

Notes on Forcing

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1 The axioms

1. Extensionality (definition of $=$): $\forall x \forall y (x = y \text{ iff } \forall z (z \in x \text{ iff } z \in y))$.
2. Pairing (pairs exist): $\forall x \forall y \exists z \forall w (w \in z \text{ iff } (w = x \text{ or } w = y))$.
3. Union (unions exist): $\forall x \exists z \forall w (w \in z \text{ iff } \exists y (w \in y \text{ and } y \in x))$.
4. Power set (power sets exist): $\forall x \exists z \forall w (w \in z \text{ iff } w \subset x)$.
5. Regularity ($\forall x (x, \in)$ has a minimal element): $\forall x \exists y (y \in x \text{ and } y \cap x = \emptyset)$.
6. Infinity (there is an infinite set): $\exists x (x \neq \emptyset \text{ and } \forall y (\text{if } y \in x \text{ then } y \cup \{y\} \in x))$.
7. Choice (choice functions exist): $\forall x \neq \emptyset (\text{if } \emptyset \notin x \text{ then } \exists f (f \text{ a function, } \text{dom } f = x \text{ and } \forall y \in x f(y) \in y))$.
8. Comprehension Schema (restricted formulas define sets): Let φ be a formula. $\forall x \exists z (y \in z \text{ iff } (y \in x \text{ and } \varphi(y)))$.
9. Replacement Schema (ranges of functions exist): Let φ be a formula. $\forall x (\text{if } \forall y \in x (\varphi(y, z) \text{ and } \varphi(y, w) \Rightarrow z = w) \text{ then } \exists u \forall v (v \in u \text{ iff } \exists y \in x \varphi(y, v)))$.¹

¹#'s 1 through 8 are due to Zermelo; #9 is due to Fraenkel.

2 AD sets and Δ systems

Definition 1. (a) $\mathcal{A} \subset [\kappa]^\kappa$ is an almost disjoint family on κ iff $\forall A, B \in \mathcal{A} |A \cap B| < \kappa$. (b) $\mathfrak{a} = \inf\{|\mathcal{A}| : \mathcal{A} \text{ is maximal almost disjoint on } \omega\}$.

By AC, there is always a maximal almost disjoint family on κ . Maximal almost disjoint families on ω are called MAD families.

Theorem 1. Let κ be regular. If $\mathcal{A} \subset [\kappa]^\kappa$ is almost disjoint and $|\mathcal{A}| = \kappa$ then there is $B \notin \mathcal{A}$ so that $|A \cap B| < \kappa$ for all $A \in \mathcal{A}$.

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$. Each $|A_\alpha \setminus (\bigcup_{\beta < \alpha} A_\beta)| = \kappa$, so pick $a_\alpha \in A_\alpha \setminus (\bigcup_{\beta < \alpha} A_\beta)$. Let $B = \{a_\alpha : \alpha < \kappa\}$. $B \in [\kappa]^\kappa$ and $|A \cap B| \leq \kappa$ for all $A \in \mathcal{A}$. \square

Corollary 1. If κ is regular there is an almost disjoint family on κ of size κ^+ .

Corollary 2. (CH) $\mathfrak{a} = \omega_1$.

Theorem 2. (MA) $\mathfrak{a} = 2^\omega$.

(to be proved later)

Theorem 3. Cons($\mathfrak{a} = \omega_1 < 2^\omega$)

(to be proved much later if we're lucky)

Theorem 4. There is an almost disjoint family on ω of size 2^ω .

Proof. Let $\{s_n : n < \omega\}$ enumerate all finite subsets of ω . If $S \in [\omega]^\omega$ and $s \in [\omega]^{<\omega}$ we define $s < S$ iff s is an initial segment of S . Define $A_S = \{n : s_n < S\}$. Then $\{A_S : S \in [\omega]^\omega\}$ is an almost disjoint family. \square

Definition 2. A family \mathcal{A} is a Δ system iff $\exists R$ (called the root) so that if $A, B \in \mathcal{A}$ then $A \cap B = R$.

Theorem 5. Let \mathcal{A} be an uncountable collection of finite sets. Then \mathcal{A} has an uncountable subfamily which is a Δ system.

This is the form we will generally use. Here is a stronger version:

Theorem 6. Let $\theta > \kappa$, θ regular, so that if $\lambda < \theta$ then $\lambda^{<\kappa} < \theta$. Suppose $|\mathcal{A}| = \theta$ and $\forall A \in \mathcal{A} |A| < \kappa$. Then there is a Δ system $\mathcal{B} \in [\mathcal{A}]^\theta$.

Proof. $|\bigcup \mathcal{A}| = \theta$ so we may assume that \mathcal{A} is a family of subsets of θ , by regularity unbounded in θ .

By regularity of θ we may assume there is $\delta < \kappa$ so that every $A \in \mathcal{A}$ has order type δ . Let $A = \{A(\xi) : \xi < \delta\}$ list A in increasing order.

Again by regularity, we may define $\xi = \inf\{\eta : \{A(\eta) : \eta < \delta\} \text{ is unbounded in } \theta\}$. Let $\lambda < \theta$ be a bound on $\{A(\eta) : A \in \mathcal{A}, \eta < \xi\}$. Since $\lambda^{<\kappa} < \theta$ there is R , order type $R = \xi$ and $\mathcal{A}_R = \{A \in \mathcal{A} : R(\eta) = A(\eta) \text{ for all } \eta < \xi\}$ has cardinality θ , hence $\{A(\xi) : A \in \mathcal{A}_R\}$ is unbounded in θ .

If $\mathcal{C} \in [\mathcal{A}]^{<\theta}$ then $\bigcup \mathcal{C}$ is bounded below θ . Construct $\{A_\alpha : \alpha < \theta\} \subset \mathcal{A}_R$ so that if $\alpha < \beta$ then $\sup A_\alpha < A_\beta(\xi)$, hence $A_\alpha \cap A_\beta = R$. \square

3 CH

Definition 3. Let $f, g \in \omega^\omega$. $f \leq^* g$ iff $\{n : f(n) > g(n)\}$ is finite.

Definition 4. (a) Let $F \subset \omega^\omega$. g dominates F iff $\forall f \in F f \leq^* g$. (b) F is dominating iff $\forall g \in \omega^\omega \exists f \in F g \leq^* f$. (c) F is bounded iff some g dominates it. (d) $\mathfrak{d} = \inf\{|F| : F \text{ is dominating}\}$. (e) $\mathfrak{b} = \inf\{|F| : F \text{ is unbounded}\}$.

Theorem 7. Every countable subset of ω^ω is bounded. (I.e., $\mathfrak{d}, \mathfrak{b} > \omega$.)

Proof. Let $F = \{f_n : n < \omega\}$ (repetitions allowed). Let $g(n) = 1 + \sum_{k < n} f_k(n)$. Then $g \geq^* f_n$ for all n . \square

Corollary 3. (CH) $\mathfrak{d} = \mathfrak{b} = \omega_1$.

Theorem 8. (MA) $\mathfrak{d} = \mathfrak{b} = 2^\omega$.

(proof later)

Theorem 9. (a) $\text{Cons}(\mathfrak{b} = \omega_1 < 2^\omega = \mathfrak{d})$ (b) $\text{Cons}(\mathfrak{d} = \omega_1 < 2^\omega)$.

(proof much later if we're lucky)

Definition 5. A scale is a dominating family well-ordered by \leq^* .

Theorem 10. (CH) There is a scale.

Proof. Let $\{f_\alpha : \alpha < \omega_1\}$ list all elements of ω^ω . By the technique of theorem 9, construct $\{g_\alpha : \alpha < \omega_1\}$ so each g_α dominates $\{f_\beta : \beta < \alpha\} \cup \{g_\beta : \beta < \alpha\}$. \square

Theorem 11. (MA) There is a scale, and every scale has order type 2^ω .

(to be proved later)

Theorem 12. $\text{Cons}(\exists \text{ scale})$

(proof much later if we're lucky)

Definition 6. (a) Let $a, b \subset \omega$. $a \subset^* b$ iff $a \setminus b$ is finite. (b) Let $\mathcal{A} \subset [\omega]^\omega, b \subset \omega$. b is a pseudo-intersection of \mathcal{A} iff b is infinite and $b \subset^* a$ for all $a \in \mathcal{A}$.

Definition 7. (a) $\{a_\alpha : \alpha < \delta\} \subset [\omega]^\omega$ is a tower iff $\alpha < \beta \Rightarrow a_\beta \subset^* a_\alpha$ and $\{a_\alpha : \alpha < \delta\}$ has no pseudo-intersection. (b) $\mathfrak{t} = \inf\{|T| : T \text{ a tower}\}$.

By AC, there is always a tower.

Theorem 13. Let \mathcal{A} be a countable filterbase of infinite sets under \subset^* . Then \mathcal{A} has a pseudo-intersection.

Proof. Since \mathcal{A} is countable, it has a cofinal descending subfamily $\{a_n : n < \omega\}$. By induction, construct $b = \{k_n : n < \omega\}$ so each $k_n \in \bigcap \{a_m : m < n\}$. \square

Corollary 4. $\mathfrak{t} > \omega$.

Theorem 14. *(CH)* $\mathfrak{t} = \omega_1$.

Proof. List the infinite subsets of ω as $\{c_\alpha : \alpha < \omega_1\}$. Construct the tower $\{a_\alpha : \alpha < \omega_1\}$ by induction so each a_α is a pseudo-intersection of $\{c_\beta : \beta < \alpha\} \cup \{a_\beta : \beta < \alpha\}$. \square

Theorem 15. *(MA)* $\mathfrak{t} = 2^\omega$.

(to be proved later)

Theorem 16. *Cons*($\mathfrak{t} < 2^\omega$)

(to be proved much later if we're lucky)

4 Martin's axiom

Theorems 1, 9 and 16 were proved by induction. There is a technique equivalent induction on ω , due to Rasiowa and Sikorski, which appears in the course of proving the Rasiowa-Sikorski theorem in Boolean algebra, so I will mis-call it the Rasiowa-Sikorski lemma:

Lemma 1. *If \mathbb{P} be a partial order, and \mathcal{D} is a countable collection of dense subsets of \mathbb{P} , then there is G a filter on \mathbb{P} so that $\forall D \in \mathcal{D} \exists g \in G \exists p \in D g \leq p$. (We say that G is \mathcal{D} -generic.)*

To understand the statement of this lemma, we need some definitions.

Definition 8. *Let \mathbb{P} is a partial order.*

- (a) $D \subset \mathbb{P}$ is dense iff $\forall p \in \mathbb{P} \exists q \in D q \leq p$.
- (b) $G \subset \mathbb{P}$ is a filter iff $\forall a \in [G]^{<\omega} \exists p \in G p \leq q$ for all $q \in a$.

It is this lemma that generalizes into Martin's axiom:

Definition 9. *MA is the following statement: If \mathbb{P} is a ccc partial order, and \mathcal{D} is a collection of dense subsets of \mathbb{P} with $|\mathcal{D}| < 2^\omega$, then there is G a filter on \mathbb{P} so that $\forall D \in \mathcal{D} \exists g \in G \exists p \in D g \leq p$.*

We will define ccc later. For now, just note that in the presence of CH, MA is a weaker form of the Rasiowa-Sikorski lemma, so $\text{CH} \Rightarrow \text{MA}$.

We prove the lemma, then prove theorems 1, 9, and 16 using it.

Proof. (a) the Rasiowa-Sikorski lemma: given \mathbb{P}, \mathcal{D} as in the hypothesis, list $\mathcal{D} = \{D_n : n < \omega\}$. By induction construct $\{g_n : n < \omega\}$ where $g_n \in D_n$ and each $g_n \geq g_{n+1}$.

Having proved the Rasiowa-Sikorski lemma in (a), we prove (b) $\mathfrak{a} > \omega$, (c) $\mathfrak{b} > \omega$, (d) $\mathfrak{t} > \omega$.

(b) theorem 1 (for $\kappa = \omega$): given \mathcal{A} a countable almost disjoint family on ω , let $\mathbb{P} = \{p = (b_p, \mathcal{B}_p) : b_p \text{ is a finite subset of } \omega, \mathcal{B}_p \text{ is a finite subset of } \mathcal{A}\}$. Define $p \leq q$ iff $b_p \supset b_q$ and $\forall A \in \mathcal{B}_q (b_p \setminus b_q) \cap A = \emptyset$, i.e., we extend the first coordinate, and what we add avoids everything in the second.

For $A \in \mathcal{A}$ let $D_A = \{p : A \in \mathcal{B}_p\}$. D_A is dense: fix $p \in \mathbb{P}$. Let $b_q = b_p \cup \mathcal{B}_q = \mathcal{B}_p \cup \{A\}$. Then $q \leq p$ and $q \in D_A$.

For $n < \omega$ let $D_n = \{p : |b_p| > n\}$. D_n is dense: fix $p \in \mathbb{P}$. Let $k \in \omega \setminus (n+1 \cup \bigcup \mathcal{B}_p)$ — we know k exists because \mathcal{A} is infinite, hence $\omega \setminus \bigcup \mathcal{B}_p$ is infinite.

Let $\mathcal{D} = \{D_A : A \in \mathcal{A}\} \cup \{D_n : n < \omega\}$. \mathcal{D} is countable. Let G be a \mathcal{D} -generic filter. Let $B = \bigcup_{p \in G} b_p$.

B is infinite: Fix n . Let $p \in G \cap D_n$. Then $B \supset b_p$ and $b_p \setminus n \neq \emptyset$. Hence $B \setminus n \neq \emptyset$ for all n .

$B \cap A =^* \emptyset$ for all $A \in \mathcal{A}$: Fix $A \in \mathcal{A}$. Let $p \in G \cap D_A$, i.e., $A \in \mathcal{B}_p$. Let $k \in B$. $\exists q \in G k \in b_q$. Since G is a filter, $\exists r \in G r \leq p, q$. So $k \in r$ and $(b_r \setminus b_p) \cap A = \emptyset$. Hence if $k \in A$ then $k \in b_p$, which is finite.

(c) theorem 9: given $F \in [\omega^\omega]^{\leq \omega}$, let $\mathbb{P} = \{p = (\sigma_p, H_p) : \sigma_p \text{ is a finite function from } \omega \text{ to } \omega, H_p \text{ is a finite subset of } F\}$. Define $p \leq q$ iff $\sigma_p \supset \sigma_q$, $H_p \supset H_q$, and $\forall f \in H_q$ if $n \in \text{dom } \sigma_p \setminus \sigma_q$ then $\sigma_p(n) > f(n)$, i.e. we extend the first coordinate and eventually dominate everything in the second.

For $f \in F$ let $D_f = \{p : f \in H_p\}$. D_f is dense: fix $p \in \mathbb{P}$. Let $\sigma_q = \sigma_p, H_q = H_p \cup \{f\}$. Then $q \leq p$ and $q \in D_f$.

For $n < \omega$ let $D_n = \{p : n \in \text{dom } \sigma_p\}$. Each D_n is dense: fix $p \in \mathbb{P}$. Let $H_q = H_p$. For $k \leq n$ with $k \notin \text{dom } \sigma_p$, define $\sigma_q(k) = 1 + \sum_{f \in H_p} f(k)$; if $k \in \text{dom } \sigma_p$, $\sigma_q(k) = \sigma_p(k)$. Then $q \leq p$ and $q \in D_n$.

Let $D = \{D_f : f \in F\} \cup \{D_n : n < \omega\}$. \mathcal{D} is countable. Let G be \mathcal{D} -generic. Let $g = \bigcup_{p \in G} \sigma_p$.
 $g \in \omega^\omega$: fix $k \in \omega$. $\exists p \in G \cap D_k$. So $k \in \text{dom } \sigma_p \subset \text{dom } g$.

$g \geq^* f$ for all $f \in F$: fix $f \in F$. Let $p \in G \cap D_f$. Let $k < \omega$. $\exists q \in G \cap D_k$, so $\sigma_q(k) = g(k)$. Let $r \leq q, p, r \in G$. Then $g(k) = \text{sigma}_r(k)$ and if $k \notin \text{dom } \sigma_p$ then $g(k) > f(k)$. So if $g(k) \leq f(k)$, $k \in \text{dom } \sigma_p$ which is finite.

(d) theorem 16: Let \mathcal{A} be a countable filterbase in $[\omega]^\omega$. Let $\mathbb{P} = \{p = (a_p, \mathcal{C}_p) : a_p \in [\omega]^{<\omega}, \mathcal{C}_p$ is a finite subset of $\mathcal{A}\}$. Define $p \leq q$ iff $a_p \supset a_q, \mathcal{C}_p \supset \mathcal{C}_q$ and for all $A \in \mathcal{C}_q$ $a_p \setminus a_q \subset A$, i.e., we extend the first coordinate and eventually stay inside everything in the second.

Define $D_A = \{p : A \in \mathcal{C}_p\}$. D_A is dense: fix $p \in \mathbb{P}$. Let $a_q = a_p, \mathcal{C}_q = \mathcal{C}_p \cup \{A\}$. $q \leq p$ and $q \in D_A$.

Define $D_n = \{p : \sup a_p > n\}$. D_n is dense: fix $p \in \mathbb{P}$. Let $k \in \bigcup \mathcal{C}_p \setminus n + 1$. (We can do this because \mathcal{A} is a non-principal filterbase.) Let $a_q = a_p \cup \{k\}, \mathcal{C}_q = \mathcal{C}_p$. $q \leq p$ and $q \in D_n$.

Let $\mathcal{D} = \{D_A : A \in \mathcal{A}\} \cup \{D_n : n < \omega\}$. \mathcal{D} is countable. Let G be a \mathcal{D} -generic filter. Let $B = \bigcup_{p \in G} a_p$.

B is infinite: fix n . Let $p \in G \cap D_n$. Then $B \supset a_p$ and $a_p \setminus n \neq \emptyset$. Hence $\forall n$ $B \setminus n \neq \emptyset$.

$B \subset^* A$ for all $A \in \mathcal{A}$: Fix A . Let $p \in G \cap D_A$. Let $k \in B$. $\exists q \in G$ $k \in a_q$. Since G is a filter, $\exists r \in G$ $r \leq p, q$. Hence $k \in a_r$ and if $k \notin a_p$ then $k \in A$. So $B \setminus A \subset a_p$, which is finite.

□

What (b), (c) and (d) have in common are: conditions in \mathbb{P} are ordered pairs; the first coordinate is a finite approximation of the final object; the second coordinate provides some kind of control, i.e., we promise that from now on we will, respectively, (a) stay away from; (b) stay above; (c) stay inside, elements in the second coordinate. Such second coordinates used to be called side conditions, now they are called promises.

Recall:

Definition 10. *MA is the following statement: If \mathbb{P} is a ccc partial order, and \mathcal{D} is a collection of dense subsets of \mathbb{P} with $|\mathcal{D}| < 2^\omega$, then there is G a filter on \mathbb{P} so that $\forall D \in \mathcal{D} \exists g \in G \exists p \in D$ $g \leq p$.*

To understand the statement of MA we need

Definition 11. *Let \mathbb{P} be a partial order.*

- (a) $p, q \in \mathbb{P}$ are compatible iff there is $r \in \mathbb{P}$ with $r \leq p, r \leq q$.
- (b) $E \subset \mathbb{P}$ is an antichain iff no two elements E of are compatible.
- (c) \mathbb{P} is ccc iff it has no uncountable antichains.

Theorem 17. (MA) (a) $\mathfrak{a} = 2^\omega$.

- (b) $\mathfrak{b} = 2^\omega$.
- (c) $\mathfrak{t} = 2^\omega$.

Proof. The proofs are straightforward generalizations of the Rasiowa-Sikorski technique and the CH situation. First we prove that a family of size $< 2^\omega$ cannot be, respectively, MAD, unbounded,

or without a pseudointersection. Hence $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 2^\omega$. Once we've done that, note that $\mathfrak{a}, \mathfrak{b}, \mathfrak{t} \leq 2^\omega$. Done.

We use the partial orders we used in the countable situation. It's clear that the sets we want to be dense are dense — that did not depend on the cardinalities of the underlying families \mathcal{A} or F . But it is not obvious that the respective partial orders are ccc. In each case, the reason will be that conditions which share the same first coordinate are compatible. Here are the details.

(a) Let \mathcal{A} be an almost disjoint family on $\omega, |\mathcal{A}| < \mathfrak{c}$, and $\mathbb{P} = \{p = (b_p, \mathcal{B}_p) : b_p \text{ is a finite subset of } \omega, \mathcal{B}_p \text{ is a finite subset of } \mathcal{A}; p \leq q \text{ iff } b_p \supset b_q \text{ and } \forall A \in \mathcal{B}_q (b_p \setminus b_q) \cap A = \emptyset\}$. As in the Rasiowa-Sikorski lemma proofs, each D_A, D_n is dense, so if G is a \mathcal{D} -generic filter then $B = \bigcup_{p \in G} a_p$ has finite intersection with every $A \in \mathcal{A}$. To show that G exists, we only have to show that \mathbb{P} is ccc.

For $a \in [\omega]^{<\omega}$, define $P_a = \{p : a = a_p\}$. If $p, q \in P_a$ then p, q are compatible (because $(a, \mathcal{B}_p \cup \mathcal{B}_q) \leq p, q$). There are only countably many P_a 's, so an uncountable subset E of \mathbb{P} must have two (in fact uncountably many) elements in the same P_a , so E could not be an antichain.

(b) $\mathfrak{b} = 2^\omega$: Let $F \subset \omega^\omega, |F| < \mathfrak{c}, \mathbb{P} = \{p = (\sigma_p, H_p) : \sigma_p \text{ is a finite function from } \omega \text{ to } \omega, H_p \text{ is a finite subset of } F, p \leq q \text{ iff } \sigma_p \supset \sigma_q, H_p \supset H_q, \text{ and } \forall n \in \text{dom } \sigma_p \setminus \text{dom } \sigma_q \forall f \in H_q \sigma_p(n) > f(n)\}$. As in the Rasiowa-Sikorski lemma proofs, each D_f, D_n is dense, so if G is a \mathcal{D} -generic filter, then $g = \bigcup_{p \in G} \sigma_p$ dominates F . To show that G exists, we only have to show that \mathbb{P} is ccc.

For σ a finite function from ω to ω define $P_\sigma = \{p : \sigma_p = \sigma\}$. If p, q are in P_σ then p, q are compatible (because $(\sigma, H_p \cup H_q) \leq p, q$). There are only countably many such P_σ 's, so an uncountable subset E of \mathbb{P} must have two (in fact uncountably many) elements in the same P_σ , so E could not be an antichain.

(c) Suppose \mathcal{A} is a filter base in $[\omega]^\omega, |\mathcal{A}| < \mathfrak{c}$, and $\mathbb{P} = \{p = (a_p, \mathcal{C}_p) : a_p \in [\omega]^{<\omega}, \mathcal{C}_p \text{ is a finite subset of } \mathcal{A}\}$, and $p \leq q$ iff $a_p \supset a_q, \mathcal{C}_p \supset \mathcal{C}_q$ and for all $A \in \mathcal{C}_q$ $a_p \setminus a_q \subset A$. As in the Rasiowa-Sikorski lemma proofs, each D_A, D_n is dense, so if G is a \mathcal{D} -generic filter then $B = \bigcup_{p \in G} a_p$ is a pseudo-intersection of \mathcal{A} . To show that G exists, we only have to show that \mathbb{P} is ccc.

For $a \in [\omega]^{<\omega}$, define $P_a = \{p : a = a_p\}$. If $p, q \in P_a$ then p, q are compatible (because $(a, \mathcal{B}_p \cup \mathcal{B}_q) < p, q$). There are only countably many P_a 's, so an uncountable subset E of \mathbb{P} must have two (in fact uncountably many) elements in the same P_a , so E could not be an antichain.

Note that the proofs of ccc do not depend on the size of the underlying sets \mathcal{A} or F . We restrict their size to restrict the number of dense sets we need to meet. \square

In the preceding proof, we showed that the partial order \mathbb{P} was the countable union of families, each of which was linked (i.e., any two elements are compatible). A partial order which is the countable union of linked families is called σ -linked. A σ -linked family is necessarily ccc. So too, a partial order which is σ -centered, i.e., the countable union of centered (= filter base) families, is ccc. The orders in theorem 21 are easily seen to be σ -centered.

