Z}_{11}$ is field.

(c) $\mathbb{Z}_{10}$ is a ring.

(d) $\mathbb{Z}_{11}$ is a ring.

Finish the sentence

1. The elements of $\sqrt[n]{1}$ form a regular _____-gon.
2. If r, s are the roots of $x^2 + 3x - 5$ then

   (a) $r + s =$_____

[1] You will be asked at least one of these on the exam.

[2] This is mostly to remind you of important theorems, but there may be some true/false questions on the exam, no guarantees.

(b) $rs =$_____

3. For any $z \in \mathbb{C}$ and $n \in \mathbb{N}$, there are _____ elements in $\sqrt[n]{z}$.
4. If $0 < k < n$ then $\binom{n}{k}$ is divisible by n if n is a _____.
5. For all $a \in \mathbb{Z}$, if p is a prime then $a^p \equiv$_____ (mod p).
6. If p is a prime and $0 < a < p$ then $o(a)$ must divide _____.

Fairly short proofs[3]

1. Prove that if n is not prime then $\sqrt[n]{1}$ has at least one element besides 1 whose order is $< n$.[4]
2. Prove that the order of every element of $\sqrt[12]{1}$ is a divisor of 60, but that not every divisor of 60 is the order of an element of $\sqrt[12]{1}$.
3. Prove that if ζ is any element of $\sqrt[7]{1}$ besides 1, then $\sqrt[7]{1} = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5, \zeta^6\}$.
4. Prove that the real number $\sqrt{5}$ is irrational.
5. Prove that every non-zero element in a field has exactly one multiplicative inverse.
6. Prove that if $o(a), o(b)$ are not relatively prime, then $o(ab) < o(a)o(b)$.
7. Prove that if $p = 2^k + 1$ is a prime, then the order of every non-zero element of $\mathbb{Z}_p$ is a power of 2.[5]
8. Prove that if p is prime and $p > 3$, then there is some element of $\mathbb{Z}_p$ whose order is neither 1 nor $p - 1$.

Proofs by induction

1. (a) Verify that the statement $\Sigma_{i=1}^{n}(2i - 1) = n^2$ holds for $n = 4$.

   (b) Give in inductive proof to show that $\Sigma_{i=1}^{n}(2i - 1) = n^2$ for all $n \in \mathbb{N}$.
2. (a) Verify that the statement $2^n > n^2$ holds when $n = 5, 6, 7$. Verify that it fails when $n = 4$.

   (b) Prove by induction that for every natural number $n \geq 5, 2^n > n^2$. [Hint: in the base case, what does n equal?]
3. (a) Verify that the statement $n^2 > 2n + 1$ holds for $n = 3$ but not for $n = 2$.

   (b) Show directly (no need for induction) that if $n > 2$ then $n^2 > 2n + 1$.

   (c) Verify that the statement $n! > n^2$ holds for $n = 4$ but not for $n = 3$.

   (d) Using (b), show by induction that if $n \geq 4$ then $n! > n^2$.

Proofs to memorize[6]

1. If p is prime then for every $a, b \in \mathbb{Z}$ $(a + b)^p \equiv a^p + b^p$ (mod p).
2. Fermat's little theorem: if p is prime and $a \in \mathbb{Z}$ then $a^p \equiv a$ (mod p).
3. If p is a prime then every non-zero element of $\mathbb{Z}_p$ is a root of unity.
4. If r is a root of unity, then $r^n = 1$ iff $o(r)$ divides n.
5. If F is a field and $a \in F$ then $0 \cdot a = 0$.

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[3] Some are very short.

[4] $\sqrt[n]{1}$ always means complex roots, i.e., roots in $\mathbb{C}$.

[5] Recall that these are the Fermat primes, that there are only five of them known, but nobody knows how many there are.

[6] You will probably be asked one of these.