Theorem 18. (MA) Let \mathcal{A} be almost disjoint, $|\mathcal{A}| < \mathfrak{c}, \mathcal{B} \subsetneq \mathcal{A}$. Then there is $B \subset \omega$ $B \cap A =^* \emptyset$ for all $A \in \mathcal{B}$ and $B \cap A \neq^* \emptyset$ for all $A \in \mathcal{A} \setminus \mathcal{B}$.

Proof. The partial order is as in (a) above, except we require that $\mathcal{B}_p \subset \mathcal{B}$. Our dense sets are now: for $A \in \mathcal{B}$ $D_A = \{p : A \in \mathcal{B}_p\}$, and for $A \in \mathcal{A} \setminus \mathcal{B}$ $D^{A,n} = \{p : |a_p \cap A| > n\}$.

Why is each $D^{A,n}$ dense? Fix p . $A \setminus \bigcup \mathcal{B}_p$ is infinite. Let $c \subset A \setminus \bigcup \mathcal{B}_p, |c| > n$. Let $q = (a_p \cup c, \mathcal{B}_p)$. Then $q \leq p$ and $q \in D^{A,n}$.

Why is \mathbb{P} ccc? As before, because \mathbb{P} is σ -centered. \square

This is an example of a common phenomenon in forcing: if you can't stop something from happening, it happens. E.g., by restricting each $\mathcal{B}_p \subset \mathcal{B}$, we can't stop the final set from having infinite intersection with each $A \in \mathcal{A} \setminus \mathcal{B}$.

Corollary 5. (MA) *Let $\omega \leq \kappa < \mathfrak{c}$. Then $2^\kappa = \mathfrak{c}$.*

Proof. Fix \mathcal{A} almost disjoint, $|\mathcal{A}| = \kappa$. Define $f : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\mathcal{A})$ by $f(B) = \{A \in \mathcal{A} : A \cap B =^* \emptyset\}$. By theorem 22, f is onto. So $2^\omega \leq 2^\kappa = |(\mathcal{P}(\mathcal{B}))| \leq |(\mathcal{P}(\omega))| = 2^\omega$. \square

Corollary 6. (MA) *\mathfrak{c} is regular.*

Proof. Let $\{a_\alpha : \alpha < \mathfrak{c}\}$ be a tower. Let $\kappa = \text{cf } \mathfrak{c}$, $f : \kappa \rightarrow \mathfrak{c}$ be the cofinal increasing map witnessing this. Then $\{a_{f(\alpha)} : \alpha < \kappa\}$ is a tower. By theorem 21(c), $\kappa = \mathfrak{c}$. \square

In the first decade or two after MA was articulated, there was a lot of work on weaker variants, e.g., MA_σ -centered where we only guarantee a generic set for partial orders which are σ -centered. Then the attention shifted to stronger variants. The main ones are PFA and MM. In some sense it is premature to mention these — I will be very happy if at the end of the course you have even a rough idea of what PFA means. But the main idea isn't difficult: we find principles more general than ccc, and guarantee generic sets for these partial orders.

Definition 12. (a) *A partial order is proper iff $\forall \kappa > \omega$ it preserves stationary subsets of $[\kappa]^\omega$. (Here "preserves" means "when you force with it"; and stationary means just what it usually means: meets every closed unbounded set. The partial order is now $[\kappa]^\omega$ instead of ω_1)*

(b) *PFA (proper forcing axiom) is the following statement: If \mathbb{P} is a proper partial order and \mathcal{D} is a family of dense subsets of size ω_1 then there is a \mathcal{D} -generic filter in \mathbb{P}*

(c) *MM (Martin's Maximum) is the following statement: If \mathbb{P} is a partial order preserving stationary subsets of ω_1 and \mathcal{D} is a family of dense subsets of size ω_1 then there is a \mathcal{D} -generic filter in \mathbb{P} .*

Both PFA and MM require large cardinals to prove their full consistency (but many results using them can be proved without the large cardinal assumption, e.g., no S spaces). The logical implications are: $\text{MM} \Rightarrow \text{PFA} \Rightarrow \text{MA}$. While MA is consistent with \mathfrak{c} being any uncountable regular cardinal, $\text{PFA} \Rightarrow 2^\omega = \omega_2$. This is due to Velickovic, and uses the combinatorics developed by Todorcevic of ladder systems and the trace function.

Theorem 19. (MA) *If \mathcal{Z} is a collection of measure zero subsets of \mathbb{R} , and $|\mathcal{Z}| < \mathfrak{c}$ then $\mu(\bigcup \mathcal{Z}) = 0$.*

Proof. It suffices to show that, for all $\varepsilon > 0$ there is u open with $Z = \bigcup \mathcal{Z} \subset u$ and $\mu(u) \leq \varepsilon$.

Let \mathbb{B} be a countable base for the topology on \mathbb{R} (e.g., intervals with rational endpoints), and let $\mathbb{U} = \{\text{finite unions of elements of } \mathbb{B}\}$. Fix $\varepsilon > 0$. Let $\mathbb{P}_\varepsilon = \{u : u \text{ an open subset of } \mathbb{R}\}$, under the order $u \leq v$ iff $u \supset v$. For each $Z \in \mathcal{Z}$ let $D_Z = \{u : Z \subset u\}$.

\mathbb{P}_ε has ccc, since \mathbb{U} is countable.

Let $Z \in \mathcal{Z}$. We prove that D_Z is dense: Let $u \in \mathbb{P}_\varepsilon$. Define $\delta = \varepsilon - \mu(u)$. There is v open with $Z \subset v, \mu(v) < \delta$. Hence $\mu(u \cup v) < \varepsilon$. Hence $u \cup v \leq u$ and $u \cup v \in D_Z$.

By MA, there is G a filterbase on \mathbb{P} with $G \cap D_Z \neq \emptyset$ for all $Z \in \mathcal{Z}$. Let $u = \bigcup G$.

$\bigcup \mathcal{Z} \subset u$: Fix $Z \in \mathcal{Z}$. There is $v \in G \cap D_Z$. $Z \subset v \subset u$.

$\mu(u) \leq \varepsilon$: If not, there is a finite set $v_0, \dots, v_n \in G$ with $\mu(v_0 \cup \dots \cup v_n) > \varepsilon$. But G is a filterbase, so $\exists v \in \mathbb{P}_\varepsilon \forall i \leq n v \leq v_i$, i.e., $v \supset v_i$ for all $i \leq n$. Hence $\mu(v_0 \cup \dots \cup v_n) \leq \mu(v) < \varepsilon$. \square

Recall that a Suslin tree is a tree with height ω_1 , no uncountable branches, and no uncountable antichains (in a tree, an antichain is a pairwise incomparable set).

Lemma 2. *If there is a Suslin tree, there is one in which every element has successors of arbitrary height.*

Proof. Let \mathbb{S} be a Suslin tree, where $\text{ht } t$ is defined as the order type of $\{s : s < t\}$. Let $E = \{s : \exists \alpha_s \text{ if } t \geq s \text{ then } \text{ht } t < \alpha_s\}$.

Suppose $\{\alpha_s : s \in E\}$ is countable. Then E is countable, and $\mathbb{S} \setminus E$ is the desired tree.

Suppose $\{\alpha_s : s \in E\}$ is uncountable. Then we can construct a sequence $\{s_\beta : \beta < \omega_1\} \subset E$ where each $\text{ht } s_\beta > \sup\{\alpha_{s_\gamma} : \gamma < \beta\}$. Hence, if $\gamma < \beta$ then s_γ and s_β are incomparable. But then $\{s_\beta : \beta < \omega_1\}$ is an antichain. \square

Theorem 20. *(MA + \neg CH) There are no Suslin trees.*

Proof. Suppose \mathbb{S} is a Suslin tree. We may assume that every element has successors of arbitrary height. Then it is a ccc partial order under the reverse ordering (i.e., $s \leq t$ iff s is above t in the tree order). Let $D_\alpha = \{s : \text{ht } s \geq \alpha\}$. Each D_α is dense, by hypothesis on \mathbb{S} . So there is a filter G meeting each of them. But then G is an uncountable chain, hence generates an uncountable branch, a contradiction. \square

Theorem 21. *(MA + \neg CH) (a) If X is ccc and \mathcal{U} is a family of uncountably many open subsets of X , then \mathcal{U} contains an uncountable filterbase.*

(b) If X and Y are ccc spaces, then so is $X \times Y$

Recall: a topological space is ccc iff it has no uncountable family of pairwise disjoint open sets.

Proof. (a) Let \mathcal{U} be a family of uncountably many open subsets of the ccc space X . We may assume $\mathcal{U} = \{u_\alpha : \alpha < \omega_1\}$. Let $\mathbb{P} = \{U : U \text{ a finite subset of } \mathcal{U}, \{v \in \mathcal{U} : v \cap \bigcap U \neq \emptyset\} \text{ is uncountable}\}$.

$\mathbb{P} \neq \emptyset$: Otherwise every element of \mathcal{U} would meet only countably many elements of \mathcal{U} , and we could easily construct an uncountable pairwise disjoint subfamily of \mathcal{U} , contradicting ccc.

If $U \in \mathbb{P}$ then $\{v \in \mathcal{U} : U \cup \{v\} \in \mathbb{P}\}$: otherwise we could construct a pairwise disjoint subfamily of $\{\bigcap U \cap v : v \in \mathcal{U}, v \cap \bigcap U \neq \emptyset\}$, contradicting ccc.

Let $D_\alpha = \{U \in \mathbb{P} : \exists \beta \geq \alpha \ u_\beta \in U\}$. By the paragraph above, D_α is dense for all α .

Let G be a $\{D_\alpha : \alpha < \omega_1\}$ -generic filter in \mathbb{P} . Then $\bigcup G$ is a filterbase, $\bigcup G \subset \mathcal{U}$.

(b) Recall that if w is an open subset of $X \times Y$ then there is u open in X , v open in Y with $u \times v \subset w$. Let X, Y be ccc. Suppose we have \mathcal{W} a pairwise disjoint open family on $X \times Y$, with \mathcal{W} uncountable. For $w \in \mathcal{W}$, let $u_w \times v_w \subset w$, where u_w, v_w are open in X, Y respectively.

By (a) we may assume that $\{u_w : w \in \mathcal{W}\}$ forms a filterbase. Since $w \neq w' \Rightarrow w \cap w' = \emptyset$, $\{v_w : w \in \mathcal{W}\}$ must be pairwise disjoint. Which contradicts Y being ccc. \square

Remark: Most of these results are due to Solovay or Martin or both; some work of Tennebaum.

Remark: Galvin proved that $\text{CH} \Rightarrow \exists X \text{ ccc with } X^2 \text{ not ccc.}$

5 A quick introduction to forcing

Suppose we have a set of reals $A \subset 2^\omega$ and we want to add a new real $x \notin A$. Here is a simple partial order to do it: $\mathbb{P} = \bigcup_{k < \omega} {}^k\omega$; $\leq = \supseteq$. (\mathbb{P} is called the Cohen partial order.) For $a \in A$ let $D_a = \{p : \exists n p(n) \neq a(n)\}$. D_a is dense. For $n < \omega$, let $D_n = \{p : n \in \text{dom } p\}$. So if G is a $\{D_a : a \in A\} \cup \{D_n : n < \omega\}$ -generic filter, then $\bigcup G \in 2^\omega \setminus A$.

In fact we can do more. Pick any κ and we can add κ reals not in A : $\mathbb{Q} = \kappa \times \mathbb{P}$. (\mathbb{Q} is the forcing adding κ many Cohen reals). For $a \in A, \alpha < \kappa$ define $D_{a,\alpha} = \{p : \exists n p(\alpha, n) \neq a(n)\}$. Each $D_{a,\alpha}$ is dense. For $n < \omega, \alpha < \kappa$ define $D_{n,\alpha} = \{p : n \in \text{dom } p\}$. Letting G be a $\{D_{a,\alpha} : a \in A, \alpha < \kappa\} \cup \{D_{n,\alpha} : n < \omega, \alpha < \kappa\}$ -generic filter, and $b_\alpha = \bigcup G|_{\{\alpha\} \times \omega}$, each $b_\alpha \notin A$ for all $a \in A$.

If $A = 2^\omega \cap M$ for some model of set theory, we've just added one, or even possibly κ , many reals to M , and we can form a new model in a fashion analogous to adding $\sqrt{2}$ to the set of rationals.

But some questions arise.

1. How do we know G exists? If M is countable, we can and will invoke Rasiowa-Sikorski. Aside: If M is countable transitive — and the theory will require it to be transitive — then our κ is really an ordinal that M thinks is a cardinal, which is not a problem as far as consistency proofs go, but you should be aware of it.

2. If we are adding, say, ω_1^M many reals, how do we know that in the new model ω_1 doesn't suddenly become countable? This is a serious question, and the issue of collapsing cardinals is an important one.

3. What other properties does the model have? For example, if $M \models CH$ then there is a MAD $\mathcal{A} \in M$ so that no matter how many Cohen reals you add, \mathcal{A} remains MAD in the new model. For another example, if you add a single Cohen real to any model of set theory, you automatically add S and L spaces and ccc spaces whose squares are not ccc. How can you understand the new models well enough to prove theorems like these?

The technicalities of forcing are designed to help us with issues such as #2 and #3. There are two approaches: forcing over countable models, and Boolean-valued models. The latter approach is easier to prove theorems about and justifies issue #1 without recourse to countable transitive models; the former approach is, for most people, more intuitive. We will prove things carefully using Boolean valued models, and then feel free to refer to adding generic sets to countable transitive models.

6 Partial orders and complete Boolean algebras

Recall that a Boolean algebra has a meet $+$ (think of \cup), a join \cdot (think of \cap) and a relative complement $-$ (think of \setminus), a 0 (think of \emptyset) and a 1 (think of X when our algebra is $\mathcal{P}(X)$) satisfying the axioms that $\cup, \cap, \setminus, \emptyset$ and X satisfy. E.g., one of De Morgan's laws reads: $a - (b \cdot c) = (a - b) + (a - c)$. Note that $-$ is relative: $-a$ is an abbreviation for $1 - a$.

Every Boolean algebra has an induced partial order \leq (think of \subset): $b \leq c$ iff $b - c = 0$.

A complete Boolean algebra is a Boolean algebra closed under arbitrary (not just finite) sums. For example, $\mathcal{P}(X)$ is a complete Boolean algebra (here X corresponds to 1) because if $\mathcal{A} \subset \mathcal{P}(X)$ then $\bigcup \mathcal{A} = \Sigma_{A \in \mathcal{A}} A \subset \mathcal{P}(X)$. For another example, the Boolean algebra $\text{FinCof}(\omega) = \{a \subset \omega : a \text{ finite or } \omega - a \text{ finite}\}$ is not complete, since, e.g., $\Sigma_{n < \omega} \{2n\} = \{2n : n < \omega\} \notin \text{FinCof}(\omega)$.

First we define: $A \subset \mathbb{P}$ is pre-dense iff $\forall p \in \mathbb{P} \exists q \in A$ p, q are compatible. Some examples: a dense set is pre-dense; a maximal antichain is pre-dense.

We write $p \perp q$ if p, q are incompatible.

If \mathbb{P} has a minimum element 0, then no two elements of \mathbb{P} are incompatible and every maximal filter in \mathbb{P} has 0 as an element, so \mathbb{P} is not very interesting in the context of forcing. A particular class of partial orders with no minimum elements is the class of separative partial orders.

Definition 13. Let \mathbb{P} be a partial order. We say \mathbb{P} is separative iff $\forall p \forall r \neq p \exists q$ either $q \leq p$ and $q \perp r$; or $q \leq r$ and $q \perp p$

Note: if \mathbb{P} is separative, then $\forall p \exists q, r$ $q, r \leq p$ and $q \perp r$.

For any separative \mathbb{P} and $p \in \mathbb{P}$, the following is a useful dense set: $D_p = \{q : q \leq p \text{ or } q \perp p\}$. Note that a filter meeting every D_p is necessarily maximal (since in \mathbb{B} it is a filterbase in which, $\forall p \in \mathbb{P}$, either $p \in G$ or $-p \in G$).

Theorem 22. Let \mathbb{P} be a separative. Then there is a complete Boolean algebra \mathbb{B} so that (a) \mathbb{P} is dense in $\mathbb{B} \setminus 0$ and (b) if A is pre-dense in \mathbb{P} , then, in \mathbb{B} , $\Sigma A = 1$.

Proof. Fix \mathbb{P} a partial order without a minimum element. Let $A \subset \mathbb{P}, p \in \mathbb{P}$. We define $p \perp A$ iff $\forall q \in A$ $p \perp q$.

For $A \subset \mathbb{P}$ define $\Sigma A = \{p : \forall q \leq p \neg(q \perp A)\}$, $1 = \mathbb{P} = \Sigma \mathbb{P}$, $0 = \emptyset = \Sigma \emptyset$. Note that $\Sigma A = \Sigma B \not\Rightarrow A = B$. For example, if A, B are pre-dense, then $\Sigma A = \Sigma B = 1$.

Let $\mathbb{B} = \{\Sigma A : A \subset \mathbb{P}\}$.

We define the Boolean operations: $\Sigma A + \Sigma B = \Sigma(A \cup B)$. $-\Sigma A = \{p : p \perp A\} = \Sigma\{p : p \perp A\}$. Necessarily, $\Sigma A \cdot \Sigma B = 1 - (-\Sigma A + -\Sigma B)$.

You can check this is a Boolean algebra. Clearly it is complete.

We embed \mathbb{P} in \mathbb{B} as follows: $e(p) = \Sigma\{p\} = \{q : q \leq p\}$.

If (a) holds, then so does (b). To show (a) it suffices to show: $\forall p \neq q \in \mathbb{P}$ either $p - q \neq 0$ or $q - p \neq 0$. By separativity, if $p \neq q$ then either $\exists r \leq p, r \perp q$ (so $0 \neq r \leq p - q$) or (ii) $\exists r \leq q, r \perp p$ (so $0 \neq r \leq q - p$).

□

We call \mathbb{B} the completion of \mathbb{P} . We will write p instead of $\Sigma\{p\}$.

An example: Let \mathbb{P} be the upside-down binary tree of height $\omega = \bigcup_{k < \omega} ({}^k 2)$. Let b be a branch. $\Pi b = 0$. Let A be a maximal antichain. $\Sigma A = 1$. Let $\sigma : 3 \rightarrow 2$, each $\sigma(i) = 1$. Let $\tau \supset \sigma, \tau : 5 \rightarrow 2, \tau(3) = \tau(4) = 0$. Then $\sigma - \tau = \{\rho : \rho \supset \sigma \text{ and } \rho(3) = 1 \text{ or } \rho(4) = 1\}$.

Fact 1. Let \mathbb{B} be the completion of \mathbb{P} .

(a) D is dense in \mathbb{B} iff $\Sigma D = 1$.

(b) Let D be dense in \mathbb{P} . Then D is dense in \mathbb{B} .

Definition 14. Let M be a transitive model of set theory, \mathbb{P} a partial order, $\mathbb{P} \in M$. We say the filter G is \mathbb{P} -generic over M iff $G \cap D \neq \emptyset$ for all dense subsets D of \mathbb{P} with $D \in M$.

Logical fine-point: For $D \in M$, D is a dense subset of \mathbb{P} iff $M \models D$ is a dense subset of \mathbb{P} .

Recall that in a transitive model M , if $x \in M$ then $x \subset M$.

Fact 2. Let M be a countable transitive model of set theory, \mathbb{P} separative, $\mathbb{P} \in M$, and let G be \mathbb{P} -generic over M . Then $G \notin M$.

Proof. If a filter $G \in M$ meets every D_p then, by separativity, $D = \{p \in \mathbb{P} : \exists q \in G \ q \perp p\}$ is dense in \mathbb{P} . So $G \cap D = \emptyset$. If $G \in M$, then $D \in M$, so either G is not \mathbb{P} -generic over M or $G \notin M$. \square

7 Names

We begin with a model M , add a generic filter G to it, and want to be able to talk about the objects in the new model $M[G]$, that is, we want to name them.

Of course our names cannot be too precise, since if we knew exactly what the new elements were, they would not be new.

So an alternative way of looking at things is to start with a model M and look at all the names for elements in some potential $M[G]$. These names do not rely on knowing exactly what G is. This construction of ambiguous names is called a Boolean-valued model. Then we can mod out by any G and get a 2-valued (i.e., standard) model. When we construct the Boolean-valued model, since we don't care whether or not a generic filter G actually exists, we can assume that M is the universe V — why not?

Definition 15. *Let \mathbb{P} be a partial order. A \mathbb{P} -name \dot{x} is a set of ordered pairs with the property that every pair in \dot{x} has the form (p, \dot{y}) where \dot{y} is either a name or $\dot{y} = \emptyset$. (When the context is clear, we just call these names.)*

Remark: This is implicitly an inductive definition. Let's work out some details.

$$\dot{\emptyset} = \emptyset.$$

$$\text{For } n \in \omega \ \dot{n} = \{(p, \dot{k}) : k < n, p \in \mathbb{P}\}.$$

$$\text{And } \dot{\omega} = \{(p, \dot{n}) : n < \omega, p \in \mathbb{P}\}.$$

$$\text{In general, if } x \in V \ \dot{x} = \{(p, \dot{y}) : y \in x, p \in \mathbb{P}\}.$$

Convention: if $x \in V$ we write \check{x} instead of \dot{x} .

Given a name \dot{x} , the object \dot{x} names in the model $V[G]$ is called \dot{x}/G , defined by: $\dot{x}/G = \{y/G : (p, \dot{y}) \in \dot{x} \text{ and } p \in G\}$. I.e., we take exactly the elements that G tells us to.

By induction, $\check{x}/G = x$ for all $x \in V$.

So much for naming elements in the ground model. What about \dot{x} when $x \notin V$?

We need a name for our generic filter: $\dot{G} = \{(p, p) : p \in \mathbb{P}\}$. That is, each p thinks "I am in G !". Let's check that $G = \dot{G}/G$: $p \in \dot{G}/G$ iff $\exists q \in G (q, p) \in \dot{G}$ iff $(p, p) \in \dot{G}$ and $p \in G$ iff $p/G \in G$ iff $p \in G$.

Consider the Cohen partial order $\mathbb{P} = \bigcup_{k < \omega} ({}^k 2)$. Recall that if G is \mathbb{P} -generic, we define the Cohen real $f_G = \bigcup G$. What is its name? $\dot{f}_G = \{(p, (n, i)) : n \in \text{dom } p \text{ and } p(n) = i\}$. Let's check that $\dot{f}_G/G = f_G$: $f_G(n) = i$ iff $\exists p \in G p(n) = i$ iff $\exists p \in G (p, (n, i)) \in \dot{f}_G$ iff $(n, i) \in \dot{f}_G/G$ iff $\dot{f}_G/G(n) = i$.

The temptation is to say that the objects in our forcing universe are all the names. But there are problems with this.

Note that even if a name names an object in the ground model (V or M) we might not know which one. For example, let $p + q = 1, p \cdot q = 0$, i.e., p, q are incompatible and $\{p, q\}$ is pre-dense. Let $\dot{x} = \{(p, \dot{2n}) : n < \omega\} \cup \{(q, \dot{2n+1}) : n < \omega\}$. If $p \in G, \dot{x}/G = \{\text{even numbers}\}$; if $q \in G, \dot{x}/G = \{\text{odd numbers}\}$. So names are highly ambiguous even when they don't have to be (because they are naming something in the ground model).

Another problem with saying that the objects in our forcing universe are all the names, is not only their ambiguity (see the preceding example), but that different names can name the same object.

Here is an example: Let $(\dot{G})^* = \{(p, \check{q}) : q \geq p\}$. The subclaim is that if G is a generic filter, then $G = (\dot{G})^*/G$: $q \in (\dot{G})^*/G$ iff $\exists p \in G (p, \check{q}) \in (\dot{G})^*$ iff $\exists p \in G q \geq p$ iff $q \in G$. But clearly $\dot{G} \neq (\dot{G})^*$.

In order to avoid at least part of the problem of more than one name for a given object, we move to names which are, in some sense, maximal.

Definition 16. (a) Let \dot{x}, \dot{y} be \mathbb{P} -names. $\llbracket \dot{y} \in \dot{x} \rrbracket = \Sigma\{p : (p, \dot{y}) \in \dot{x}\}$.

(b) If \dot{x} is a \mathbb{P} -name, then $\dot{x}^f = \{(q, \dot{y}^f) : q \leq \llbracket \dot{y} \in \dot{x} \rrbracket\}$.

(c) A \mathbb{P} -name \dot{x} is full iff $\dot{x} = \dot{x}^f$.

(b) looks circular, but it is not. It is, instead, implicitly inductive. Note: This is *not* Kunen's use of the phrase "full name."

The names we defined before definition 16 are not full. It is left to the reader to turn them into full names.

If we are working in the completion \mathbb{B} of \mathbb{P} , just substitute \mathbb{B} for \mathbb{P} above. As noted below, it doesn't matter.

Note that a full \mathbb{P} -name is not full in \mathbb{B} , where \mathbb{B} is the completion of \mathbb{P} . As we will learn when we define forcing, going to the full \mathbb{B} name doesn't really add anything, so we will use whichever name is more convenient.

Fact 3. (a) For all generic filters G , $\dot{x}/G = (\dot{x}^f)/G$,

(b) $(\dot{x}^f)^f = (\dot{y}^f)^f$ iff $\forall G$ a generic filter $\dot{x}/G = \dot{y}/G$.

A word about the limitations of fact 3. Recall the name $\dot{x} = \{p\} \times \{\check{2}n : n < \omega\} \cup \{q\} \times \{2n + 1 : n < \omega\}$, where $p \perp q$ and $p + q = 1$. Note that if \check{E} is the name for the set of even numbers, and \check{O} is the name for the set of odd numbers, then for every G generic, either $\dot{x}/G = \check{E}/G$ or $\dot{x}/G = \check{O}/G$. But it is not true that $\forall G$ generic $\dot{x}/G = \check{E}/G$, and it is not true that $\forall G$ generic $\dot{x}/G = \check{O}/G$.

Exercise: Let $\dot{G} = \{(p, p) : p \in \mathbb{P}\}$. What is \dot{G}^f ? Let $\dot{f}_{\dot{G}}$ be the name for $\bigcup \dot{G}$ where $\mathbb{P} = \bigcup_{k < \omega} ({}^k 2)$ defined above. What is $(\dot{f}_{\dot{G}})^f$?

From now on we will not worry unduly about which name we use, or whether we are in \mathbb{P} or its completion. As long as $(\dot{x}^f)^f = (\dot{y}^f)^f$ we will use \dot{x}, \dot{y} interchangeably and may even refer to *the* name for an object, knowing full well there isn't a single one.

We are now ready to define the various universes we will talk about:

Definition 17. Let \mathbb{P} be a separative partial order.

(a) $V^{\mathbb{P}} = \{\dot{x}^f : \dot{x} \text{ a } \mathbb{P}\text{-name}\}$.

(b) If \mathbb{B} is the completion of \mathbb{P} , then $V^{\mathbb{B}} = \{\dot{x}^f : \dot{x} \text{ a } \mathbb{B}\text{-name}\}$.

(c) If M is a model and $\mathbb{P} \in M$ we define $M^{\mathbb{P}} = \{\dot{x}^f : \dot{x} \text{ a } \mathbb{P}\text{-name in } M\}$, $M^{\mathbb{B}} = \{\dot{x}^f : \dot{x} \text{ a } \mathbb{B}\text{-name in } M\}$.

(d) If G is \mathbb{P} -generic over M a model, $M[G] = \{\dot{x}/G : \dot{x} \in M^{\mathbb{P}}\}$.

The following fact is an immediate consequence from \mathbb{P} dense in its completion.

Fact 4. If \mathbb{B} is the completion of \mathbb{P} and G is \mathbb{P} -generic over M a model, $M[G] = M[G]^{\mathbb{B}}$.

Example: Let $p : 3 \rightarrow 2$, $p(0) = p(1) = 1$; $p(2) = 0$. Then if $p \in G$, $\dot{f}_{\dot{G}}/G(0) = \dot{f}_{\dot{G}}/G(1)$; $\dot{f}_{\dot{G}}/G(2) = 0$.

8 Forcing

The previous exercise gives us an idea of what forcing is about: certain conditions $p \in \mathbb{P}$ know something about the generic object. Other conditions know something else. We need to define what we mean by “know something about.”

Definition 18. Let \mathbb{P} be a partial order, $p \in \mathbb{P}$.

- (a) If \dot{x}, \dot{y} are \mathbb{P} -names, $p \Vdash \dot{y} \in \dot{x}$ iff $p \leq \llbracket \dot{y} \in \dot{x} \rrbracket$. (\Vdash is pronounced: “forces.”)
- (b) If φ, θ are formulas, $p \Vdash \varphi \wedge \theta$ iff $p \Vdash \varphi$ and $p \Vdash \theta$.
- (c) If φ is a formula, $p \Vdash \neg \varphi$ iff $p \cdot \Sigma\{q : q \Vdash \varphi\} = 0$.
- (d) If $\varphi(v)$ is a formula with free variable v , $p \Vdash \exists v \varphi(v)$ iff $\exists \dot{x}$ a \mathbb{P} -name $p \Vdash \varphi(\dot{x})$.

Definition 19. Let φ be a formula. $\llbracket \varphi \rrbracket = \Sigma\{p : p \Vdash \varphi\}$. ($\llbracket \varphi \rrbracket$ is called the Boolean value of φ).

Note that we earlier defined $\llbracket \dot{y} \in \dot{x} \rrbracket$.

Exercise: Let $\mathbb{P} = \bigcup_{k < \omega} ({}^k 2)$. Let $\dot{f}_{\dot{G}}$ be the name for $\bigcup \dot{G}$, where \dot{G} names the generic filter. What is $\llbracket \dot{f}_{\dot{G}}(5) = 0 \rrbracket$? What is $\llbracket \dot{f}_{\dot{G}}(5) \neq 0 \rrbracket$? Fix $g \in M \cap 2^\omega$. What is $\llbracket \exists \check{n} \dot{f}_{\dot{G}}(\check{n}) = \check{g}(\check{n}) \rrbracket$? What is $\llbracket \neg \exists \check{n} \dot{f}_{\dot{G}}(\check{n}) = \check{g}(\check{n}) \rrbracket$?

We write $M^{\mathbb{P}} \Vdash \varphi$ iff $\llbracket \varphi \rrbracket = 1$. (Sometimes $M^{\mathbb{P}} \Vdash \varphi$ is written as $1 \Vdash \varphi$, sometimes as $\Vdash_{\mathbb{P}} \varphi$.)

Fact 5. (a) $M^{\mathbb{P}} \Vdash \varphi$ iff $\forall G$ generic $M[G] \models \varphi$.

(b) $M^{\mathbb{P}} \Vdash \varphi$ iff $\exists D$ dense in \mathbb{P} $\forall p \in D$ $p \Vdash \varphi$.

Proof. (a) $1 \in G$ for all filters G .

(b) D is pre-dense iff $\Sigma D = 1$; every pre-dense set extends to a dense set by closing downward. \square

This simple fact is extremely powerful. Since D dense in $\mathbb{P} \Rightarrow D$ dense in the completion of \mathbb{P} , and D dense in the completion of \mathbb{P} iff $D \cap \mathbb{P}$ is dense in \mathbb{P} , this fact, like fact 4, allows us to move back and forth between \mathbb{P} and its completion.

It also gives us our main tool for deciding when a statement is forced in $V^{\mathbb{P}}$. Here is an example:

Fact 6. Let M be a transitive model of set theory. Let $\dot{f}_{\dot{G}}$ be the name for $\bigcup \dot{G}$ where G is $\bigcup_{k < \omega} ({}^k 2)$ -generic. Then $\forall g \in M \cap 2^\omega$ $M^{\mathbb{P}} \Vdash \{\check{n} : \dot{f}_{\dot{G}}(\check{n}) = \check{g}(\check{n})\}$ is infinite. (Note that, since $\dot{f}_{\dot{G}} \notin M$, necessarily $\{\check{n} : \dot{f}_{\dot{G}}(\check{n}) \neq \check{g}(\check{n})\}$ is infinite.)

Proof. By fact 5, we just need to find the right dense set.

Given \check{g} , let $D_n = \{p : \exists m \geq n$ $p(m) = \check{g}(m)\}$. Each D_n is easily seen to be dense. Hence for all $n < \omega$, $M^{\mathbb{P}} \Vdash \exists m \geq n$ $\dot{f}_{\dot{G}}(\check{n}) = \check{g}(\check{n})$. Hence $\forall n < \omega$ $M^{\mathbb{P}} \Vdash \exists m \geq n$ $\dot{f}_{\dot{G}}(\check{n}) = \check{g}(\check{n})$. I.e., $M^{\mathbb{P}} \Vdash \{\check{n} : \dot{f}_{\dot{G}}(\check{n}) = \check{g}(\check{n})\}$ is infinite. \square

Note: In some sense this is a silly example: If $\{n : g(n) \neq f(n)\}$ is co-finite, then $f \in M$ for any model with $g \in M$. But $f \notin M$. However this proof is an example of an important technique, which you will use in a moment.

From now on, we will simply write x instead of \check{x} , for all $x \in V$.

In the next few examples, we let $\mathbb{P} = \bigcup_{k < \omega} ({}^k \omega)$, let \dot{G} name the \mathbb{P} -generic filter, and let $\dot{f}_{\dot{G}}$ name $\bigcup \dot{G}$.

1. Show that $\forall g \in M \cap \omega^\omega \ M^{\mathbb{P}} \Vdash \{n : g(n) = \dot{f}_{\dot{G}}(n)\}$ is infinite.
2. Show that $\forall g \in M \cap \omega^\omega \ M^{\mathbb{P}} \Vdash \{n : g(n) < \dot{f}_{\dot{G}}(n)\}$ is infinite.
3. Show that $\forall g \in M \cap \omega^\omega$ if $\exists^\infty n \ g(n) \neq 0$ then $M^{\mathbb{P}} \Vdash \{n : g(n) > \dot{f}_{\dot{G}}(n)\}$ is infinite.
4. Show that $\forall n < \omega \ M^{\mathbb{P}} \Vdash \exists^\infty k \ \dot{f}_{\dot{G}}(k) = n$.

As a corollary to exercise 2 we have: in $M \cap \omega^\omega$ is not cofinal in $M^{\mathbb{P}} \cap \omega^\omega$, where $\mathbb{P} = \bigcup_{k < \omega} ({}^k \omega)$.

Theorem 23. *Let \mathbb{P} be a separative partial order.*

- (a) $V^{\mathbb{P}} \Vdash ZFC$.
- (b) If $\mathbb{P} \in M$ a transitive model of ZFC and G is a \mathbb{P} -generic filter over M , $M[G] \models ZFC$

Proof. By fact 5 (a) we just have to prove (a); by fact 5(b) we just need to find the right dense sets. Most of the proof is fairly tedious and best left to the reader — if you work through the details you will have a very good understanding of how forcing works. So we just provide a few samples.

Most of the axioms are existence axioms: power sets exist; an infinite set exists; unions exist; pairs exist; subsets defined by a formula exist; ranges of functions defined by a formula exist; choice functions exist. To show something exists, all we need to do is find a name for it.

Union axiom Let $\dot{x} \in V^{\mathbb{P}}$. Define $\dot{z} = \{(p, \dot{w}) : \exists \dot{y} \ p \Vdash \dot{w} \in \dot{y} \in \dot{x}\}$. It is left to the reader to prove that $V^{\mathbb{P}} \Vdash \forall \dot{w} \ (\dot{w} \in \dot{z} \text{ iff } \exists \dot{y} \ \dot{w} \in \dot{y} \in \dot{x})$.

Axiom of choice It's easier to prove the well-ordering principle, which is equivalent since all of the other axioms hold in $V^{\mathbb{P}}$. So let $\dot{x} \in V^{\mathbb{P}}$. Let $Y = \{\dot{y} : \exists p \ (p, \dot{y}) \in \dot{x}\}$. Since $V \models WO, \exists \kappa$ a cardinal and $f : \kappa \rightarrow Y, f$ 1-1, onto. Define $\dot{h} = \{(p, (\alpha, \dot{y})) : p \Vdash \dot{y} \in \dot{x} \text{ and } f(\alpha) = \dot{y}\}$. Then $V^{\mathbb{P}} \Vdash \dot{h}$ is a partial function from κ to \dot{x} . But every partial function from an ordinal induces a well-ordering of the range, so $V^{\mathbb{P}} \Vdash \dot{x}$ has a well-ordering.

Technical point: The function \dot{h} need not be 1-1: there may be p, \dot{y}, \dot{y}' with $p \Vdash \dot{y} = \dot{y}' \in \dot{x}$. But this doesn't matter. The theorem that every partial function from an ordinal induces a well-ordering of the range does not require that the partial function is 1-1.

The two non-existence axioms are extensionality and regularity. Extensionality is left to the reader. Regularity follows from the inductive definition of names. Define the complexity $c(\dot{x}) = \sup\{c(\dot{y}) : \exists p \ p \Vdash \dot{y} \in \dot{x}\}$, where $c(\emptyset) = 0$. For any \dot{x} , we need to find a term \dot{w} so $V^{\mathbb{P}} \Vdash \dot{w} \cap \dot{x} = \emptyset$.

Fix \dot{x} and define $D = \{p : \exists \dot{y} \ p \Vdash \dot{y} \in \dot{x} \text{ so that if } \dot{z} \in \dot{x} \text{ then } c(\dot{z}) \geq c(\dot{y})\}$. D is dense (proof left to the reader). If $p \in D$ there is \dot{y}_p with $p \Vdash \dot{y}_p \in \dot{x}$ and $p \Vdash$ if $\dot{z} \in \dot{x}$ then $c(\dot{z}) \geq c(\dot{y}_p)$. By the induction definition of names, $p \Vdash \dot{y}_p \cap \dot{x} = \emptyset$. Let A be a maximal antichain in D . Define $\dot{w} = \{(q, \dot{z}) : \exists p \in A, q \leq p, q \Vdash \dot{z} \in \dot{y}_p\}$. $V^{\mathbb{P}} \Vdash |A \cap \dot{G}| = 1$. Let $p \in A \cap \dot{G}$. $p \Vdash (\dot{y}_p \cap \dot{x} = \emptyset \text{ and } \dot{w} = \dot{y}_p)$. Since A is pre-dense, $V^{\mathbb{P}} \Vdash \dot{w} \cap \dot{x} = \emptyset$. \square

Theorem 29 is what makes everything work: We want to prove that $ZFC + \varphi$ is consistent. So we force with some \mathbb{P} with $V^{\mathbb{P}} \Vdash \varphi$. Since $V^{\mathbb{P}} \Vdash ZFC$, we're done.

A quick summary: $V^{\mathbb{P}}$ is the set of full \mathbb{P} -names, so $V^{\mathbb{P}} \subset V$. But not everything that $V^{\mathbb{P}}$ names is in V . For example, \dot{G} names an object which is necessarily not in V . The name is in V , but the object named is not. The object named doesn't even exist until we know exactly what G is. But even though we don't know exactly what it is, we can know a lot about it, e.g., we know a lot about a Cohen real (see the exercises above).

Comment on notation: If we say $Y = \{\dot{x} : \varphi(\dot{x})\}$ then Y is a set of names, but is not itself a name. If we say $\dot{x} \subset \omega$ we mean that $\llbracket \dot{x} \subset \omega \rrbracket = 1$; the name \dot{x} is not a subset of ω ; it is a subset of $\mathbb{P} \times V$.

Another convention is the phrase “In $V^{\mathbb{P}}$, φ used instead of $V^{\mathbb{P}} \Vdash \varphi$.”

9 Preserving and collapsing cardinals

Suppose M is a transitive model, $M \models \kappa$ a cardinal.² Since M is transitive, we know that in any $M^{\mathbb{P}}$ κ will still be an ordinal. But could we have added a function from some smaller ordinal α onto κ ? In this case, $M^{\mathbb{P}} \Vdash \kappa$ is not a cardinal.

Here is an example:

Definition 20. *The standard forcing $Col(\kappa, \lambda)$ collapsing κ to $\lambda < \kappa$ is defined as follows: $Col(\kappa, \lambda) = \bigcup_{\alpha < \lambda} \alpha \kappa$. The order is: $p \leq q$ iff $p \supset q$.*

Fact 7. *Let $\mathbb{P} = Col(\kappa, \lambda)$. Then $V^{\mathbb{P}} \Vdash \exists \dot{f} : \lambda \rightarrow \kappa, \dot{f}$ onto. (Hence $V^{\mathbb{P}} \Vdash |\kappa| = |\lambda|$.)*

Proof. For $\beta < \kappa$ define $D_\beta = \{p \in \mathbb{P} : \beta \in \text{range } p\}$. D_β is dense for all $\beta < \kappa$. For $\alpha < \lambda$ define $D^\alpha = \{p \in \mathbb{P} : \alpha \in \text{dom } p\}$. Each D^α is dense. So if $\dot{f} = \bigcup \dot{G}$, where \dot{G} is \mathbb{P} -generic, then $\text{dom } \dot{f} = \lambda$ and $\text{range } \dot{f} = \kappa$. \square

There are times when we want to collapse cardinals, but usually we don't.

Definition 21. \mathbb{P} has κ -cc iff it has no antichain of size κ .

We write ccc instead of ω_1 -cc.

Note that if $\kappa < \lambda$ then κ -cc \Rightarrow λ -cc.

Fact 8. *Suppose κ is regular and \mathbb{P} has κ -cc. Suppose \dot{f} is a \mathbb{P} -name for a function where $\text{range } \dot{f} \subset V$. Then $\forall \dot{y} |\{x : \llbracket \dot{f}(\dot{y}) = x \rrbracket \neq 0\}| < \kappa$.*

In particular, if $\text{range } \dot{f} \subset ON$, the hypothesis holds.

Proof. Fix \dot{y} . $\forall x \in V$ let $b_x = \Sigma\{p : p \Vdash \dot{f}(\dot{y}) = x\} = \llbracket \dot{f}(\dot{y}) = x \rrbracket$. If $x \neq z$ then $b_x \cdot b_z = 0$, so $\{b_x : b_x \neq 0\}$ is an antichain, hence has size $< \kappa$. Hence $|\{x : b_x \neq 0\}| < \kappa$, as desired. \square

Corollary 7. *If κ is regular and \mathbb{P} has κ -cc, then $V^{\mathbb{P}} \Vdash \kappa$ is a cardinal.*

Proof. Suppose $\dot{f} : \lambda \rightarrow \kappa$ where $\lambda < \kappa$. For $\alpha < \lambda$ let $A_\alpha = \{\beta : \llbracket \dot{f}(\alpha) = \beta \rrbracket \neq 0\}$. Then $|A_\alpha| < \kappa$, hence $|\bigcup_{\alpha < \lambda} A_\alpha| < \kappa$ and $\kappa \setminus \bigcup_{\alpha < \lambda} A_\alpha \neq \emptyset$. But $V^{\mathbb{P}} \Vdash \text{range } \dot{f} \subset \bigcup_{\alpha < \lambda} A_\alpha$, so $V^{\mathbb{P}} \Vdash \dot{f}$ not onto. \square

Definition 22. *We say that \mathbb{P} is κ -closed iff all descending chains of non-zero elements $\{p_\alpha : \alpha < \beta\}$, where $\beta < \kappa$, have a non-zero lower bound.*

We write countably closed for ω_1 -closed.

If \dot{f} names a function with $\text{range } \dot{f} \subset V$, we write $p \Vdash \dot{f}(\dot{x}) = y$ iff $\exists y p \Vdash \dot{f}(\dot{x}) = y$.

Fact 9. *If \mathbb{P} is κ -closed, and $\dot{f} : \lambda \rightarrow V$ where $\lambda < \kappa$ then $V^{\mathbb{P}} \Vdash \exists g \in V \dot{f} = g$.*

²if M is countable and $M \models \kappa > \omega$ then κ is not a cardinal in V . But we don't care — we are looking through M 's eyes.

Proof. Suppose $\lambda \leq \kappa$ and \dot{f} is a name for a function from λ to V .

Let $D = \{p : p \Vdash \exists g \in V \dot{f} = g\}$.

Fix p , and let $\{p_\alpha : \alpha < \lambda\}$ be a descending chain of non-zero elements below p so $p_\alpha \parallel \dot{f}(\alpha)$ — we can do this because \mathbb{P} is κ -closed. Let q be a non-zero lower bound for $\{p_\alpha : \alpha < \rho\}$. Let $g(\alpha) = \beta$ iff $p_\alpha \Vdash \dot{f}(\alpha) = \beta$. Then $q \Vdash \dot{f} = g$. So D is dense and we are done. \square

As a corollary

Fact 10. *If \mathbb{P} is κ -closed, then no cardinal $\leq \kappa$ is collapsed in $V^{\mathbb{P}}$.*

Proof. Suppose $\lambda < \rho \leq \kappa$, $V \models \lambda, \rho, \kappa$ are cardinals. $V^{\mathbb{P}} \Vdash$ if $\dot{f} : \lambda \rightarrow \kappa$, then there is $g \in V \dot{f} = g$. But no ground model function can take λ onto ρ , so ρ is not collapsed. \square

10 \neg CH at last, and more

Here we show how to prove the consistency of $\lambda = \kappa$ for any κ with $\text{cf } \kappa > \lambda$.

First we focus on $\lambda = \omega, \kappa > \omega_1$, i.e., destroying CH.

Definition 23. *The Cohen partial order on κ is $\mathbb{C}_\kappa = \{p : p \text{ a partial function from } \kappa \text{ to } 2, \text{ dom } p \text{ finite}\}$.*

Fact 11. *For all κ \mathbb{C}_κ is ccc.*

Proof. Let $P \in [\mathbb{C}_\kappa]^{\omega_1}$. By the Δ -system lemma we may assume $\{\text{dom } p : p \in P\}$ is a Δ -system with root r . Since r is finite and there are only finitely many functions from r to 2, we may assume there is $q : r \rightarrow 2$ so that if $p \in P$ then $p \leq q$. Hence $\{p \setminus q : p \in P\}$ has pairwise disjoint domain, so P is linked (in fact, centered). \square

Thus, forcing with \mathbb{C}_κ does not collapse cardinals.

Theorem 24. *Assume GCH. $V^{\mathbb{C}_\kappa} \Vdash \mathfrak{c} = \kappa$.*

Proof. For each δ a limit ordinal, let $E_\delta = \{\delta + n : n < \omega\}$. Let \dot{G} name the \mathbb{C}_κ -generic filter over V , and \dot{f} name $\bigcup \dot{G}$. Let $\dot{g}_\delta : \omega \rightarrow 2, \dot{g}_\delta(n) = \dot{f}(\delta + n)$. We need to show that $V^\mathfrak{c} \Vdash$ the \dot{g}_δ 's are distinct, (hence $\mathfrak{c} \geq \kappa$), and that there are at most κ many \mathbb{C}_κ -names for subsets of ω (hence $V^{\mathbb{C}_\kappa} \Vdash \mathfrak{c} \leq \kappa$).

Fix $\delta \neq \delta'$. Let $D = \{p : \exists n p(\delta + n) \neq p(\delta' + n)\}$. We show that D is dense. Fix $q \in \mathbb{C}_\kappa$. Since $\text{dom } q$ is finite, there is n with $\delta + n, \delta' + n \notin \text{dom } q$. Define p as follows: $\text{dom } p = \text{dom } q \cup \{\delta + n, \delta' + n\}, p \supset q, p(\delta + n) = 0, p(\delta' + n) = 1$. Then $p \leq q$ and $p \Vdash g_\delta(n) \neq g_{\delta'}(n)$. So $V^\mathfrak{c} \Vdash \mathfrak{c} \leq \kappa$.

Since $|\mathbb{C}_\kappa| = \kappa$, \mathbb{C}_κ is ccc, GCH holds, and $\text{cf } \kappa > \omega$, there are at most κ many maximal antichains. And since there are at most κ many maximal antichains in \mathbb{C}_κ , there are at most κ many sequences of antichains $\{A_n : n < \omega\}$. Fix \dot{k} a \mathbb{C}_κ -name for a function from ω to 2. Let $A_{\dot{k}, n}$ be a maximal antichain deciding $\dot{k}(n)$, i.e. if $p \in A_{\dot{k}, n}$ then $p \Vdash \dot{k}(n) = i$. Let $\dot{k}^* = \{(q, (n, i)) : \exists p \exists n q \leq p \in A_{\dot{k}, n} p \Vdash \dot{k}(n) = i\}$. Then $\dot{k}^f = (\dot{k}^*)^f$. So \dot{k} is determined by a countable sequence of antichains, and there are only κ many of these. \square

The reader will note that we did not need full GCH. The pieces of GCH we used were: if $\lambda < \kappa$ then $[\lambda]^\omega < \kappa$ (to limit the number of names) and (to name the obvious) $[\omega]^{<\omega} = \omega$ (to use the Δ -system lemma and show \mathbb{C}_κ is ccc).

We now generalize this construction.

Definition 24. *Let ρ be a cardinal. $F_n(A, B, \rho) = \{f : \text{dom } f \in [A]^{<\rho}, \text{ range } f \subset B\}; \leq = \supset$.*

Remark: \mathbb{C}_κ from definition 23 is $F_n(\kappa, 2, \omega)$.

Fact 12. *Let ρ be a regular cardinal, $|B| \leq \rho$.*

- (a) *if $[\rho]^{<\rho} \leq \rho$, then $F_n(A, B, \rho)$ has the ρ^+ -cc.*
- (b) *$F_n(A, B, \rho)$ is ρ -closed.*

Proof. (b) is immediate from the definition. (a) follows from the Δ -system lemma: our hypothesis implies that if $\lambda < \rho^+$ then $[\lambda]^{<\rho} < \rho^+$, which is the hypothesis needed for the Δ -system lemma. So given $P \in [Fn(A, B, \rho)]^{\rho^+}$, there is $P' \in [P]^{\rho^+}$ with $\{\text{dom } p : p \in P'\}$ a Δ -system with root r . Since $|r| < \rho$, $|B^{r}| \leq \rho$, so there is $P^* \in [P']^{\rho^+}$ and q with $p|_r = q$ for all $p \in P^*$. Then $\{p \setminus q : p \in P^*\}$ is pairwise compatible (because its domains are pairwise disjoint). Hence P^* is pairwise compatible, and P was not an antichain. \square

Fact 13. *Let ρ be a regular cardinal, $|B| \leq \rho$. If GCH holds below ρ ,³ then $Fin(A, B, \rho)$ does not collapse cardinals*

Proof. ρ regular and GCH below ρ imply that $\rho^{<\rho} = \rho$, so by fact 12(a), no cardinals above ρ^+ are collapsed. By fact 12(b), no cardinals below ρ are collapsed. \square

Theorem 25. *Assume GCH. Let $cf \kappa > \lambda$. Let $\mathbb{P} = Fn(\kappa, 2, \lambda)$. Then $V^{\mathbb{P}} \Vdash |2^\lambda| = \kappa$.*

To understand what this theorem means, let \mathbb{P} be as in the theorem. By fact 11, for all ordinals ρ , $V \models \rho$ a cardinal iff $V^{\mathbb{P}} \Vdash \rho$ a cardinal. So if, for example, $\kappa = \omega_2$ and $\lambda = \omega$, then $V^{\mathbb{P}} \Vdash 2^\omega = \omega_2$. If $\kappa = \omega_{17}$ and $\lambda = \omega_3$, then $V^{\mathbb{P}} \Vdash 2^{\omega_3} = \omega_{17}$. If $\kappa = \aleph_{\omega_3}$ and $\lambda = \omega_2$, then $V^{\mathbb{P}} \Vdash 2^{\omega_2} = \aleph_{\omega_3}$.

Later we will state (and hopefully prove) a theorem using iterated forcing which simultaneously changes the size of power sets. But for now we settle for doing one at a time. The proof closely follows the proof of theorem 31.

Proof. Let κ be partitioned into $\{E_\alpha : \alpha < \kappa\}$ where each $|E_\alpha| = \lambda$, and let $h_\alpha : \lambda \rightarrow E_\alpha$ be 1-1 onto. Let \dot{G} name the generic filter, and let \dot{f} name $\bigcup \dot{G}$. Define $\dot{g}_\alpha = \dot{f}|_{E_\alpha} \circ h_\alpha : \lambda \rightarrow 2$.

First we show that $V^{\mathbb{P}} \Vdash \forall \alpha \neq \beta \dot{g}_\alpha \neq \dot{g}_\beta$ (hence $V^{\mathbb{P}} \Vdash 2^\lambda \geq \kappa$). Pick $\alpha \neq \beta$. Let $D = \{p : \text{dom } p \cap E_\alpha \neq \emptyset \neq \text{dom } p \cap E_\beta \text{ and } \exists \gamma < \lambda p(h_\alpha(\gamma)) \neq p(h_\beta(\gamma))\}$.

We subclaim that D is dense: Fix $p \in \mathbb{P}$. Since $|\text{dom } p| < \lambda$ there is γ with $h_\alpha(\gamma), h_\beta(\gamma) \notin \text{dom } p$. Let $\text{dom } q = \text{dom } p \cup \{h_\alpha(\gamma), h_\beta(\gamma)\}$, $q \supset p$, $q(h_\alpha(\gamma)) = 0$, $q(h_\beta(\gamma)) = 1$. Then $q \in D$.

But if $q \in D$ then $q \Vdash \dot{g}_\alpha \neq \dot{g}_\beta$, as desired.

Why does $V^{\mathbb{P}} \Vdash 2^\lambda \leq \kappa$? Let \dot{g} be a name for a function from λ into 2. For all $\alpha < \lambda$ there is a maximal antichain A_α deciding $\dot{g}(\alpha)$. By λ^+ -cc, $|A_\alpha| \leq \lambda$. Let $\dot{g}' = \{(p, (\alpha, i)) : p \in A_\alpha, p(\alpha) = i\}$. Since $\dot{g}^f = \dot{g}'^f$, it suffices to ask: how many conditions are there of the form \dot{g}' ? By GCH, there are κ many antichains. \dot{g}' is essentially determined by picking λ many maximal antichains. By GCH and $cf \kappa > \lambda$, $\kappa^\lambda = \kappa$. So there are at most κ names for elements of 2^λ . Hence $V^{\mathbb{P}} \Vdash 2^\lambda \leq \kappa$. \square

³i.e., if $\lambda < \rho$ then $2^\lambda < \rho$

11 Some other forcings and technical results

Before moving on to iterated forcing, we introduce the measure algebra and the Sacks algebra and some technical matters.

Definition 25. A partial order \mathbb{P} is ω^ω -bounding iff $\forall \dot{f} : \omega \rightarrow \omega \exists g \in V \dot{f} < g^4$

Fact 14. . Assume CH. If \mathbb{P} is ω^ω -bounding, then $V^{\mathbb{P}} \Vdash \mathfrak{d} = \omega_1$.

Thus an ω^ω -bounding partial order which adds $> \omega_1$ reals over a model of CH shows $\text{Cons}(\mathfrak{c} > \mathfrak{d})$. We are about to meet such a partial order.

Definition 26. Let κ be infinite.

(a) We define μ be the probability measure on the product space 2^κ as follows: If σ is a finite function from κ to 2 , and $[\sigma] = \{f \in 2^\kappa : \sigma \subset f\}$, then $\mu([\sigma]) = 2^{-|\sigma|}$. We extend μ to a measure on the Borel subsets⁵ of 2^κ as usual: $\mu(B) = \inf\{\mu(u) : u \text{ open and } B \subset u\}$.

(b) If B, B' are Borel, then $B \equiv B'$ iff $\mu((B \setminus B') \cup (B' \setminus B)) = 0$. The measure algebra $\mathbb{M}_\kappa = \{B / \equiv : B \text{ Borel, } \mu(B) > 0\}$. $B / \equiv \leq B' / \equiv$ iff $\exists C \in B / \equiv, C' \in B' / \equiv \ C \subset C'$.

We defined \mathbb{M}_κ in terms of $/ \equiv$ in order to make \mathbb{M}_κ separative, but it is customary to talk about B instead of B / \equiv to make notation friendlier. Just be aware that we are speaking of B / \equiv instead of B .

The following is an easy consequence of the definition of measure:

Fact 15. (a) $\mu(2^\omega) = 1$.

(b) If \mathcal{B} is a family of sets of positive measure whose pairwise intersections have measure zero, then $\mu(\bigcup \mathcal{B}) = \sum_{B \in \mathcal{B}} \mu(B)$.

(c) $\mu(2^\omega \setminus B) = 1 - \mu(B)$.

An important corollary to fact 14 (b) is

Corollary 8. \mathbb{M}_κ is ccc.

Theorem 26. Assume GCH, cf $\kappa > \omega$. $V^{\mathbb{M}_\kappa} \Vdash 2^\omega = \kappa$.

Proof. First we show that $V^{\mathbb{M}_\kappa} \Vdash 2^\omega \geq \kappa$. (This part does not use GCH.) Let $\dot{f} = \bigcup\{\sigma : [\sigma] \in \dot{G}\}$. For δ a limit $< \kappa$, define $\dot{x}_\delta = \dot{f}|_{\{\delta+n:n<\omega\}}$. We show that if $\delta \neq \delta'$ then $V^{\mathbb{M}_\kappa} \Vdash \dot{x}_\delta \neq \dot{x}_{\delta'}$. The proof uses the following

Fact 16. $V^{\mathbb{M}_\kappa} \Vdash \forall \beta < \kappa \exists \sigma [\sigma] \in \dot{G}$ and $\beta \in \text{dom } \sigma$.

Hence $\text{dom } \dot{f} = \kappa$ and each \dot{x}_δ is defined.

It also uses the following

Fact 17. : Suppose $\{s_n : n < \omega\} \subset [\kappa]^2$ is a pairwise disjoint family. Then $V^{\mathbb{M}_\kappa} \Vdash \exists^\infty n : \dot{f}|_{s_n}$ is not constant.

⁴Recall that $g \geq f$ iff $\forall n : g(n) > f(n)$.

⁵The Borel sets are generated from the open sets by complements and countable unions. The important fact here is that the family of measurable sets is closed under complements and unions.

Assuming fact 16, fix $\delta \neq \delta'$. Let $D = \{B : B \Vdash \dot{x}_\delta \neq \dot{x}_{\delta'}\}$. For $n < \omega$ let $s_n = \{\delta + n, \delta' + n\}$. Fix $C \in \mathbb{M}_\kappa$. By the fact, $\exists B \leq C B \Vdash \dot{f}(\delta + n) \neq \dot{f}(\delta' + n)$. So $B \in D$, hence D is dense.

For the proof of $V^{\mathbb{M}_\kappa} \Vdash 2^\omega \leq \kappa$, closely imitate the proof of theorem 31. \square

An important corollary to fact 14 (a) is

Corollary 9. $V^{\mathbb{M}_\kappa} \Vdash \varphi$ iff $\forall \varepsilon \in (0, 1) \exists B \mu(B) > \varepsilon$ and $B \Vdash \varphi$.

Proof. \Rightarrow is clear. We show \Leftarrow via the contrapositive.

If $V^{\mathbb{M}_\kappa} \nVdash \varphi$ then $\exists B B \Vdash \neg\varphi$, and $\exists \delta \in (0, 1] \mu(B) = \delta$. If $\delta = 1$ we are done. Otherwise, let $\varepsilon = 1 - \frac{\delta}{2}$. If $\mu(C) = \varepsilon$, then $\mu(B \cap C) > 0$, so $C \nVdash \varphi$. \square

Theorem 27. \mathbb{M}_κ is ω^ω -bounding.

Proof. Fix $\dot{f} : \omega \rightarrow \omega$. By corollary 34, it suffices to show that $\forall \varepsilon > 0 \exists B \in \mathbb{M}_\kappa \exists g \in V B \Vdash g = \dot{f}$ and $\mu(B) > \varepsilon$.

Fix ε . Let A_0 be a maximal antichain deciding $\dot{f}(0)$. Since $\mu(\bigcup A_0) = 1$ there is $E_0 \in [A_0]^{<\omega}$ with $\mu(\bigcup E_0) > \varepsilon$. Define $n_0 = \sup\{k : \exists B \in E_0 B \Vdash \dot{f}(0) = k\}$. (Since E_0 is finite, $n_0 < \omega$). Let A_1 be a maximal antichain below $\bigcup E_0$ (i.e., if $C \in A_1$ then $C \leq \bigcup E_0$) deciding $\dot{f}(1)$. Since $\mu(\bigcup E_0) > \varepsilon$, there is $E_1 \in [A_1]^{<\omega}$ with $\mu(\bigcup E_1) > \varepsilon$. Define $n_1 = \sup\{k : \exists B \in E_1 B \Vdash \dot{f}(1) = k\}$. And so on. In this way we get a descending sequence $\{\bigcup E_m : m < \omega\}$ of conditions and a sequence $\{n_m : m < \omega\}$ where each $\mu(\bigcup E_m) > \varepsilon$ and each $\bigcup E_m \Vdash \dot{f}(m) \leq n_m$. Define $B = \bigcap_{m < \omega} B_m$, $g(m) = 1 + n_m$. Then $B \Vdash \dot{f} < g$ and $\mu(B) \geq \varepsilon$. \square

If κ is regular, then $V^{\mathbb{M}_\kappa} \Vdash 2^\omega = \kappa$ (proof left to the reader). But \mathbb{M}_κ is also useful for the study of measure and category, defined forthwith.

Recall that an ideal is a family of sets closed under finite union and subset.

An important ideal in the set-theoretic study of the reals is the ideal of \mathcal{M} of meager sets, where a set is meager iff it is the union of countably many nowhere dense sets.⁶ Another important ideal is the ideal of \mathcal{N} of measure zero (a.k.a. null) sets.

Whenever we have an ideal \mathcal{I} on a set X some natural questions come to mind:

(a) What is the smallest κ so that the union of κ many elements of \mathcal{I} is not in \mathcal{I} ? (This is called the additivity $Add(\mathcal{I})$)

(b) What is the smallest κ so that the union of κ many elements of \mathcal{I} is X ? (This is called the covering number $Cov(\mathcal{I})$.)

(c) What is the smallest κ so that some set of size κ is not an element of \mathcal{I} ? (This is called $Non(\mathcal{I})$.)

(d) What is the smallest κ so that there is $\mathcal{A} \subset \mathcal{I}$ with $|\mathcal{A}| = \kappa$ and $\forall B \in \mathcal{I} \exists A \in \mathcal{A} B \subset A$? (This is called the cofinality $Cof(\mathcal{I})$.)

The following facts are well-known.

Fact 18. (a) The above cardinal invariants are the same for \mathcal{M} whether the underlying space is \mathbb{R} , the Cantor space 2^ω , or the product space ω^ω —which is homeomorphic to the irrationals.

⁶A set is nowhere dense iff its closure contains no open set iff the complement of its closure is open dense.

(b) The above cardinal invariants are the same for \mathcal{N} whether the underlying space is \mathbb{R} (under Lebesgue measure), the Cantor space 2^ω (under the product measure above), or the product space ω^ω — which is homeomorphic to the irrationals (under Lebesgue measure).

(c) All of these cardinal invariants with respect to \mathbb{R} are uncountable, so under CH they all equal ω_1 .

Fact 19. Let $g : \omega \rightarrow \omega$. Then $L_g = \{f : f < g\} \in \mathcal{M}$.

Proof. In fact, L_g is nowhere dense. We prove this by showing that its complement is nowhere dense.

Suppose σ is a finite function from ω to ω . If $\exists n \in \text{dom } \sigma$ with $\sigma(n) > g(n)$, let $\tau = \sigma$. Otherwise let $n \notin \text{dom } \sigma$ and set $\tau = \sigma \cup \{(n, g(n) + 1)\}$. In either case, $[\tau] \subset [\sigma]$ and $[\tau] \cap D_g = \emptyset$.

□

Corollary 10. Assume CH. Then $V^{\mathbb{M}_\kappa} \Vdash \mathfrak{d} = \omega_1$ hence $\text{Cov}(\mathcal{M}) = \omega_1$.

Proof. By theorem 35, $\{L_g : g \in V \cap \omega^\omega\}$ covers ω^ω .

□

This theorem is of interest, because if κ is regular uncountable, and CH holds in the ground model, then $V^{\mathbb{M}_\kappa} \Vdash \text{Cov}(\mathcal{N}) = \kappa$. Hence $\text{Cons}(\text{Cov}(\mathcal{M}) < \text{Cov}(\mathcal{N}))$. We may prove this when we discuss iterated forcing.

A theorem we will not prove is the dual:

Theorem 28. Let κ be regular uncountable. Then $V^{\mathbb{C}_\kappa} \Vdash \text{Cov}(\mathcal{N}) = \omega_1$ and $\text{Cov}(\mathcal{M}) = \kappa$.

If you are interested in these matters, the standard reference is Bartosynski's and Judah's book *Set theory on the structure of the real line*.

A relatively short introduction to random and Cohen reals which proves theorem 37 and its dual is Kunen's paper *Random and Cohen reals* in the *Handbook of Set-Theoretic Topology*. We mention from that paper, without proof, the characterization of random and Cohen reals.

Definition 27. A real x is Cohen (resp. random) over a model M iff $x = \bigcup G$ where G is \mathbb{C}_ω -generic (resp. \mathbb{M}_ω -generic) over M .

Theorem 29. A real x is Cohen (resp. random) over a model M iff $x \in E$ for every open dense (resp. measure 1) set $E \in M$.

We turn now to another way of not collapsing cardinals, known as fusion. Rather than give a general definition of fusion, we will give an example of its use.

Definition 28. The Sacks partial order \mathbb{S} is the set of all nonempty perfect (= closed with no isolated points) subsets of the Cantor set 2^ω ; $\leq = \subset$.

\mathbb{S} was first used to prove a result in recursion theory. But it is also of interest for cardinal invariants of the reals.

Note that every nonempty perfect set has size 2^ω .

\mathbb{S} is provably equivalent to $\{S : S \text{ is a splitting tree, } S \subset \mathbb{T}\}$ where \mathbb{T} is the binary tree of height ω , $\leq = \supset$. (A splitting tree is a separative tree in the upside down order.) The equivalence is: the tree S corresponds to the set of F_S functions determined by its branches. As for the other direction, a perfect set $S \subset 2^\omega$ corresponds to the tree $T_S = \{\sigma : \exists f \in S \ f \supset \sigma\}$.

Theorem 30. (a) Assume CH. \mathbb{S} does not collapse cardinals.

(b) \mathbb{S} is ω^ω -bounding.

Proof. Since $|\mathbb{S}| = 2^\omega$, under CH $|\mathbb{S}| = \omega_1$, hence \mathbb{S} has ω_2 -cc and preserves cardinals $\geq \omega_2$. So for (a) we only have to show that it preserves ω_1 , i.e., if $\dot{f} : \omega \rightarrow \omega_1$ then $\text{range } \dot{f}$ is bounded below ω_1 .

In fact we prove (a) and (b) simultaneously by proving: (*) if $\dot{f} : \omega \rightarrow V$ then $D_{\dot{f}} = \{S \in \mathbb{S} : \text{for each } n \text{ there is } s_n \text{ with } |s_n| \leq 2^n \text{ and } S \Vdash \dot{f}(n) \in s_n\}$ is dense.

Suppose we've proved (*). For (a): Fix $\dot{f} : \omega \rightarrow \omega_1$. Let $S \in D_{\dot{f}}$ via $\{s_n : n < \omega\}$. Each s_n is a finite subset of ω_1 , hence $\bigcup s_n$ is a countable subset of ω_1 and $S \Vdash \text{range } \dot{f} \subset \bigcup s_n$. For (b): Fix $\dot{f} : \omega \rightarrow \omega$. Let $S \in D_{\dot{f}}$ via $\{s_n : n < \omega\}$. $\forall n$ let $g(n) = \sup s_n + 1$. Then $S \Vdash \dot{f}(n) < g(n)$.

So let's prove (*). We use the following fact: Every uncountable closed set of reals contains a perfect set.

Some notation: We denote the finite function $\sigma \cup \{(|\sigma|, i)\}$ by σi .

Let $S \in \mathbb{S}$. Let $S_\emptyset \leq S$ with $S_\emptyset \Vdash \dot{f}(0)$. Define $s_0 = \{k\}$ where $S_\emptyset \Vdash \dot{f}(0) = k$. Now let $S_0, S_1 < S_\emptyset$ with $S_i \Vdash \dot{f}(1)$, and let $s_1 = \{k : S_0 \Vdash \dot{f}(1) = k \text{ or } S_1 \Vdash \dot{f}(1) = k\}$. Etc. I.e., given $\sigma : m-1 \rightarrow 2$ and S_σ we define $S_{\sigma 0}, S_{\sigma 1} \leq S_\sigma$, each $S_{\sigma i} \Vdash \dot{f}(m)$, and define $s_m = \{k : \exists \tau : m \rightarrow 2 \ S_\tau \Vdash \dot{f}(m) = k\}$.

This gives us a branching set of conditions $\{S_\tau : \tau \in \bigcup_{m < \omega} {}^m 2\}$ so that $S_\tau \leq S_\sigma$ iff $\tau \supset \sigma$, and $S_\tau \perp S_\sigma$ iff $\tau \perp \sigma$.

We define the fusion S^* as follows: Let $S_n = \bigcup \{S_\sigma : |\sigma| = n\}$. Define $S^* = \bigcap_{n < \omega} S_n$. Since 2^ω is compact, S^* is closed uncountable. By the fact there is a perfect set $S^\dagger \subset S^*$. $S^\dagger \Vdash \forall n \ \dot{f}(n) \in s_n$. \square

Theorem 31. Assume CH, let κ be regular uncountable and let \mathbb{P} be a countable iteration of length κ (whatever that is) of Sacks forcing. Then $V^\mathbb{P} \Vdash 2^\omega = \kappa$ and $\text{Inv}(\mathcal{N}) = \omega_1$, where Inv is any cardinal invariant of \mathcal{N} defined above.

Proof. By a fusion argument, every new measure zero set sits inside an old measure zero set.

Two hints on the proof: 1. Let \dot{N} name a null set in $V^\mathbb{S}$, and $\varepsilon \in (0, 1)$. There is a sequence of names $\{\dot{u}_n : n < \omega\}$ where each \dot{u}_n is basic open (hence in V),⁷ $\mu(\dot{u}_n) \leq \varepsilon \cdot 2^{-(2n+1)}$ and $V^\mathbb{S} \Vdash \dot{N} \subset \bigcup_{n < \omega} \dot{u}_n$. Below any fixed element of \mathbb{S} we construct a splitting tree $\{S_\sigma : \sigma \in \bigcup_{k < \omega} {}^k 2\}$ of conditions as before, where each S_σ decides $\dot{u}_{|\sigma|} = v_\sigma$ and let $\dot{w}_n = \bigcup_{|\sigma|=n} v_n$. Then $\mu(w_n) = \varepsilon \cdot 2^{-n}$ and the fusion $S \Vdash \dot{N} \subset \bigcup_{n < \omega} w_n$; $\bigcup_{n < \omega} w_n \leq \varepsilon$.

2. Using #1, below any fixed element of \mathbb{S} we construct a splitting tree $\{S_\sigma : \sigma \in \bigcup_{k < \omega} {}^k 2\}$ of conditions as before, where this time for each S_σ there is v_σ with $\mu(v_\sigma) \leq 2^{-(2n+1)}$, if $\sigma \subset \tau$ then $v_\sigma \supset v_\tau$, and $S_\sigma \Vdash \dot{N} \subset v_\sigma$. Let S be the fusion, and let $W = \bigcap_{n < \omega} \bigcup_{|\sigma|=n} v_n$. $\mu(W) = 0$, and $S \Vdash \dot{N} \subset W$. \square

Gerald Sacks' original motivation for Sacks forcing was to prove:

Theorem 32. Let M be a countable transitive model of set theory, let G be \mathbb{S} -generic over M , and let $x = \bigcap G$. Then for all $y \in M[G] \setminus M$, $M[y] = M[x]$; in fact $\forall y \in M[G] \setminus M$, $M[y] = M[x]$.

Hence, if x is a Sacks real over M , then there are no models between M and $M[x]$.

⁷Each \dot{u}_n names an element of V , we just aren't told which one.

When are two partial orders \mathbb{P}, \mathbb{Q} essentially (in the sense of forcing) the same? Clearly if they have isomorphic Boolean completions (we say then that $\mathbb{P} \cong \mathbb{Q}$). But when does that happen? Also, when does forcing with one partial order automatically add a generic object you get by forcing with another partial order?

Definition 29. An embedding $e : \mathbb{P} \rightarrow \mathbb{Q}$ is a complete embedding iff for all $p, p' \in \mathbb{P}$ and all $q \in \mathbb{Q}$

1. $p \perp p'$ iff $e(p) \perp e(p')$
2. if $p \leq p'$ then $e(p) \leq e(p')$
3. $\exists r \in \mathbb{P} q \cdot e(r') \neq 0$ for all $r' \leq r$.

Note that condition (3) follows from (2) and “ $e[\mathbb{P}]$ is dense in \mathbb{Q} .”

For example, let \mathbb{B} be the Boolean completion of \mathbb{P} . Then the identity map is a complete embedding from \mathbb{P} to \mathbb{B} . Similarly, if \mathbb{B} is the smallest Boolean sub-algebra of the completion of \mathbb{P} containing \mathbb{P} , then the identity map is a complete embedding from \mathbb{P} to \mathbb{B} .

We write $\mathbb{P} \geq \mathbb{Q}$ iff there is some $e : \mathbb{P} \rightarrow \mathbb{Q}$ a complete embedding.

Why are complete embeddings important?

Theorem 33. If $\mathbb{P} \geq \mathbb{Q}$, and \dot{G} is a \mathbb{Q} -generic filter, then $e^{-1}[\dot{G}]$ is a \mathbb{P} -generic filter.

I.e., forcing with \mathbb{Q} necessarily adds a \mathbb{P} -generic object.

Proof. Let $e : \mathbb{P} \rightarrow \mathbb{Q}$ be a complete embedding. Let D be pre-dense in \mathbb{P} . It suffices to show that $e[D]$ is pre-dense in \mathbb{Q} .

By the contrapositive, suppose $e[D]$ is not pre-dense in \mathbb{Q} . Then there is $q \in \mathbb{Q}$ with $q \perp e(p)$ for all $p \in D$. Let r be as in (3): if $r' \leq r$ then $q \cdot r' \neq 0$ for all $r' \leq r$. If $r \cdot s \neq 0$ for some $s \in D$, then $q \cdot (r \cdot s) \neq 0$, a contradiction. So $r \perp D$, and D is not pre-dense. □

Fact 20. $\mathbb{P} \cong \mathbb{Q}$ iff $\mathbb{P} \geq \mathbb{Q} \geq \mathbb{P}$.

Proof. The composition of complete embeddings is a complete embedding, and the image of \mathbb{P} under the first embedding is dense in \mathbb{Q} whose image is in turn dense in \mathbb{P} under the second embedding. Hence the completion of \mathbb{Q} is squeezed above and below by the completion of \mathbb{P} , so they are the same. □

Here is a not obvious application of theorem 41.

Fact 21. A countable weak product of partial orders necessarily adds a Cohen real. More precisely, for $n < \omega$ let \mathbb{P}_n be any partial order. Define $\mathbb{P} = \oplus_{n < \omega} \mathbb{P}_n = \bigcup_{k < \omega} \prod_{n < k} \mathbb{P}_n$, where we say $\bar{p} = (p_0, \dots, p_n) \leq \bar{q} = (q_0, \dots, q_m)$ iff $n \geq m$ and for $i < m$ $p_i \leq q_i$. Then $\mathbb{C}_\omega = Fn(\omega, 2, \omega)$ embeds completely in the Boolean completion \mathbb{B} of \mathbb{P}

Note that if some $p_i \perp q_i$, then $\bar{p} \perp \bar{q}$.

Proof. For $\sigma \in \mathbb{C}_\omega$ define $e(\sigma) = (p_i^{\sigma(i)} : i < |\sigma|)$ where we define $p^0 = -p, p^1 = p$. Then (1) and (2) are clearly satisfied. What about (3)?

Suppose $q \in \mathbb{B}$. Without loss of generality, $q \in \mathbb{P}$. Suppose $q = (q_i : i < n)$. Define $\sigma : n \rightarrow 2$ as follows: $\sigma(i) = 1$ iff $q_i \cdot p_i \neq 0$. Suppose $\tau \leq \sigma$. Then $e(\tau)|_n = e(\sigma)|_n$ so $e(\tau) \cdot q \neq 0$.

□

This application is not that useful — we almost never use products. But the method of proof can be generalized to show that any countable iteration of ccc partial orders with finite support — whatever those are — necessarily adds a Cohen real, and we will do this later.

As long as we're on the subject of adding a single Cohen real:

Fact 22. *Let \mathbb{P} be a countable separative partial order. Then $\mathbb{P} \cong \bigcup_{k < \omega} ({}^k 2)$, i.e., any countable separative partial order essentially adds a Cohen real.*

Proof. Here is a sketch of the proof: 1. Any two countable atomless Boolean algebras are isomorphic. (This is similar to the proof that any two countable dense linear orders are isomorphic.) 2. If \mathbb{P} is a countable separative partial order, consider the smallest Boolean algebra in which \mathbb{P} embeds densely. This is not the completion: it is much smaller, in fact it is countable. Hence any two of them are the same. But they all embed densely in their completions, hence their completions are isomorphic. □

Corollary 11. *Let \dot{x} be a \mathbb{C}_ω -name for a real not in V . If M is countable transitive and G is \mathbb{C}_ω -generic over M , then $M[\dot{x}/G]$ (= the smallest model extending M containing \dot{x}/G) adds a Cohen real.*

Proof. Let \mathbb{P} be the partial order in the completion of \mathbb{C}_ω generated by all $\llbracket \dot{x}(n) = i \rrbracket$ where $n < \omega, i < 2$ (or, equivalently, $i < \omega$). Since $\dot{x} \notin V$, \mathbb{P} is a countable separative partial order, hence adds a Cohen real. The reader is invited to show that $\mathbb{P} \geq \mathbb{C}_\omega$. (Hint: property (3) is the only not-entirely-trivial property.) □

The situation for Sacks reals contrasts with the Cohen situation: If $y \in M[x] \cap 2^\omega \setminus M$, where x is Cohen over M , then $M[y]$ adds some Cohen real, but it need not be x . There are many models between M and $M[x]$. For example, if $A \subsetneq^* B, A, B \in M \cap \mathcal{P}(\omega)$ and x is Cohen over M then $M[x|_A] \subsetneq M[x|_B] \subsetneq M[x]$.

12 Two-stage iteration

A two-stage product is easy: $\mathbb{P} \times \mathbb{Q}$ has the partial order $(p, q) \leq (p', q')$ iff $p \leq p'$ and $q \leq q'$.

But iteration is different from product. In product forcing, all the partial orders live in the ground model. But in iterated forcing, the second partial order does not live in the ground model, even though V may recognize a partial order with the “same” definition (e.g., “all perfect sets” or “all positive measure sets”...) something with the same definition might.

To make this concrete, let’s consider first adding a Cohen real and then adding a Sacks real..

This means first forcing with \mathbb{C}_ω and then with whatever the new model thinks is the set of all perfect subsets of \mathbb{R} , i.e., with \dot{S} . Note that there are perfect sets in $V^{\mathbb{C}_\omega} \setminus V$, so \dot{S} is not a subset of V . In general, product forcing is not iteration.⁸

To force first with \mathbb{C}_ω and then \dot{S} our conditions have the form (p, \dot{S}) where $p \Vdash \dot{S} \in \dot{S}$, and the partial order is $(p, \dot{S}) \leq (p', \dot{S}')$ iff $p \leq p'$ and $p \Vdash \dot{S} \leq \dot{S}'$.

As an exercise, let’s give some specific examples. Let \dot{G} name the $\dot{\mathbb{C}}_\omega$ -generic filter. We use the characterization of \dot{S} as the set of splitting subtrees of $\bigcup_{k < \omega} ({}^k \omega)$. Here are four elements of \dot{S} . (Note that $\emptyset \in S$ for all $S \in \dot{S}$.)

1. Let \dot{a} name $\{n : \bigcup \dot{G}(n) = 0\}$. \dot{S} names the following binary tree: if $\sigma \in \dot{S}$ and $|\sigma| \in \dot{a}$ then $\sigma 0 \in \dot{S}$ and $\sigma 1 \in \dot{S}$. Otherwise $\sigma 0 \in \dot{S}$ and $\sigma 1 \notin \dot{S}$.
2. Let \dot{R} name the following binary tree: if $\sigma \in \dot{R}$ and $2|\sigma| \in \dot{a}$ then $\sigma 0 \in \dot{R}$ and $\sigma 1 \in \dot{R}$. Otherwise $\sigma 0 \in \dot{R}$ and $\sigma 1 \notin \dot{R}$.
3. Let \dot{T} name the following binary tree: if $\sigma \in \dot{T}$ and $|\sigma| \notin \dot{a}$ then $\sigma 0 \in \dot{T}$ and $\sigma 1 \in \dot{T}$. Otherwise $\sigma 0 \in \dot{T}$ and $\sigma 1 \notin \dot{T}$.

As an example, let p be the following condition (listed as a sequence rather than a function): $p = (0101)$. Then p puts the following elements in \dot{a} : 0, 2. And p puts the following elements in $\omega \setminus \dot{a}$: 1, 3. So p puts the following sequences in \dot{S} : $\emptyset, (0), (1), (00), (10), (000), (001), (100), (101), (0000), (0010), (1000), (1010)$.

Exercise:

Let $p = (0101)$. What does p put into \dot{R}, \dot{T} ?

As an example, we’ll show that $V^{\mathbb{C}_\omega} \Vdash \dot{S} \perp \dot{T}$: Suppose some $p \Vdash \sigma \in \dot{S} \cap \dot{T}$. If $\sigma(k) = 1$ for some k then, by definition of \dot{S}, \dot{T} , $p \Vdash \sigma|_{k-1}$ splits in both trees, so $p \Vdash (k-1 \in \dot{a} \text{ and } k-1 \notin \dot{a})$, a contradiction. Hence $\forall n \in \text{dom } \sigma \sigma(n) = 0$. So $\dot{S} \perp \dot{T} = \{\sigma : \forall n \in \text{dom } \sigma \sigma(n) = 0\}$, which is not a splitting tree.

Exercise:

Show that $V^{\mathbb{C}_\omega} \Vdash \dot{S}, \dot{R}$ are not comparable.

Now for the general definition:

Definition 30. Let \mathbb{P} be a partial order in V , and suppose $V^{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is a partial order. We define $\mathbb{P} * \dot{\mathbb{Q}} = \{(p, \dot{q}) : p \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$ under the order $(p, \dot{q}) \leq (p', \dot{q}')$ iff $p \leq p'$ and $p \Vdash \dot{q} \leq \dot{q}'$.

Here is an interesting example, known as Mathias forcing:⁹

$\mathbb{P} = [\omega]^\omega; \leq = \subset^*$. Let \dot{G} be a \mathbb{P} -generic filter. Note that $A \Vdash C \in \dot{G}$ iff $C \supset^* A$.

⁸In rare cases it is, as we’ll see later.

⁹developed to discuss some questions in descriptive set theory

Fact 23. 1. \mathbb{P} is countably closed, hence adds no new reals.

2. $V^{\mathbb{P}} \Vdash \dot{G}$ is a non-principal ultrafilter on ω .

3. $V^{\mathbb{P}} \Vdash \dot{G}$ is a Ramsey ultrafilter, i.e., $V^{\mathbb{P}} \Vdash$ if $\vec{A} = \{A_n : n < \omega\} \in [\omega]^\omega$ pairwise disjoint and $\bigcup \vec{A} = \omega$ then there is $B \in \dot{G}$ either $\exists n A_n \in \dot{G}$ or $\forall n |B \cap A_n| \leq 1$ (we say that B is good for \vec{A}).¹⁰

Proof. For #1: This is because any countable non-principal filterbase on ω has a lower bound.

For #2: \dot{G} is a filter by definition.

For non-principal: Fix $a \in [\omega]^{<\omega}$. Given $A \in \mathbb{P}$, let $B = A \setminus a$. Then $B \leq A$ and $B \Vdash a \notin \dot{G}$.

For ultrafilter: We use the fact that there are no new subsets of ω in $V^{\mathbb{P}}$. So it suffices to show that $V^{\mathbb{P}} \Vdash \forall C \subset \omega$ either $C \in \dot{G}$ or $\omega \setminus C \in \dot{G}$. Fix $C \subset \omega$ and let $A \in \mathbb{P}$. If $C \cap A$ is finite, let $B = A \setminus C$. Then $B \leq A$ and $B \Vdash \omega \setminus C \in \dot{G}$. If $C \cap A$ is infinite, let $B = A \cap C$. Then $B \leq A$ and $B \Vdash C \in \dot{G}$. So $\{B : B \Vdash (\omega \setminus C \in \dot{G} \text{ or } B \Vdash C \in \dot{G})\}$ is dense, hence \dot{G} is an ultrafilter.

For #3: First note that, by #1, if $V^{\mathbb{P}} \Vdash \vec{A}$ is a countable sequence of subsets of ω , then $V^{\mathbb{P}} \Vdash$ there is some $\vec{B} \in V$ with $\vec{A} = \vec{B}$. So it suffices to consider $\vec{A} \in V$.

Let $\vec{A} = \{A_n : n < \omega\} \in [\omega]^\omega$ pairwise disjoint, $\bigcup \vec{A} = \omega$. Let $A \in \mathbb{P}$. If $\exists n A \cap A_n$ is infinite, let $B = A \cap A_n$. If $\forall n A \cap A_n$ is finite, there is $B \in [A]^\omega$ with $|B \cap A_n| \leq 1$ for all n . In either case B is good for \vec{A} and $B \Vdash B \in \dot{G}$. So $\{B \in \mathbb{P} : B \Vdash (B \text{ is good for } \vec{A} \text{ and } B \in \dot{G})\}$ is dense. \square

Now define $\dot{Q} = \{q = (a_q, A_q) : a_q \in [\omega]^{<\omega}, A_q \in \dot{G}, \sup a_q < \inf A_q; q \leq q' \text{ iff } a_q \supset a'_q, A_q \subset A'_q, \sup a'_q < \inf(a_q \setminus a'_q), (a_q \setminus a'_q) \subset A'_q\}$.¹¹

Exercises: 1. $V^{\mathbb{P}} \Vdash \dot{Q}$ is ccc, in fact σ -centered. (Hint: Fix a . In $V^{\mathbb{P}}$, what can you say about $\{q : a_q = a\}$? Use that $V^{\mathbb{P}} \Vdash \dot{G}$ a filter.)

2. If $(A, q) \in \mathbb{P} * \dot{Q}$, then $A_q \supset^* A$.

Let \dot{H} be the \dot{Q} -generic filter over $V^{\mathbb{P}}$. Let $\dot{C} = \bigcup_{q \in \dot{H}} a_q$

Exercise: $V^{\mathbb{P} * \dot{Q}} \Vdash \dot{C} \subset^* A$ for all $A \in \dot{G}$.

Note that while the elements of \dot{Q} are in V , \dot{Q} itself is not in V , since its definition depends on \dot{G} .

Mathias forcing is the forcing $\mathbb{P} * \dot{Q}$. It first adds a Ramsey ultrafilter, and then destroys it by finding a pseudointersection.

Fact 24. Mathias forcing adds a function $\dot{f} : \omega \rightarrow \omega$ which dominates $V \cap \omega^\omega$.

Proof. First, some notation: If $A \subset \omega$, f_A denotes the enumerating function of A , i.e., $f_A(n)$ is the n^{th} element of A .

Let \dot{C} be the generic set added by \dot{Q} , and define $\dot{h} = f_{\dot{C}}$. Suppose $g \in V, g : \omega \rightarrow \omega, g$. We show $\{(B, q) : (B, q) \Vdash \dot{h} >^* g\}$ is dense.

Pick $(A, q) \in \mathbb{P} * \dot{Q}$. Define B as follows: Let $n = |a_q|$. For $m \geq n, k_m =$ the least $i \in A \cap A_q$ with $i > g(m)$. $B = a_q \cup \{k_m : m \geq n\}$. By definition $f_B|_{\omega \setminus \sup a_q} > g|_{\omega \setminus \sup a_q}$. $(B, (a_q, B)) \leq (A, q)$ and $(B, (a_q, B)) \Vdash \dot{h} \geq f_B >^* g$. \square

¹⁰Equivalently: F is a Ramsey ultrafilter iff for all $f : [\omega]^m \rightarrow n$ for some $m, n < \omega$ then there is $A \in F, A$ homogeneous for f .

¹¹Note that $V^{\mathbb{P}} \Vdash \dot{Q} \subset V$, even though $V^{\mathbb{P}} \Vdash \dot{Q} \notin V$ (the name \dot{Q} is, of course, in V).

Here is another example of two-step iterated forcing:

Theorem 34. *Cons($2^{\omega_1} = \omega_{17}$ and $2^{\omega_2} = \omega_{42}$).*

Proof. Start with a model of V of GCH.

If your instinct is to force with $F_n(\omega_{17}, 2, \omega_1) * \dot{F}_n(\omega_{42}, 2, \omega_2)$ — forget it. In order to have $\dot{F}(\omega_{42}, 2, \omega_2)$ have the ω_3 -cc and not collapse cardinals, we would need that $V^{F_n(\omega_{17}, 2, \omega_1)} \Vdash [\omega_2]^{\omega_1} = \omega_2$. Which it doesn't.

So instead we switch: let $\mathbb{P} = F_n(\omega_{42}, 2, \omega_2)$, and in $V^{\mathbb{P}}$ let $\dot{\mathbb{Q}} = \dot{F}_n(\omega_{42}, 2, \omega_2)$. Since \mathbb{P} adds no subsets of ω , $V^{\mathbb{P}} \Vdash [\omega_1]^\omega = \omega_1$, which is what we need to show that $V^{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ has the ω_2 -cc, hence does not collapse cardinals. \square

Let's carefully examine the definition of $\dot{\mathbb{Q}}$ in the proof of theorem 25. Because \mathbb{P} is ω_2 -closed, $V^{\mathbb{P}} \Vdash \forall \dot{f} \in V^{<\omega_1} \exists g \in V \cap V^{<\omega_1} \dot{f} = g$. So $V^{\mathbb{P}} \Vdash V \cap F_n(\omega_{17}, 2, \omega_1) = F_n(\omega_{17}, 2, \omega_1)$. I.e., $\mathbb{P} * \dot{\mathbb{Q}} \cong F_n(\omega_{42}, 2, \omega_2) \times F_n(\omega_{17}, 2, \omega_1)$. This iteration is actually a product.

There are two other standard instances of this phenomenon.

First a definition: Given a set I , \mathbb{M}_I is the measure algebra on 2^I ; $\mathbb{C}_I = F_n(I, 2, \omega)$. We will not prove

Theorem 35. *Let I, J be disjoint. Then $\mathbb{M}_{I \cup J} = \mathbb{M}_I \times \mathbb{M}_J \cong \mathbb{M}_I * \dot{\mathbb{M}}_J$.*

The interested reader can look this up in Kunen's article. It is essentially the Fubini theorem, and its proof is not nearly as trivial as the proof of

Theorem 36. *Let I, J be disjoint. Then $\mathbb{C}_{I \cup J} = \mathbb{C}_I \times \mathbb{C}_J \cong \mathbb{C}_I * \dot{\mathbb{C}}_J$. In fact, for any J , for any partial order $\mathbb{P}, \mathbb{P} * \dot{\mathbb{C}}_J \cong \mathbb{P} * \mathbb{C}_J$.*

Proof. $V^{\mathbb{P}} \Vdash$ if \dot{p} is a finite function into 2, then there is $q \in V$ with $q = \dot{p}$. So $V^{\mathbb{P}} \Vdash V \cap \mathbb{C}_J = \dot{\mathbb{C}}_J$. \square

A proof similar to the proof of fact 21 gives:

Theorem 37. *Let \mathbb{P} be a ccc iterated forcing with finite support of length γ . Then $\mathbb{C}_\gamma \geq \mathbb{P}$.*

Theorem 38. *If \mathbb{P} is ccc and $V^{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is ccc, then $\mathbb{P} * \dot{\mathbb{Q}}$ is ccc.*

Before proving this theorem, let's put it in context.

Theorem 21 stated that $\text{MA} + \neg\text{CH} \Rightarrow$ the product of two ccc spaces is ccc. But this is not always true.

What can go wrong is that $V \models \mathbb{P}$ is ccc, but $V^{\mathbb{P}} \Vdash \mathbb{P}$ is not ccc. When this happens \mathbb{P}^2 is not ccc.

For example, the partial order sending a branch through a Suslin tree is ccc but forces its own lack of ccc. Back in the mid-1970's, Galvin proved that under CH there is a separative ccc partial order whose product is not ccc. Fleissner then got the same conclusion in any $V^{\mathbb{C}_{\omega_1}}$, and then Roitman got the same conclusion in any $V^{\mathbb{C}_\omega}$ or $V^{\mathbb{M}_\omega}$. We may prove the \mathbb{C}_ω result later.

Proof. Let $V^{\mathbb{P}} \Vdash \dot{Q}$ is ccc. Suppose $A = \{(p_\alpha, \dot{q}_\alpha) : \alpha < \omega_1\}$ is an antichain in $\mathbb{P} * \dot{Q}$. If $\{p_\alpha : \alpha < \omega_1\}$ is countable, then there is p and uncountable E so $\alpha \in E \Rightarrow p = p_\alpha$. Hence $p \Vdash \dot{q}_\alpha \perp \dot{q}_\beta$ for $\alpha \neq \beta \in E$, which contradicts $V^{\mathbb{P}} \Vdash \dot{Q}$ is ccc

Now suppose $\{p_\alpha : \alpha < \omega_1\}$ is uncountable. Let \dot{G} be the \mathbb{P} -generic filter, and define $\dot{E} = \{(p_\alpha, \alpha) : \alpha < \omega_1\}$, i.e., $p_\alpha \in \dot{G} \Rightarrow \alpha \in \dot{E}$. If some $p_\alpha \Vdash \dot{E}$ countable, then $H_\alpha = \{\beta : p_\alpha \text{ is compatible with } p_\beta\}$ is countable. If $\{\alpha : H_\alpha \text{ is countable}\}$ is uncountable, then by induction we can construct an uncountable set $W \subset \omega_1$ so that if $\alpha < \beta \in W$ then $p_\alpha \notin H_\beta$, i.e., W is an uncountable antichain. But \mathbb{P} ccc, so there are at most countably many such p_α with H_α countable. By throwing them away we may assume that each $p_\alpha \Vdash \dot{E}$ is uncountable, i.e., $V^{\mathbb{P}} \Vdash \{\alpha : p_\alpha \in \dot{G}\}$ is uncountable.

If $p_\alpha \cdot p_\beta \neq 0$ then, since A is an antichain, $p_\alpha \cdot p_\beta \Vdash \dot{q}_\alpha \perp \dot{q}_\beta$. And $V^{\mathbb{P}} \Vdash$ if $p_\alpha, p_\beta \in \dot{G}$ then $p_\alpha \cdot p_\beta \neq 0$. i.e. $V^{\mathbb{P}} \Vdash \{\dot{q}_\alpha : \alpha \in \dot{E}\}$ is an antichain in \dot{Q} . Since $V^{\mathbb{P}} \Vdash \dot{Q}$ is uncountable, this contradicts the hypothesis that $V^{\mathbb{P}} \Vdash \dot{Q}$ is ccc. \square

By closely imitating the proof of theorem 38, we have

Theorem 39. *For any regular κ , if \mathbb{P} is κ -cc and $V^{\mathbb{P}} \Vdash \dot{Q}$ is κ -cc, then $\mathbb{P} * \dot{Q}$ is κ -cc.*

13 Iterated forcing

Definition 31. An iterated forcing of length γ is a sequence of the form $\{\mathbb{P}_\alpha : \alpha < \gamma\}$ where $|\mathbb{P}_0| = 1$, and each $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.¹² Given the iteration $\{\mathbb{P}_\alpha : \alpha < \gamma\}$ and $\alpha < \gamma$, we write $\mathbb{P}_\alpha = \{\mathbb{P}_\beta : \beta < \alpha\}$.

A word about how the iteration starts off. \mathbb{P}_0 is the trivial partial order. $\mathbb{Q}_0 \in V^{\mathbb{P}_0} = V$. $\mathbb{P}_1 = \mathbb{P}_0 * \mathbb{Q}_0$.¹³

Note that when $\alpha > 0$ is a limit, then \mathbb{P}_α is also a limit, that is, each $p \in \mathbb{P}_\alpha$ is the union of all $p_\gamma, \gamma < \alpha$.

A condition in the iterated forcing $\{\mathbb{P}_\alpha : \alpha < \gamma\}$ is necessarily a function p with domain γ , each $p(\alpha) = p_\alpha \in \mathbb{P}_\alpha$. Hence $p_0 = 1_{\mathbb{P}_0}, p_1 \in \mathbb{Q}_0$, and each $p_\alpha \Vdash p_{\alpha+1} = p_\alpha * \dot{q}_\alpha$, where $p_\alpha \Vdash \dot{q}_\alpha \in \dot{\mathbb{Q}}_\alpha$. This is not, however, sufficient — not all such functions are forcing conditions in a particular iteration, and part of the art of constructing any particular iterated forcing is constraining which functions are conditions.

The order is: $p \leq p'$ iff each $p_\alpha \Vdash \dot{q}_{\alpha+1} \leq \dot{q}'_{\alpha+1}$.

We have $\mathbb{P}_\alpha \geq \mathbb{P}_\beta$ if $\alpha < \beta$ by the embedding $e(p)(\delta) = p_\delta$ if $\delta < \alpha$; otherwise $e(p)(\delta) = 1_{\mathbb{P}_\delta}$. (Note: From now on we will dispense with the subscript on 1.)

It is very important to note that conditions in each \mathbb{P}_α are sequences of names, and hence elements of V , so we can talk about these names even though we don't know exactly what these names signify.

Definition 32. An iterated forcing $\{\mathbb{P}_\alpha : \alpha < \gamma\}$ is said to have finite support (respectively, countable support) (respectively, support of size λ) iff for every condition p at most finitely many (respectively countably many) (respectively λ many) $p_\alpha \Vdash \dot{q}_\alpha = 1$.

We define the support of a condition p to be $\{\alpha + 1 : p_\alpha \Vdash \dot{q}_\alpha = 1\}$.

Let P be an arbitrary property of partial orders (e.g., ccc, or countably closed). We say that an iterated forcing is an iteration of forcings with property P iff each $V^{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha$ has property P .

Here are two examples:

Adding a scale Given a model V , let \mathbb{Q}_V be the standard ccc partial order forcing some $\dot{g} : \omega \rightarrow \omega$ which dominates $V \cap \omega^\omega$. (See theorem 21(b) for the definition of \mathbb{Q}_V .) Let $\text{cf } \kappa > \omega$. We define a finite support iteration $\mathbb{P}_{\text{scale}} = \{\mathbb{P}_\alpha : \alpha < \kappa\}$ where $\mathbb{P}_0 = \mathbb{Q}_V$, and each $V^{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}_{V^{\mathbb{P}_\alpha}}$. We denote by \dot{g}_α the generic function added by $\dot{\mathbb{Q}}_\alpha$. By definition, if $\dot{f} \in (\omega^\omega)^{\mathbb{P}_\alpha}$ then $\dot{g}_\alpha \geq^* \dot{f}$.

$\mathbb{P}_{\text{scale}}$ adds a κ -scale. To show this we will need: Theorem 40: a finite support iteration of ccc forcings is ccc; and the following corollary of theorem 42: if κ has uncountable cofinality, and \mathbb{P} is a ccc iteration of length κ with finite support, then any name for a countable subset of V in $V^{\mathbb{P}}$ is an element of some $V^{\mathbb{P}_\alpha}$ where $\alpha < \kappa$. We will prove these theorems later.

Using theorems 40 and 42, let's prove that this forcing adds a κ -scale, i.e., that $\{\dot{g}_\alpha : \alpha < \kappa\}$ is a dominating family well-ordered by \leq^* : If $\dot{h} : \omega \rightarrow \omega$,¹⁴ then there is α with $\dot{h} \in V^{\mathbb{P}_\alpha}$. By definition of $\dot{\mathbb{Q}}_\alpha, \dot{h} <^* \dot{g}_\alpha$. So $\{\dot{g}_\alpha : \alpha < \omega\}$ is a dominating family. Since each $\dot{g}_\alpha \in V^{\mathbb{P}_{\alpha+1}}$, if $\alpha < \beta$, then $\dot{g}_\alpha <^* \dot{g}_\beta$. So $\{\dot{g}_\alpha : \alpha < \kappa\}$ is well-ordered by \leq^* .

¹²It is possible to have iterations along other sorts of directed systems, but we will not deal with this here.

¹³when we refer to all $\dot{\mathbb{Q}}_\alpha$, the reader is reminded that we are including \mathbb{Q}_0

¹⁴this includes $\dot{h} = \dot{f}$ for some $f \in V$

A well-ordered base for the club filter Given a model V of GCH, let $\mathcal{C}_V = \{C \in V : C \text{ is a closed unbounded subset of } \omega_1\}$. Recall that \mathcal{C}_V is closed under countable intersection, i.e., the countable intersection of clubs is club. We define the forcing $\mathbb{Q}_V = \{(c, C) : c \in V \text{ is a countable closed subset of } \omega_1, C \in \mathcal{C}_V\}$. The order is: $(c, C) \leq (c', C')$ iff c' is an initial segment of c , $C \subset C'$ and $c \setminus c' \subset C'$. If \dot{G} is the \mathbb{Q}_V -generic filter, we define $\dot{D} = \bigcup \{c : \exists C (c, C) \in \dot{G}\}$.

The following is assigned for homework, due orally on the Tuesday after break:

1. \mathbb{Q}_V is countably closed.
2. \dot{D} is uncountable.
3. Define $A \subset^{\text{ctble}} B$ iff $A \setminus B$ is countable. Then $\dot{D} \subset^{\text{ctble}} C$ for all $C \in \mathcal{C}_V$.

We proved in class that, assuming #1, \dot{D} is closed, hence, by #2, \dot{D} is club. Proof that \dot{D} is closed: It suffices to show that if $p \Vdash E \subset \dot{D}$ for some countable E , then $\exists q \leq p \ q \Vdash \sup E \in \dot{D}$. (By #1, no countable subsets of V are added, so it suffices to consider ground model sets E .) So suppose some $p \Vdash E$ is a countable subset of \dot{D} . Since E is countable, there is $f : \omega \rightarrow E$, f is 1-1 onto. Let $q_n = (c_n, C_n) \leq p$ be a descending chain of conditions, where each $f(n) \in c_n$ — we can do this because $p \Vdash E \subset \dot{D}$. Let $q = (c, C)$ be a lower bound of all q_n . By definition, $E \subset c$. Since c is closed, $\sup E \in c$. And $q \Vdash c \subset \dot{D}$.

Later we will need the following: if CH holds in V , then \mathbb{Q}_V is ω_2 -cc (because two conditions with the same first coordinate are compatible). Note that $\text{cf } \omega_2 > \omega_1$.

For uncountable sets A, B we write $A \leq_c B$ iff $A \setminus B$ is countable. Then $V^{\mathbb{Q}_V} \Vdash \dot{C} \leq_c D$ for all $D \in \mathcal{C}_V$. I.e., \dot{C} is a lower bound for the club subsets of V .

We define a countable support iteration $\mathbb{P}_{\text{club}} = \{\mathbb{P}_\alpha : \alpha < \omega_2\}$ where $\mathbb{P}_0 = \mathbb{Q}_V$ and each $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}_{V^{\mathbb{P}_\alpha}}$. This forcing adds a \leq_c -decreasing ω_2 sequence of club sets $\{\dot{C}_\alpha : \alpha < \omega_2\}$ so that if \dot{D} is club in ω_1 then there is some α with $\dot{D} \geq_c \dot{C}_\alpha$. So in $V^{\mathbb{P}_{\text{club}}}$ we have an ω_2 -descending family (under \leq_c) of club sets which generates the club filter.

To show this we need: Theorem 41: a countable support iteration of countably closed forcings is countably closed; and the following corollary of theorem 42: if CH holds, $\text{cf } \kappa > \omega_1$, \mathbb{P} is a countably closed iteration of length κ and if \dot{D} names a subset of ω_1 in $V^{\mathbb{P}}$ then there is $\alpha < \kappa$ with $\dot{D} \in V^{\mathbb{P}_\alpha}$.

Given theorem 41 and theorem 43, the proof that \mathbb{P}_{club} adds a well-ordered base for the club filter is similar to the proof that $\mathbb{P}_{\text{scale}}$ adds a scale.

So we need two types of theorems: 1. cardinals are not collapsed; 2. small sets are added at early stages.

First, we give sufficient conditions for not collapsing cardinals:

Theorem 40. *Iterated ccc forcing with finite support is ccc, hence does not collapse cardinals.*

Proof. Suppose that $\{\mathbb{P}_\alpha : \alpha < \gamma\}$ is a finite support iteration of ccc forcings. Suppose we know that each \mathbb{P}_α is ccc for $\alpha < \beta$. If $\beta = \alpha + 1$, then we are done by 2-stage iteration. Otherwise β is a limit. Suppose A is an uncountable antichain in \mathbb{P}_β . Without loss of generality $|A| = \omega_1$. Let $S = \{\text{support}(p) : p \in A\}$. Since S is a family of ω_1 many finite sets, without loss of generality S is a Δ -system with root r . For $p \in A$, let $p'_\alpha = p_\alpha$ if $\alpha \notin r$; otherwise $p'_\alpha = 1$. Then $p \neq q \in A \Rightarrow p', q'$ compatible (because if $p'_\alpha \neq 1$ then $q'_\alpha = 1$ and vice versa). Hence if $p \neq q \in A$ then $\exists \alpha \in r \ p_\alpha \perp q_\alpha$. But $\sup r < \beta$, so $\mathbb{P}_{\sup r}$ is ccc, hence A is not an antichain. □

Theorem 41. *Iterated countably closed forcing with countable support does not collapse cardinals.*

Proof. Let $\mathbb{P} = \{\mathbb{P}_\alpha : \alpha < \gamma\}$ be an iteration of countably closed forcings with countable support. Suppose $\{p^n : n < \omega\}$ is a descending sequence of conditions, support $p^n = S_n$. Note that each $S_n \subset S_{n+1}$. Let $S = \bigcup_{n < \omega} S_n$. We define p a lower bound for $\{p^n : n < \omega\}$ by induction on S :

Suppose we know $p_{\alpha+1}$ for $\alpha \in S$. Let $\beta = \inf S \setminus \alpha + 1$. Let $\dot{q}_\delta = 1$ if $\alpha < \delta < \beta$, and for all $\delta \in (\alpha, \beta)$, $p_\delta \Vdash p_{\delta+1} = p_\delta * \dot{q}_\delta$; hence we have defined p_β . Each $(p^n)_{\beta+1} = r^n * \dot{q}^n$ for some r^n . Define $p_{\beta+1} = p_\beta * \dot{q}$ where $p_\beta \Vdash \dot{q}$ is a lower bound for $\{\dot{q}^n : n < \omega\}$. Etc.

Support $p = S$, which is countable, so $p \in \mathbb{P}$. By construction, $p \leq p^n$ for all n . □

Now to give sufficient conditions for small sets being added at early stages. We use the fact that sets in forcing extensions are determined by antichains

Theorem 42. *Let $\{\mathbb{P}_\alpha : \alpha < \gamma\}$ be a κ -ccc iteration with support $\leq \lambda$. Suppose $\kappa \leq \lambda < \text{cf } \gamma$. If $\dot{x} \in [V]^{\leq \lambda}$ then there is $\delta < \gamma, \dot{x} \in V^{\mathbb{P}_\delta}$.*

Proof. We work in $V^{\mathbb{P}_\gamma}$. Since $\dot{x} \subset V$ and $|\dot{x}| \leq \lambda$, there is $\dot{f} : \lambda \rightarrow V, \dot{x} = \text{range } \dot{f}$. Since \mathbb{P}_γ is κ -ccc, for each $\alpha < \lambda$ there is $A_\alpha \in V$, A_α a maximal antichain deciding $\dot{f}(\alpha)$, $|A_\alpha| < \lambda$. Each $E_\alpha = \bigcup_{p \in A_\alpha} \text{support}(p)$ has size $\leq \lambda$, as does $E = \bigcup_{\alpha < \lambda} E_\alpha$. So $\text{sup } E < \gamma$. Let $\delta > \text{sup } E$. Then $\exists \dot{y} \in \mathbb{P}_\delta \dot{x} \cong \dot{y}$. □

In particular, the hypotheses of theorem 42 hold if \mathbb{P} is a ccc iteration with finite support and $\text{cf } \gamma > \omega$; or \mathbb{P} is a countable support ω_2 -iteration of countably closed forcings with $\omega_2 = \text{cc}$, $\text{cf } \gamma > \omega_1$.

These three theorems complete the proofs that $\mathbb{P}_{\text{scale}}$ adds a scale, and that \mathbb{P}_{club} adds a well-ordered base for the club filter on ω_1 .

14 MA + ¬CH is consistent

We are about to prove the consistency of MA + ¬CH.

Some useful notation: if X is definable from parameters in M a model (hence $X \in M$), then $X^M = \{x : M \models x \in X\}$. Similarly, if \dot{X} is definable from parameters in $M^{\mathbb{P}}$ a model (hence $\dot{X} \in M^{\mathbb{P}}$), then $\dot{X}^{M^{\mathbb{P}}} = \{\dot{x} : M^{\mathbb{P}} \Vdash \dot{x} \in \dot{X}\}$ ¹⁵

First, a technical lemma.

Lemma 3. *Let \mathbb{P} be a ccc separative partial order of size κ .*

(a) *For all $\lambda < \kappa$, $(\lambda^\omega)^{V^{\mathbb{P}}} \leq (\kappa^\omega)^V$.*¹⁶

(b) *For all $\lambda < \kappa$, $(2^\lambda)^{V^{\mathbb{P}}} \leq (\kappa^\lambda)^V$.*

Proof. (a) In $V^{\mathbb{P}}$, let $\dot{f} : \omega \rightarrow \lambda$. Each $\dot{f}(n)$ is determined by a countable antichain $A_n \subset \mathbb{P}$. In V , how many such antichains are there? κ^ω . In V , how many sequences of such antichains are there? $(\kappa^\omega)^\omega = \kappa^\omega$

(b) In $V^{\mathbb{P}}$, let $\dot{f} : \lambda \rightarrow 2$. Each $\dot{f}(\alpha)$ is determined by a countable antichain $A_\alpha \subset \mathbb{P}$. In V , how many such antichains are there? κ^ω . In V , how many λ -sequences of such antichains are there? $(\kappa^\omega)^\lambda = \kappa^\lambda$

□

Corollary 12. *Assume GCH, and let \mathbb{P} be a ccc separative partial order of size κ , where κ is regular. Then $\forall \lambda < \kappa$ $(2^\lambda)^{V^{\mathbb{P}}} \leq \kappa$.*

Proof. If GCH holds and κ is regular, then $\kappa^{<\kappa} = \kappa$. □

Theorem 43. *Let κ be regular in a model V of GCH. Then there is a partial order \mathbb{P}_{MA} so that $V^{\mathbb{P}_{\text{MA}}} \Vdash \text{MA} + \mathfrak{c} = \kappa$.*

Proof. We want to find a model of: $\mathfrak{c} = \kappa$ and if \mathbb{Q} is a ccc separative partial order, and \mathcal{D} is a family of dense sets, where $|\mathcal{D}| < \kappa$, then there is a filter $G \subset \mathbb{Q}$, $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

First, note that if D is dense, and A is a maximal antichain of D , then $\Sigma A = \Sigma D = 1$, and a filter meeting A also meets D . So it suffices to require: if \mathcal{A} is a family of maximal antichains where $|\mathcal{A}| < \kappa$ then there is a filter G so if $A \in \mathcal{A}$ then $G \cap A \neq \emptyset$.

Suppose such a G exists. Then there is a partial order \mathbb{Q}' of size $|\mathcal{A}|$, $\mathbb{Q}' \subset \mathbb{Q}$, with $G \subset \mathbb{Q}'$ and $\bigcup \mathcal{A} \subset \mathbb{Q}'$ (namely the Boolean algebra generated by $\bigcup \mathcal{A}$). A filter in \mathbb{Q}' meeting every $A \in \mathcal{A}$ is a filter in \mathbb{Q} meeting every $A \in \mathcal{A}$. So it suffices to look at ccc separative partial orders of size $< \kappa$, and families of maximal antichains of size $< \kappa$.

We will use ccc iterated forcing that works its way through every ccc separative partial order of size $< \kappa$ that arises at earlier stage. I.e., our final partial order \mathbb{P}_{MA} will be a ccc iteration with finite support $\{\mathbb{P}_\alpha : \alpha < \kappa\}$ where each $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha$ is a ccc separative partial order of size $< \kappa$.

Why does it suffice to consider only κ many such partial orders? And how can we make sure that we take care of each one?

¹⁵for those who know what elementary submodels are: if M is an elementary submodel of a sufficiently large $H(\theta)$, then $X^M = X \cap M$; a similar statement holds for $M^{\mathbb{P}}$

¹⁶Here “ λ^ω ” denotes the cardinal, not the set of functions, so we don’t need a $\dot{\lambda}^\omega$.

Since every partial order of size λ is isomorphic to a partial order on λ , it suffices to consider partial orders on each $\lambda < \kappa$.

So we consider only partial orders of the form $\dot{Q} = (\lambda, \leq_{\dot{Q}})$, where $\lambda < \kappa$.

How big is our final iteration \mathbb{P}_{MA} ? Each condition is a sequence of functions from κ with finite support, and if $\alpha + 1 \in \text{support } p$ then $p_\alpha \Vdash \dot{q}_\alpha \in \lambda$ for some $\lambda < \kappa$, so $|\mathbb{P}_{\text{MA}}| = [\kappa]^{<\omega} = \kappa$.

What about the $\leq_{\dot{Q}}$'s? Each is a subset of λ^2 for some $\lambda < \kappa$. So by corollary 12 there are at most κ many. And by theorem 42, each is an element of some (hence all but $< \kappa$ -many) $V^{\mathbb{P}_\alpha}$.

Also, suppose $\dot{\mathcal{A}}$ is a family of maximal antichains of some \dot{Q} a partial order on some $\lambda < \kappa$, $|\dot{\mathcal{A}}| < \kappa$. If $\dot{A} \in \dot{\mathcal{A}}$ then \dot{A} is a countable subset of λ , so by theorem 42 $\dot{A} \in \mathbb{P}_{\alpha_{\dot{A}}}$ for some $\alpha_{\dot{A}} < \kappa$. Since $|\dot{\mathcal{A}}| < \kappa$, $\sup\{\alpha_{\dot{A}} : \dot{A} \in \dot{\mathcal{A}}\} < \kappa$, hence $\exists \alpha \forall \gamma > \alpha \dot{\mathcal{A}} \subset V^{\mathbb{P}_\gamma}$. If $\dot{Q} = \dot{Q}_\beta$ for some $\gamma > \alpha$ then any \dot{Q}_γ -generic filter meets every element of $\dot{\mathcal{A}}$.

Now we know how to proceed.

Let $h : \kappa \rightarrow \kappa^2$ so that if $h(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$. For each β , let $\{\dot{Q}_{\beta, \gamma} : \gamma < \kappa\}$ list all ccc separative partial orders in $V^{\mathbb{P}_\alpha}$ of size $< \kappa$ so each partial order is listed κ many times.¹⁷ Let \mathbb{P}_{MA} be the iteration with finite support of $\{\mathbb{P}_\alpha : \alpha < \kappa\}$ where each $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{Q}_{h(\alpha)}$.¹⁸

Now we show that $\text{MA} + \mathfrak{c} = \kappa$ holds in $V^{\mathbb{P}_{\text{MA}}}$.

First, since cofinally many $\dot{Q}_\alpha = \mathbb{C}_\omega$, we have added κ many Cohen reals, so $\mathfrak{c} \geq \kappa$. Second, by lemma 3(b), $\mathfrak{c} \leq \kappa$. Hence $(\mathfrak{c})^{V^{\mathbb{P}_{\text{MA}}}} = \kappa$.

Now for MA. Suppose \dot{Q} is a \mathbb{P}_{MA} -name for a partial order on some $\lambda < \kappa$, and $\dot{\mathcal{A}}$ is a \mathbb{P}_{MA} -name for a family of maximal antichains of \dot{Q} . $|\dot{\mathcal{A}}| < \kappa$. Then $\dot{Q} = \dot{Q}_\alpha$ for a cofinal set of α 's and there is α so if $\gamma > \alpha$ then $\dot{\mathcal{A}} \subset V^{\mathbb{P}_\gamma}$. Let $\gamma > \alpha$ with $\dot{Q} = \dot{Q}_\alpha$. The \dot{Q}_γ -generic filter over $V^{\mathbb{P}_\gamma}$ is a filter in \dot{Q} meeting each element of $\dot{\mathcal{A}}$.

□

¹⁷the convention we use is that each $\dot{Q}_{\alpha, \beta}$ names a partial order in the ground model V

¹⁸this is often shortened to “by a bookkeeping argument...”

15 Easton forcing

Recall König's theorem: $\text{cf } 2^\kappa > \kappa$. Clearly $2^\kappa \leq 2^\lambda$ if $\kappa \leq \lambda$. The purpose of this section is to show that these are, effectively, the only constraints on (simultaneously) all 2^κ for κ regular.

We define REG = the class of regular cardinals. The class ¹⁹ function $v : \text{REG} \rightarrow \text{CARD}$ is Easton iff $\kappa \leq \lambda \Rightarrow v(\kappa) \leq v(\lambda)$ and each $\text{cf } v(\kappa) > \kappa$.

Theorem 44. (*Easton*) Assume GCH. Let v be an Easton function. Then there is partial order \mathbb{P}_v where $V^{\mathbb{P}_v} \Vdash 2^\kappa = v(\kappa)$

Proof. In the 2-step iteration of theorem 34, in order to avoid collapsing cardinals we needed to change 2^{ω_2} before 2^{ω_1} . This downgraded our iteration to a product of partial orders in the ground model. This is the insight that allows Easton's construction to go forward.

So let v be an Easton function. We define the elements of \mathbb{P}_v as follows: $p \in \mathbb{P}_v$ iff $\text{dom } p$ is an initial segment of REG , each $p(\kappa) \in \text{Fn}(v(\kappa), 2, \kappa)$, and for all cardinals $\lambda \leq \sup \text{dom } p$, $|\{\kappa \in \lambda \cap \text{REG} : p(\kappa) \neq \emptyset\}| < \lambda$. (This last clause is only meaningful when λ is a regular limit cardinal, i.e. when $\lambda = \aleph_\lambda$; otherwise it is automatically satisfied.)

\mathbb{P}_v is also a class. Define $V^{\mathbb{P}_v} = \bigcup_{\kappa \in \text{REG}} V^{\mathbb{P}_{v,\kappa}}$, where $\mathbb{P}_{v,\kappa} = \{p \in \mathbb{P} : \text{dom } p \subset \kappa\}$.

We define the order: $p \leq p'$ iff $\text{dom } p \supset \text{dom } p'$ and if $\alpha \in \text{dom } p'$ then $p(\alpha) \supset p'(\alpha)$.

Define $\mathbb{P}_v^\kappa = \{p \in \mathbb{P}_v : p(\lambda) \neq \emptyset \text{ then } \lambda \geq \kappa\}$. Then, for each κ , $\mathbb{P}_v = \mathbb{P}_{v,\kappa} \times \mathbb{P}_v^\kappa = P_v^\kappa * (\mathbb{P}_{v,\kappa})^V$.

Each \mathbb{P}_v^λ is, by definition, λ -closed, so it preserves cardinals $\leq \lambda$.

In general, if \mathbb{Q} is a class forcing, then $V^{\mathbb{Q}}$ need not be a model of ZF: consider $\mathbb{Q} = \text{Fn}(\omega, \text{ORD}, \omega)$, which makes ORD countable. In $V^{\mathbb{Q}}$, replacement fails. Luckily that doesn't happen here. It is a useful exercise (left to the reader) to show that $V^{\mathbb{P}_v} \Vdash \text{ZFC}$. This relies heavily on the fact that each \mathbb{P}_v^λ is λ closed, so adds no sets of size $< \lambda$.

We show that each \mathbb{P}_{v,λ^+} is λ^+ -cc:

Let $p \in \mathbb{P}_{v,\lambda^+}$, and let $I_p = \{\kappa \in \text{REG} : p(\kappa) \neq \emptyset\}$. Each $|I_p| < \lambda$ (because $I_p \subset (I_p \cap \lambda) \cup \{\lambda\}$ and $|I_p \cap \lambda| < \lambda$). For all $\kappa \in I_p$ let $E_{p,\kappa} = \text{dom } p(\kappa)$. Each $|E_{p,\kappa}| < \kappa \leq \lambda$. Let $E_p = \bigcup_{\kappa \in I_p} E_{p,\kappa} \times \{\kappa\}$. Each $|E_p| < \lambda$. Note that $p \perp q$ iff there is $(\gamma, \kappa) \in E_p \cap E_q$ with $p(\kappa)(\gamma) \neq q(\kappa)(\gamma)$.

Suppose $\{p_\alpha : \alpha < \lambda^+\}$ are distinct elements of \mathbb{P}_{v,λ^+} . By GCH $\lambda^{<\lambda} = \lambda < \lambda^+$, so by the generalized Δ -system lemma there is $X \in [\lambda^+]^{\lambda^+}$ with $\{E_{p_\alpha} : \alpha \in X\}$ a Δ -system with root E . $|E| < \lambda$, so $2^{|E|} \leq \lambda$, so there is $Y \in [X]^{\lambda^+}$ so that for all $\alpha, \beta \in Y$ and all $(\gamma, \kappa) \in E$ $p_\alpha(\kappa)(\gamma) = p_\beta(\kappa)(\gamma)$. Hence if $\alpha, \beta \in Y$ then p_α, p_β are compatible. So $\{p_\alpha : \alpha < \lambda^+\}$ is not an antichain.

Hence no λ^+ is collapsed: $\mathbb{P}_v^{\lambda^+}$ is λ^+ -closed, so does not collapse λ^+ ; $(\mathbb{P}_{v,\lambda^+})^V$ has $\lambda^+ - \text{cc}$, so does not collapse λ^+ , and $\mathbb{P}_v = P_v^{\lambda^+} * (\mathbb{P}_{v,\lambda^+})^V$.

What about other cardinals?

Fact 25. Suppose $\forall \alpha \in \text{ON } V \models \text{cf } \alpha = \lambda$ iff $V^{\mathbb{Q}} \Vdash \text{cf } \alpha = \lambda$. Then \mathbb{Q} does not collapse cardinals.

Proof. Suppose κ is regular in V . If $V^{\mathbb{Q}} \Vdash |\kappa| = \lambda < \kappa$, then $V^{\mathbb{Q}} \Vdash \text{cf } \kappa \leq \lambda < \kappa$, a contradiction. Suppose κ is a limit, i.e., $\kappa = \sup\{\kappa_\alpha : \alpha < \text{cf } \kappa\}$ an increasing sequence, where each κ_α is regular. Since no κ_α is collapsed, κ is the sup of an increasing sequence of cardinals, so it must be a cardinal. \square

¹⁹ v is a class, not a set

So we need to show that cofinalities are preserved by \mathbb{P}_v .

First, note that if $V \models \kappa = \sup\{\kappa_\alpha : \alpha < \mu\}$ an increasing sequence cofinal in κ , $\mu = (\text{cf } \kappa)^V$, and $V^{\mathbb{P}_v} \Vdash \text{cf } \kappa = \lambda < \mu$, then $V^{\mathbb{P}_v} \Vdash \text{cf } \mu = \lambda$: let $\{\lambda_\beta : \beta < \lambda\}$ be a $V^{\mathbb{P}_v}$ -sequence cofinal in κ . In $V^{\mathbb{P}_v}$ define $\dot{f} : \lambda \rightarrow \mu$ by $\dot{f}(\beta) = \inf\{\kappa_\alpha : \kappa_\alpha > \lambda_\beta\}$. Then \dot{f} is non-decreasing and cofinal in μ .

Since $V \models \mu$ regular, it suffices to show that the cofinalities of regular cardinals are preserved.

Suppose $\dot{f} : \lambda \rightarrow \kappa$ is increasing, $\lambda < \kappa$, $V \models \kappa$ is regular. $\mathbb{P}_v^{\lambda^+}$ is λ^+ -closed, so adds no new functions with domain λ . Hence $\dot{f} \in V^{\mathbb{P}_v, \lambda^+}$. $\mathbb{P}_{v, \lambda^+}$ has λ^+ -cc, so for every $\alpha < \lambda$ there is a maximal antichain A_α of size $\leq \lambda$ so $A_\alpha \parallel \dot{f}(\alpha)$. Define $F \in V$, $F : \lambda \rightarrow \kappa$, $F(\alpha) = \{\beta : \exists p \in A_\alpha \ p \Vdash \dot{f}(\alpha) = \beta\}$. Each $|F(\alpha)| \leq \lambda$, and $\forall \alpha < \lambda \ V^{\mathbb{P}_v, \lambda} \Vdash \dot{f}(\alpha) \in F(\alpha)$. But $V \models \bigcup_{\alpha < \lambda} F(\alpha)$ is not cofinal in κ , so range \dot{f} is not cofinal in κ , and κ remains regular.

Finally, we show that $V^{\mathbb{P}_v} \Vdash 2^\kappa = v(\kappa)$ for every $\kappa \in \text{REG}$: Since $\text{Fn}(v(\kappa), 2, \kappa) \geq \mathbb{P}_v$, ${}^{20}V^{\mathbb{P}_v} \Vdash 2^\kappa \geq v(\kappa)$. Since GCH holds, there are at most $v(\kappa)$ many \mathbb{P}_{v, κ^+} -names for subsets of κ , so $V^{\mathbb{P}_{v, \kappa^+}} \Vdash 2^\kappa = v(\kappa)$. Since $V_v^{\kappa^+}$ is κ^+ -closed, it adds no new subsets of κ . Hence $V^{\mathbb{P}_v} = V^{\mathbb{P}_v^{\kappa^+} * \mathbb{P}_{v, \kappa^+}} \Vdash 2^\kappa = v(\kappa)$.

□

Note that nothing like Easton's theorem is true for singular cardinals: there are serious constraints on 2^κ for κ singular. Early on, Silver showed that if κ is singular of uncountable cofinality then 2^κ depends deeply on 2^λ for $\lambda < \kappa$. E.g., if $\kappa = \aleph_\gamma$ singular, $\text{cf } \gamma > \omega$ and for some fixed n , $2^{\aleph_\alpha} = \aleph_{\alpha+n}$ for all regular $\aleph_\alpha < \kappa$, then $2^\kappa = \aleph_{\gamma+n}$. Uncountable cofinality is essential here: early on, Magidor showed that GCH could hold below \aleph_ω but $2^{\aleph_\omega} > \aleph_\omega^+$. Magidor's techniques do not extend to making 2^{\aleph_ω} as large as you want: Shelah showed that $(\aleph_\omega)^\omega \leq \sup\{\aleph_{\omega_4}, \mathfrak{c}\}$. Another theorem of Shelah's is that if κ is the first cardinal with $\kappa^{\text{cf } \kappa} > \sup\{\kappa^+, 2^{\text{cf } \kappa}\}$ then $\text{cf } \kappa = \omega$.²¹

²⁰strictly speaking this does not make sense, since \mathbb{P}_{\geq} is a class, but it's easy to adapt the definition of \geq to make sense of it

²¹this is an extremely brief summary of the singular cardinals problem, and there are many more profound and beautiful results in this area

16 Kurepa's hypothesis

The point of this section is to give you a sense of how collapsing a large cardinal can give you a result on small cardinals.

Kurepa's hypothesis says that there is no Kurepa tree. A Kurepa tree is, in some sense, an anti-Suslin tree. Define an ω_1 -tree to be a tree of height ω_1 in which every level is countable. A Suslin tree is an ω_1 -tree with no uncountable branch and no uncountable antichain. A Kurepa tree is an ω_1 -tree with at least ω_2 uncountable branches (hence a splitting tree, hence with at least ω_2 uncountable antichains).

It is known that in L , the constructible universe, there are Kurepa trees. (That is what the combinatorial principle \square is about.) If there is an inaccessible cardinal in a model V , then you can define a partial order adding a Kurepa tree to V . This is an exercise in Kunen and I will leave it as such. Instead we will prove:

Theorem 45. *Let κ be inaccessible in V . Then there is a partial order \mathbb{P} with $V^{\mathbb{P}} \Vdash (\kappa = \omega_2$ and there are no Kurepa trees).*

Proof. \mathbb{P} is defined as follows: $p \in \mathbb{P}$ iff p is a countable function, $\text{dom } p \subset (\kappa \setminus \{0\}) \times \omega_1$, and each $p(\beta, \alpha) < \beta$. $p \leq q$ iff $p \supset q$.

If $\{p_n : n < \omega\}$ is a countable descending sequence of elements of \mathbb{P} , then $\bigcup_{n < \omega} p_n$ is a lower bound, so \mathbb{P} is countably closed and does not collapse ω_1 .

Since κ is inaccessible, if $\lambda < \kappa$ then $(\lambda^\omega)^+ < \kappa$. So, by a Δ -system argument, \mathbb{P} has κ -cc, hence does not collapse cardinals $\geq \kappa$.

Define \dot{G} to be the \mathbb{P} -generic filter, $\dot{f} = \bigcup \dot{G}$ and, for $\beta < \kappa$, $\dot{f}_\beta : \omega_1 \rightarrow \beta$ by: $\dot{f}_\beta(\alpha) = \dot{f}(\beta, \alpha)$. By a genericity argument, each \dot{f}_β is onto. Hence if $\beta < \kappa$ then $V^{\mathbb{P}} \Vdash |\beta| \leq \omega_1$, so $V^{\mathbb{P}} \Vdash \kappa = \omega_2$.

Why then does \mathbb{P} destroy Kurepa trees?

Suppose \dot{T} is an ω_1 tree. We want to show that we can factor \mathbb{P} in such a way that we can assume \dot{T} is in an intermediate model, and that we are forcing with \mathbb{P} over this model.

Without loss of generality, $\dot{T} = (\omega_1, \leq_{\dot{T}})$, so we know \dot{T} if we know $\leq_{\dot{T}}$. For each $\alpha < \beta < \omega_1$, let $A_{\alpha, \beta}$ be a maximal antichain deciding $\leq_{\dot{T}} \upharpoonright_{\{\alpha, \beta\}}$, i.e., if $p \in A_{\alpha, \beta}$ then either $p \Vdash \alpha \leq_{\dot{T}} \beta$ or $p \Vdash \alpha \geq_{\dot{T}} \beta$ or $p \Vdash \alpha, \beta$ are not \dot{T} -compatible.

By κ -cc, each $|A_{\alpha, \beta}| < \kappa$. There are ω_1 many $A_{\alpha, \beta}$. And the $A_{\alpha, \beta}$'s decide \dot{T} . So there is $\lambda < \kappa$ with $\dot{T} \cong$ to a \mathbb{P}_λ -name, where $\mathbb{P}_\lambda = \{p \in \mathbb{P} : \text{dom } p \subset \lambda \times \omega_1\}$. Define $\mathbb{P}^\lambda = \{p \in \mathbb{P} : \text{dom } p \cap \lambda = \emptyset\}$. $\mathbb{P} = \mathbb{P}_\lambda \times \mathbb{P}^\lambda$. \mathbb{P}_λ also preserves ω_1 . It collapses something to ω_2 — for our purposes, we don't care what, as long as it leaves κ strongly inaccessible. Does it? \mathbb{P}_λ has at most 2^λ -cc, so there is $\eta \leq (2^\lambda)^V$ with $V^{\mathbb{P}_\lambda} \Vdash \omega_2 = \eta$. κ remains a regular strong limit cardinal.

We show $\mathbb{P}^\lambda \cong \mathbb{P}$: Let $E = \{(\beta, \gamma) : 0 \leq \gamma \leq \beta \leq \lambda\}$. Let $\varphi : E \rightarrow \lambda, \varphi$ 1-1 onto. For $p \in \mathbb{P}^\lambda$ we define $p' \in \mathbb{P}$ as follows: $p(\beta, \alpha) = p'(\beta, \alpha)$ for all $\beta > \lambda$. If $\beta \leq \lambda, p'(\beta, \alpha) = \gamma$ iff $p(\lambda, \alpha) = \varphi(\beta, \gamma)$. The map $p \mapsto p'$ is an order isomorphism under the order $\leq = \supset$.

So to show there are no Kurepa trees in the generic extension, it suffices to consider T an ω_1 -tree in V . And, since $|(2^{\omega_1})^V|^{V^{\mathbb{P}}} = \omega_1^{22}$, it suffices to show that \mathbb{P} adds no new branches to T .

We write $T(\delta) =$ the δ^{th} level of T .

Suppose \dot{b} is a new uncountable branch of T . Since it is a new branch, if $p \in \mathbb{P}$ then $p \Vdash "T(\alpha) \cap \dot{b}"$

²²i.e., the old 2^{ω_1} is collapsed to ω_1

for at most countable many α . I.e., $\forall p \exists \delta_p < \omega_1 \forall \gamma \geq \delta_p p \Vdash "T(\gamma) \cap \dot{b}"$. But then if $\gamma \geq \delta_p$ there are $\beta_0 \neq \beta_1 \in T(\gamma)$, $q_{p,0,\gamma}, q_{p,1,\gamma} \leq p$ with $q_{p,0,\gamma} \Vdash \beta_0 \in \dot{b}$ and $q_{p,1,\gamma} \Vdash \beta_1 \in \dot{b}$.

So fix p . For each $k < \omega$ and $\sigma : k \rightarrow 2$ we define p_σ as follows: $p_\emptyset = p$. If we know p_σ , let δ_σ be as above, $p_{\sigma \frown 0} = q_{p_\sigma,0,\delta_\sigma}, p_{\sigma \frown 1} = q_{p_\sigma,1,\delta_\sigma}$.

Finally, for each $f : \omega \rightarrow 2$, let p_f be a lower bound for $\{p_\sigma : \sigma \subset f\}$. Let $\delta = \sup \{\delta_{p_\sigma} : \sigma \in \bigcup^{<\omega} \}$.

$\dot{b} \cap T(\delta) \neq \emptyset$. Each $p_f \Vdash$ if $\gamma = \{\dot{b} \cap T(\delta)\}$ then $\gamma \geq \beta_{p_\sigma}$ for all $\sigma \subset f$. Let $q_f \leq p_f$ so that $\exists \beta_f q_f \Vdash T(\delta) \dot{b} = \beta_f$. If $f \neq g$ then $\beta_f \neq \beta_g$.

The set $\{\beta_f : f \in 2^\omega\} \in V$, and $V^{\mathbb{P}}$ preserves the order on T . So $|T(\delta)| \geq 2^\omega$, a contradiction.

□

17 More on Cohen forcing

In this section we present two results on Cohen forcing.

Theorem 46. (Shelah) *Let \mathbb{C} be a countable separative partial order. Then in $V^{\mathbb{C}}$ there is a Suslin tree.*

This is a stronger version of $V^{\mathbb{C}} \Vdash \exists \text{ccc } \dot{\mathbb{P}}$ with \mathbb{P}^2 not ccc. The proof I will give is due to Todorćević.

The point of this theorem is that there even if there is no Suslin tree in V , a single Cohen real is enough to add one. So adding a single Cohen real is a little like \diamond , i.e., combinatorially a single Cohen real is quite strong.

Before giving the proof, note that if T is a Suslin tree in V , then $V^{\mathbb{C}} \Vdash T$ is Suslin. This is because of the following fact:

Fact 26. *Let \dot{E} be an uncountable set in $V^{\mathbb{C}} \cap \mathcal{P}(V)$. Then $V^{\mathbb{C}} \Vdash \exists H \in V, H$ uncountable, $H \subset \dot{E}$.*

Proof. Let $q \Vdash \dot{E} = \{\dot{e}_\alpha : \alpha < \omega_1\} \subset V$ where the \dot{e}_α 's are distinct. We show that there is $p \leq q$ and $H \in V$ uncountable with $p \Vdash H \subset \dot{E}$.

For each α there is $p_\alpha \leq q, p_\alpha \parallel \dot{e}_\alpha$, say $p_\alpha \Vdash \dot{e}_\alpha = x_\alpha$ for some $x_\alpha \in V$. Since \mathbb{C} is countable, there is $Y \in [\omega_1]^{\omega_1}$ and p with $p = p_\alpha$ for all $\alpha \in Y$. Hence $H = \{x_\alpha : \alpha \in Y\}$ is uncountable, is in V , and $p \Vdash H \subset \dot{E}$. \square

By the fact, if \dot{T} is a tree indexed by a set in V , an uncountable chain or antichain \dot{A} in $V^{\mathbb{C}}$ would have an uncountable subset $\dot{B} = \{\dot{a}_i : i \in I\}$ where $I \in V$. \dot{B} would also be a chain or antichain, but the fact that $I \in V$ will enable us to use genericity to show this doesn't happen.

Let's prove theorem 46.

Proof. First we show that there is a family of functions $\{f_\alpha : \omega \leq \alpha < \omega_1\}$ where

1. each $f_\alpha : \alpha \rightarrow \omega$
2. each f_α is 1-1
3. each $\omega \setminus \text{range } f_\alpha$ is infinite.
4. if $\alpha < \beta$ then $\{\gamma < \alpha : f_\alpha(\gamma) \neq f_\beta(\gamma)\}$ is finite. (We say that $f_\alpha \subset^* f_\beta$.)

We begin by constructing $\{f_\alpha : \omega \leq \alpha < \omega \cdot \omega\}$ so $\alpha < \beta \Rightarrow f_\alpha \subset f_\beta$: let $a_n \subset a_{n+1} \subset \omega$ for all finite n , with each $a_{n+1} \setminus a_n$ infinite; construct $f_{\omega \cdot n}$ 1-1 with domain a_n so each $f_{\omega \cdot n} \subset f_{\omega \cdot (n+1)}$, and for $k < \omega$ let $f_{\omega \cdot n + k} = f_{\omega \cdot (n+1)} \upharpoonright_{\omega \cdot n + k}$.

Now suppose we have $\{f_\beta : \omega \cdot \omega \leq \beta < \alpha\}$ where each $|\omega \setminus \text{dom } f_\beta| = \omega$. If $\alpha = \beta + 1$ for some β , then pick $n \in \omega \setminus \text{dom } f_\beta$, and set $f_\alpha = \{(\beta, n)\} \cup f_\beta$.

Otherwise there is $\{\beta_n : n < \omega\}$ an increasing sequence with $\sup \alpha$. We construct g_n as follows: $g_0 = f_{\beta_0}$. At stage $n + 1$, we have, for all $m \leq n, g_m = *f_{\beta_m}$, $\text{dom } g_m = \beta_m, g_{m-1} \subset g_m, k_m \in \omega \setminus \text{range } g_m$. Note that $S = \omega \setminus (\text{range } g_n \cup \text{range } f_{\beta_{n+1}})$ is infinite. Let $k_{n+1} \in S \setminus \{k_m : m \leq n\}$. Let $a = \{\rho \in [\beta_n, \beta_{n+1}) : \exists \gamma < \beta_n, g_n(\gamma) = f_\alpha(\rho)\}$, i.e., a is the set of ρ that might give us trouble trying to extend g_n to a 1-1 function $=* f_{\beta_{n+1}}$. Since $f_{\beta_{n+1}}$ is 1-1 and $f_{\beta_{n+1}} \supset^* g_n$, a is finite.

Let $e \in [S \setminus \{k_m : m \leq n+1\}]^{|a|}$ and define g_{n+1} as follows: $g_{n+1}|_{\beta_n} = g_n$; $g_{n+1}|_{[\beta_n, \beta_{n+1}) \setminus a} = f_{\beta_{n+1}}|_{[\beta_n, \beta_{n+1}) \setminus a}$, and $g_{n+1}|_a$ is 1-1 onto e . Finally, let $f_\alpha = \bigcup_{n < \omega} g_n$. By construction f_α is a 1-1 function from α to ω with $f_\alpha \supset^* f_\beta$ for all $\beta < \alpha$. Since $\{k_n : n < \omega\} \cap \text{range } f_\alpha = \emptyset$, condition (3) is met. [Exercise to understand the construction a bit better: What happens when $\alpha = \beta + \omega$?]

Recall that $\{f_\alpha : \alpha < \omega_1\}$ generates an Aronszajn tree $T = \{f_\alpha|_\gamma : \gamma < \alpha < \omega_1\}$, under the partial order $s \leq t$ iff $s \subset t$.

We will turn this into a Suslin tree.

Let $\dot{x} : \omega \rightarrow 2$ be the generic Cohen real added by $\dot{\mathbb{C}}$. For $t \in T$ define $\dot{g}_t = t \circ \dot{x}$. We show that $\dot{T}^* = \{\dot{g}_t : t \in T\}$ is a Suslin tree by showing that no uncountable subset of \dot{T}^* is a chain or an antichain.

By fact 26, it suffices to show that if $E \in V \cap [T]^\omega$ then $\dot{E}^* = \{\dot{g}_t : t \in E\}$ is neither a chain nor an antichain.

So let $p \in Fn(\omega, 2, \omega)$, $\text{dom } p = a$. There is $F \in [E]^\omega$ with $\{t \upharpoonright a : t \in F\}$ a Δ -system with root b . By another reduction we may assume there is $\sigma : b \rightarrow a$ with $t|_b = \sigma$ for all $b \in F$.

To show \dot{E}^* is not a chain: Pick $s, t \in F$ with $\text{dom } s \subset \text{dom } t$, $s \not\subset t$ (we can do this because F is not a chain). Then there is $\alpha \in \text{dom } s$ with $s(\alpha) \neq t(\alpha)$. Since $\alpha \notin b$ at least one of $s(\alpha), t(\alpha) \notin a$. Extend p to $q \leq p$ so $\text{dom } q = a \cup \{s(\alpha), t(\alpha)\}$ and $q(s(\alpha)) = 1 - q(t(\alpha))$. Then $q \Vdash \dot{g}_s(\alpha) \neq \dot{g}_t(\alpha)$, so $q \Vdash \dot{g}_s \perp \dot{g}_t$.

To show \dot{E}^* is not an antichain: Pick $s, t \in F$ with $\text{dom } s \subset \text{dom } t$. If $s \subset t$ we're done, so suppose F is an uncountable antichain in T . Then $s \not\subset t$. Let $c = \{\alpha \in \text{dom } s : s(\alpha) \neq t(\alpha)\}$. c is finite, and $c \cap b = \emptyset$. If $\alpha \in c$ and $s(\alpha) \in a$ then, by the Δ -system, $t(\alpha) \notin a$. Similarly, if $t(\alpha) \in a$ then $s(\alpha) \notin a$. So at least one of $s(\alpha), t(\alpha) \notin a$. Extend p to $q \leq p$ with $\text{dom } q = a \cup c$ and for all $\alpha \in c$, $q(s(\alpha)) = q(t(\alpha))$. Then $q \Vdash \dot{g}_s \subset \dot{g}_t$. \square

Theorem 47. (Harrington) Assume CH. Let $\mathbb{P} = \mathbb{C}_\kappa$ where $\kappa \geq \omega_1$. Then in $V^\mathbb{P}$ there is a MAD family of size ω_1 .

I.e., \mathfrak{c} can be anything and $\mathfrak{a} = \omega_1$ can hold.

Proof. Let $\mathbb{C} = Fn(\omega, 2, \omega)$. By CH, there is a sequence $\{\dot{x}_\alpha : \alpha < \omega_1\}$ listing $[\omega]^\omega \cap V^\mathbb{C}$. Every new subset of ω is added by a single Cohen real, so if $\dot{x} \in V^\mathbb{P} \cap [\omega]^\omega$ then there is $\alpha < \omega_1$ with $\dot{x} \cong \dot{x}_\alpha$.

Suppose we can construct, in V , an almost disjoint family $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ so that for all \dot{x}_α there is β with $V^\mathbb{C} \Vdash a_\beta \cap \dot{x}_\alpha$ is infinite. Consider $\dot{x} \in [\omega]^\omega$. There is E countable with $\dot{x} \in \mathbb{Q} = Fn(E, 2, \omega) \cong \mathbb{C}$. If G is a \mathbb{P} -generic filter then $G \cap \mathbb{Q}$ is a \mathbb{Q} -generic filter, hence is isomorphic to some \mathbb{C} -generic filter H , and, for some α , $\dot{x} \cong \dot{x}_\alpha$ by the same isomorphism. So $V^\mathbb{P} \Vdash \dot{x} \cap a_\beta$ infinite iff $V^\mathbb{Q} \Vdash \dot{x} \cap a_\beta$ infinite iff $V^\mathbb{C} \Vdash \dot{x}_\alpha \cap a_\beta$ infinite. I.e., there will be β so $\dot{x} \cap a_\beta$ is infinite. Since \dot{x} was arbitrary, \mathcal{A} is MAD.

Suppose we have $\{a_\beta : \beta < \alpha\}$. Consider \dot{x}_α . Let $\mathbb{C} = \{p_n : n < \omega\}$. Rewrite $\{a_\beta : \beta < \alpha\} = \{c_n : n < \omega\}$. We construct a_α by induction on ω .

At stage n we have disjoint sets $\{k_m^\alpha : m < n\}, \{j_m^\alpha : m < n\}$ with each $k_m^\alpha, j_m^\alpha \in \omega \setminus \bigcup_{i \leq m} c_i$. Let $d_m = \{k_i^\alpha : i < m\} \cup \{j_i^\alpha : i < m\} \cup \bigcup_{i \leq m} c_i$. Also at stage n , for each $m < n$ we have some $q_m \leq p_m$ with $q_m \Vdash$ either $(k_m^\alpha, j_m^\alpha \in \dot{x}_\alpha \setminus d_m)$ or $(\dot{x}_\alpha \cap d_m$ is infinite and $k_m^\alpha, j_m^\alpha \notin d_m)$. At stage n , if $p_n \Vdash \dot{x}_\alpha \cap d_n$ is infinite, pick k_n^α, j_n^α distinct elements of $\omega \setminus d_n$, and set $q_n = p_n$. Otherwise, since $p_n \Vdash (\dot{x}_\alpha$ is infinite and $\dot{x}_\alpha \cap d_n$ is finite), there is $q_n \leq p_n$ and distinct k_n^α, j_n^α with $q_n \Vdash k_n^\alpha, j_n^\alpha \in \dot{x}_\alpha \setminus d_n$. Let $a_\alpha = \{k_m^\alpha : m < \omega\}$ and note that $\{j_m^\alpha : m < \omega\} \subset \omega \setminus a_\alpha$. Also, note that $a_\alpha \cap c_n \subset \{k_m^\alpha : m < n\}$ hence is finite.

It remains to show that $a_\beta \cap \dot{x}_\alpha$ is infinite for some $\beta \leq \alpha$: Suppose $p \Vdash \forall \beta < \alpha \dot{x}_\alpha \cap a_\beta$ is finite. We show that $p \Vdash \dot{x}_\alpha \cap a_\alpha$ is infinite. By contradiction, assume this fails. By extending p we may assume there is $n < \omega$ $p \Vdash |a_\alpha \cap \dot{x}_\alpha| = n < \omega$. Construct a chain of conditions $\{q_m : m \leq n\}$ as follows: $q_0 \leq p$ with $q_0 \Vdash \exists i_0 k_{i_0}^\alpha \in \dot{x}_\alpha$; given q_m there is $q_{m+1} \leq q_m$ with $q_{m+1} \Vdash \exists k_{i_{m+1}}^\alpha \in \dot{x}_\alpha$ where $k_{i_{m+1}}^\alpha > k_{i_m}^\alpha$. Then $q_n \Vdash |a_\alpha \cap \dot{x}_\alpha| \geq n+1$, which contradicts $q_{n+1} \leq p$. Hence no $p \Vdash \forall \beta \leq \alpha a_\beta \cap \dot{x}_\alpha$ is finite, so $V^C \Vdash \exists \beta \leq \alpha a_\beta \cap \dot{x}_\alpha$ is infinite. \square

18 PFA

In this section we will content ourselves with defining proper forcing, giving some examples, stating (but not proving) a general iteration theorem, and stating the proper forcing axiom. We will not give the proof of its consistency, but will mention that the proof involves large cardinals, to wit, supercompact cardinals (which are stronger than measurable cardinals). It is known that the proof of the consistency of PFA necessarily involves at least a weakly compact cardinal²³. Thus, there is no proof in ZFC of the consistency of the proper forcing axiom.

Recall the statement of the proper forcing axiom:

Definition 33. *PFA is the statement: if \mathbb{P} is proper and \mathcal{D} is a family of at most ω_1 dense subsets of \mathbb{P} , then there is a \mathcal{D} -generic filter in \mathbb{P} .*

Note the similarity to MA — we have simply enlarged the set of forcings we can use. The astute reader will note the ω_1 instead of $< \mathfrak{c}$, but this is moot since Velickovic and Todorcevic proved that $\text{PFA} \Rightarrow \mathfrak{c} = \omega_2$.

The astute reader may also worry: when we invoke PFA are we necessarily invoking a large cardinal in our hypothesis? The answer is: no. When the actual forcings we are looking at do not collapse cardinals, then no large cardinal hypothesis is needed, since we could simply have iterated just these forcings to get the model we are interested in.

We defined proper forcing formally in section 4. Now let's unpack the definition.

18.1 Clubs and stationary sets

Definition 34. *Let X be an uncountable set, and let $\mathcal{A} \subset [X]^\omega$.*

- (a) *We say that \mathcal{A} is unbounded iff for all $B \in [X]^\omega$ there is $A \in \mathcal{A}$ $B \subset A$.*
- (b) *We say that \mathcal{A} is closed iff $\forall \{B_i : i < \omega\} \subset \mathcal{A}$ where each $B_i \subset B_{i+1}$ $\bigcup_{i < \omega} B_i \in \mathcal{A}$.*
- (c) *We say that \mathcal{A} is stationary iff $\forall \mathcal{C}$ closed unbounded (= club) $\mathcal{A} \cap \mathcal{C} \neq \emptyset$.*

As usual, club = closed unbounded.

Exercises 1. Using the identity $\alpha = \{\beta : \beta < \alpha\}$ for each $\alpha \in ON$, prove that A contains a club in $[\omega_1]^\omega$ iff $A \cap \omega$ contains a club in ω_1 . Then prove that A is stationary in $[\omega_1]^\omega$ iff $A \cap \omega$ is stationary in ω_1 .

2. Suppose Y is an uncountable subset of X . Prove that if A contains a club (respectively is stationary) on $[X]^\omega$, then $A|_Y = \{a \cap Y : a \in A\}$ contains a club (respectively is stationary) on $[Y]^\omega$. Conversely, if A contains a club (respectively is stationary) of $[Y]^\omega$ then there is $B, B|_Y = A$, B contains a club (respectively is stationary) of $[X]^\omega$.

Definition 35. *\mathbb{P} is proper iff for all uncountable X and every $\mathcal{A} \subset [X]^\omega$ with $\mathcal{A} \in V$, $V \models \mathcal{A}$ is stationary iff $V^{\mathbb{P}} \models \mathcal{A}$ is stationary.*

We say that \mathbb{P} preserves stationary sets.

Fact 27. *If \mathbb{P} is proper, then it preserves ω_1 .*

²³ κ is weakly compact iff $\kappa \rightarrow (\kappa)_2^2$, i.e., iff a form of Ramsey's theorem holds for κ

Proof. Suppose $V^{\mathbb{P}} \Vdash |(\omega_1)^V| = \omega$. Working in $V^{\mathbb{P}}$, there is an increasing $\dot{f} : \omega \rightarrow (\omega_1)^V$ with range \dot{f} cofinal in $(\omega_1)^V$. Consider $\{\dot{f}(n) + 1 : n < \omega\} \in V^{\mathbb{P}}$. This is club in $(\omega_1)^V$, but it misses the ground model stationary set $S = \{\alpha < (\omega_1)^V : \alpha \text{ a limit}\}$. So $V^{\mathbb{P}} \Vdash S$ is not stationary, hence \mathbb{P} is not proper. \square

Note that a proper forcing need not preserve other cardinals. For example, $F_n(\omega_1, \omega_2, \omega_1)$ collapses ω_2 , but it is countably closed and hence, as we shall shortly learn, proper.

Theorem 48. (*Baumgartner, Harrington, Kleinberg 1976*) *There is a forcing which preserves ω_1 which is not proper.*²⁴

Proof. Let S be any stationary set. We define $\mathbb{P} = \{a \subset S : a \text{ is closed}\}$. The order is: $a \leq b$ iff $a \supset b$ and $\forall \alpha \in a \setminus b \forall \beta \in b \alpha > \beta$ (i.e., a is an end-extension of b).

Clearly, if \dot{G} is \mathbb{P} -generic, $V^{\mathbb{P}} \Vdash \bigcup \dot{G} \subset S$ and $\bigcup \dot{G}$ is a club (because if it weren't, some $a \in \dot{G}$ would know that $\bigcup \dot{G} \cap \sup a$ is not closed). So $V^{\mathbb{P}} \Vdash S$ contains a club.

Lemma 4. \mathbb{P} adds no new functions from ω to V (hence does not collapse ω_1).

Proof. Suppose $b \Vdash \dot{f} : \omega \rightarrow V$. For each $a \leq b$ and each $\gamma < \omega_1$ we fix a set $A_{a,\gamma}$ so that

1. $\forall n < \omega$ there is $c \in A_{a,\gamma}$ with $c \parallel \dot{f}(n)$
2. if $c \in A_{a,\gamma}$ then $c \leq a$ and $\sup c > \gamma$
3. $A_{a,\gamma}$ is minimal with respect to properties (1) and (2).

Note that, by (3), each $A_{a,\gamma}$ is countable.

We also define, for each countable $A \subset \mathbb{P}$, $\text{ht } A = \sup \bigcup A$

We construct sets A_α as follows: $A_0 = A_{b, \sup b}$. If β is a limit, $A_\beta = \bigcup_{\gamma < \beta} A_\gamma$. $A_{\beta+1} = \bigcup_{a \in A_\beta} A_{a, \text{ht } A_\beta}$. Setting $\gamma_\beta = \text{ht } A_\beta$, note that $C = \{\gamma_\beta : \beta < \omega_1\}$ is club. So there is some $\gamma \in S \cap C$.

We construct a descending chain $b \geq a_0 \geq a_1 \geq \dots a_n \geq a_{n+1} \dots$ where each $a_n \parallel \dot{f}(n)$ as follows: let $\{\alpha_n : n < \omega\}$ be an increasing sequence of successor ordinals converging to γ . Let $a_0 \in A_{a_0, \text{ht } A_{\alpha_0}}$, $a_0 \parallel \dot{f}(0)$. Given a_n , let $a_{n+1} \in A_{\alpha_{n+1}} \cap A_{a_n, \text{ht } A_{\alpha_n}}$, $a_{n+1} \parallel \dot{f}(n+1)$. Define $a = \bigcup_{n < \omega} a_n \cup \{\gamma\}$.

$a \in \mathbb{P}$: Since $\gamma \in S$, $a \subset A$. By end-extension, each proper initial segment of a is closed. Since $\gamma = \sup \{\sup c : c \text{ a proper initial segment of } a\}$, a is closed.

$a \Vdash \dot{f} \in V$: $\forall n < \omega$ $a \leq a_n \parallel \dot{f}(n)$. So $\forall n < \omega$ $a \parallel \dot{f}(n)$. \square

Hence \mathbb{P} preserves ω_1 . But note that if S is stationary co-stationary, then $V^{\mathbb{P}} \Vdash \omega_1 \setminus S$ is not stationary, so \mathbb{P} is not proper. \square

²⁴of course, since this result came about at least five years before the definition of proper forcing, this is not how they stated their theorem

Fact 28. (a) If \mathbb{P} is proper and $V^{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is proper, then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper.

(b) An iteration of proper forcings with countable support is proper.²⁵

The proof of (b) is highly technical, so we only prove (a).

First a useful lemma:

Lemma 5. $(V^{\mathbb{P}})^{\dot{\mathbb{Q}}} = V^{\mathbb{P} * \dot{\mathbb{Q}}}$.

Note that the lemma has nothing to do with proper forcing, but is simply true for any two-step iteration.

An alternate statement of this lemma takes the generic filter point of view: If K is $\mathbb{P} * \dot{\mathbb{Q}}$ -generic over M , then $M[K] = M[G][H]$ where G is \mathbb{P} -generic over M , and H is $\dot{\mathbb{Q}}/G$ -generic over $M[G]$.

Proof. We work by induction on the level of construction of \dot{x} . Suppose $(V^{\mathbb{P}})^{\dot{\mathbb{Q}}} \cap \dot{x} = V^{\mathbb{P} * \dot{\mathbb{Q}}} \cap \dot{x}$. $\dot{x} \in V^{\mathbb{P} * \dot{\mathbb{Q}}}$ iff \dot{x} is a set of pairs $((p, \dot{q}), \dot{y})$ where $(p, \dot{q}) \Vdash \dot{y} \in \dot{x}$. But $(p, \dot{q}) \Vdash \dot{y} \in \dot{x}$ iff $p \Vdash (\dot{q} \Vdash \dot{y} \in \dot{x})$. So letting $\dot{x}' = \{(p, (\dot{q}, \dot{y})) : p \Vdash (\dot{q} \Vdash \dot{y} \in \dot{x}')\}$, by extensionality $\dot{x} = \dot{x}'$; $\dot{x}' \in (V^{\mathbb{P}})^{\dot{\mathbb{Q}}}$. □

Now to prove fact 28:

Proof. Let $S \in [X]^\omega$ where X is uncountable, and suppose $V \models S$ is stationary. Then $V^{\mathbb{P}} \Vdash S$ is stationary. Since $V^{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is proper, $(V^{\mathbb{P}})^{\dot{\mathbb{Q}}} \Vdash S$ is stationary. □

While definition 35 is the simplest definition of “proper” to state, it is not the easiest one to apply, and for that definition we have to look more closely at pre-dense sets in elementary submodels.

18.2 Elementary submodels

Definition 36. If M, N are models of the same language, then $M \prec N$ (read: M is an elementary submodel of N) iff $M \subset N$ and for any sentence φ with parameters in M , $M \models \varphi$ iff $N \models \varphi$.

Here’s a negative example: Consider the language of fields (with operation symbols $+$, \cdot and constant symbols $0, 1$). Let \mathbb{Q} = the rationals under the usual interpretation of these symbols, and \mathbb{R} = the reals under the usual interpretation of these symbols. Then $\mathbb{Q} \not\prec \mathbb{R}$ because the sentence $\exists x x^2 = 2$ is true in \mathbb{R} but not in \mathbb{Q} .

Here’s a positive example: Again, consider the language of fields, and note that every sentence in this language is equivalent to some kind of quantified polynomial equation in several variables. For example $\exists x \forall y \exists z xyz = x^3 + y^2 + z$ is such a sentence (and a true one at that: let $x = 0, z = -y^2$). Let \mathbb{Q}^* be the algebraic closure of \mathbb{Q} , and let \mathbb{C} = the complex numbers under the usual interpretation.. Then, by definition of algebraic closure, $\mathbb{Q}^* \prec \mathbb{C}$.

Definition 37. Let θ be a cardinal. $H(\theta) = \{x : |TC x| < \theta\}$.

²⁵i.e., each $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha$ is proper

I.e. $H(\omega) = \{x : |\text{TC } x| \text{ is finite}\} = V_\omega$. This is a special case of: if κ is strongly inaccessible, then $H(\kappa) = V_\kappa$. In general, $H(\theta) \neq V_\theta$. For example, $\mathcal{P}(\omega) \in V_{\omega_1} \setminus H(\omega_1)$.

We are interested in countable elementary submodels of $H(\theta)$'s. It is important to realize that these models have major gaps — they are very far from transitive.

Here's an example of such a gap. If M is a countable elementary submodel of some $H(\theta)$ where $\theta > \omega_1$, then $\omega_1 \in M$ (because ω_1 is definable), but clearly $\omega_1 \not\subset M$. Let $\delta_M = \omega_1 \cap M \subset M$. Then by definition $\delta_M = \sup \omega_1 \cap M$, $\delta_M \subset M$. But $\delta_M \notin M$, since if $\delta_M \in M$ then by elementarity $M \models \delta_M$ is countable, so $\delta_M \in M \cap \omega_1 = \delta_M$, a contradiction. In fact, since $M \models \omega_1 = \sup\{\alpha : \alpha \text{ a countable ordinal}\}$, $[\delta_M, \omega_1) \cap M = \emptyset$.

Let's relate section 18.2 to section 18.1: By basic facts of model theory, for all cardinals $\theta > \omega$ $\{M : M \text{ countable}, M \prec H(\theta)\}$ is club in $[H(\theta)]^\omega$, and $\{M \cap \theta : M \text{ countable}, M \prec H(\theta)\}$ is club in $[\theta]^\omega$.

Exercise Prove the preceding statement using the following facts from model theory: If $x \in H(\theta)$ and $M \prec H(\theta)$ then there is $N \prec H(\theta)$ with $x \in N$, $M \subset N$ and $|M| = |N|^{26}$; if $x \in [H(\theta)]^{\leq |M|}$ and $M \prec H(\theta)$ then there is $N \prec H(\theta)$ with $M \cup x \subset N$ and $|M| = |N|$; if $\{M_n : n < \omega\}$ is an increasing sequence of elementary submodels of some $H(\theta)$ then $\bigcup_{n < \omega} M_n \prec H(\theta)$.

Which H_θ 's are we interested in? That depends on the context.

Definition 38. Let $\mathbb{P} = (P, \leq)$ be a partial order. We say that θ is sufficient for \mathbb{P} iff $\mathcal{P}(P) \in H(\theta)$.

Note that if $\mathcal{P}(P) \in H(\theta)$ then so is every subset of P and every subset of P^2 (hence \leq). This means that $H(\theta)$ knows everything that V knows about P . For example, every subset of P is an element of $H(\theta)$, and if $D \subset P$, then the sentence $\forall p \in P \exists q \in D q \leq p$ holds in $H(\theta)$ iff it holds in V . I.e. (and crucially) $H(\theta) \models D$ is dense in \mathbb{P} iff $V \models D$ is dense in \mathbb{P} . Finally, $H(\theta)$ can well-order the dense subsets of \mathbb{P} .

Now suppose $\mathbb{P} \in M \prec H(\theta)$ where θ is sufficient for \mathbb{P} . Suppose $D \in M$ is dense in \mathbb{P} . Then, even though D need not be a subset of M , $M \models \forall p \in \mathbb{P} \exists q \in D q \leq p$. Let's unpack this: for all $p \in \mathbb{P} \cap M \exists q \in \mathbb{P} \cap M \cap D q \leq p$.

If $\mathbb{P} = (P, \leq) \in M$ we write $\mathbb{P}^M = (P \cap M, \leq|_{P \cap M})$ — this is what M thinks \mathbb{P} is.²⁷

Some quick examples (where M denotes a countable elementary submodel of some $H(\theta)$):

- If $\theta > 2^\omega$ then θ is sufficient for $Fn(\omega, 2, \omega)$. Because it is definable $Fn(\omega, 2, \omega) \in M$ for all $M \prec H(\theta)$, and $Fn(\omega, 2, \omega) \subset M$ — this is essentially the only example where a separative partial order is a subset, since no uncountable partial order can be a subset of M .
- if $\theta > 2^{\omega_1}$, then it is sufficient for $Fn(\omega_1, 2, \omega)$ and again by definability $Fn(\omega_1, 2, \omega) \in M$ for all $M \prec H(\theta)$. $Fn(\omega_1, 2, \omega)^M = Fn(M \cap \omega_1, 2, \omega)$.
- Let $\mathbb{S} = \text{Sacks forcing}$. If $\theta > 2^c$ then θ is sufficient for \mathbb{S} and, again by definability, $\mathbb{S} \in M$ for all $M \prec H(\theta)$. But, since M is countable and $|\mathbb{S}| = c$, \mathbb{S}^M falls far short of the real \mathbb{S} .
- Let $S = (\omega_1, \leq)$ be a Suslin tree, and let $\mathbb{P} = (\omega_1, \geq)$, i.e., the forcing that adds an uncountable branch to S . Then any $\theta > 2^{\omega_1}$ is sufficient for S , but note that since S is not definable it is not automatically an element of every $M \prec H(\theta)$.²⁸

²⁶note: x could equal M

²⁷note that if $(a, b) \in M$ and $a, b \in H(\theta)$ then by elementarity $a \in M$ and $b \in M$ since $M \models (a, b)$ is an ordered pair — working out the details is a good exercise.

²⁸If $V \models \exists$ Suslin tree, then this is moot, although since “ \exists a Suslin tree” is consistent, there will be countable

18.3 Generic conditions

From now on, when we say that M is an elementary submodel, it is understood that it is an elementary submodel of some $H(\theta)$ where θ is sufficient for a relevant \mathbb{P} with $\mathbb{P} \in M$ (hence in $H(\theta)$).

Definition 39. Let M be a countable elementary submodel, $q \in \mathbb{P} \in M$. q is M -generic iff $\forall D$ pre-dense in \mathbb{P} D is pre-dense below q , i.e., $\forall r \leq q \exists p \in M \cap D$ p, r are compatible.

Note that q itself does not necessarily code a generic filter over M , i.e. it is not true that for every dense $D \in M \exists p \in D \cap M$ $p \leq q$. Also, q need not be an element of M . Finally, note that if $r \leq q$ and q is M -generic, so is r .

Definition 40. Let M be a countable elementary submodel, $\mathbb{P} \in M$. A filter G is M -generic iff $\forall D \in M$ with D dense, $G \cap D \cap M \neq \emptyset$.

Fact 29. Let M be a countable elementary submodel, $\mathbb{P} \in M$. q is M -generic iff $\forall r \leq q \exists G$ M -generic, $r \in G$.

Proof. Necessity is obvious. For sufficiency: Given $r \leq q$ we construct an M -generic filter G with $r \in G$, as follows: list $\{D \in M : D \text{ dense}\}$ as $\{D_n : n < \omega\}$. Fix $r \leq q$. Let $p_0 \in D_0 \cap M$ be compatible with $r, r_0 \leq r, p_0$. Given $r_n \leq r_{n-1}$ and $p_n \in D_n$ with $r_n \leq p_n$, let $p_{n+1} \in D_{n+1} \cap M$ be compatible with r_n , and let $r_{n+1} \leq p_{n+1}, r_n$. In this way we construct a descending sequence $r \geq r_0 \geq r_1 \dots$ and a sequence of p_n 's with $p_n \in D_n \cap M, p_n \geq r_n$. Let $G = \{q : \exists n q \geq r_n\}$. G is a filter meeting each $D_n \cap M$ and $r \in G$. \square

Note that $G \notin M$. Similarly we have

Fact 30. Let M be a countable elementary submodel, $\mathbb{P} \in M, \dot{G}$ names the \mathbb{P} -generic filter over V . q is M -generic iff $q \Vdash \dot{G} \cap M$ is $\mathbb{P} \cap M$ -generic over M .

Proof. $q \Vdash \dot{G} \cap M$ is $\mathbb{P} \cap M$ -generic over M iff $\forall r \leq q$ r is compatible with $\Sigma(M \cap D)$ for all $D \in M, D$ pre-dense below q iff q is M -generic. \square

Fact 31. Let M be a countable elementary submodel, $q \in \mathbb{P} \in M$. Then q is M -generic iff $\forall \alpha \in M$ if $q \Vdash \alpha$ an ordinal then $q \Vdash \exists \beta \in M$ $\dot{\alpha} = \beta$.

Proof. \Rightarrow : Suppose q is M -generic, $\alpha \in M, q \Vdash \dot{\alpha} \in ON$. Let $D = \{p : \exists \beta p \Vdash \dot{\alpha} = \beta\} \cup \{p : p \Vdash \dot{\alpha} \notin ON\}$. D is definable from $\dot{\alpha}$, and $\dot{\alpha} \in M$, so $D \in M$. D is dense, so $D \cap M$ is dense in $\mathbb{P} \cap M$. Let $p \in D \cap M, p \Vdash \dot{\alpha} \in ON$. $\exists \beta_p p \Vdash \dot{\alpha} = \beta_p$. Since β_p is definable from $\dot{\alpha}$, $p \in M, \beta_p \in M$. Let $r \leq q$ so that $\exists \beta \in ON$ $r \Vdash \dot{\alpha} = \beta$. Since q is M -generic, there is $p \leq q, p \in D \cap M, p, r$ compatible. Hence $p \Vdash \dot{\alpha} \in ON$, so $p \Vdash \dot{\alpha} = \beta_p$. By compatibility, $\beta = \beta_p$.

\Leftarrow : Since $H(\theta) \Vdash WO$, there is $h \in M$ $h : \mathbb{P} \rightarrow |\mathbb{P}| = \kappa, h$ 1-1 onto. Let $D \in M, D$ predense. Let $\dot{\alpha} = \inf(h[\dot{G}] \cap h[D])$. Then $\dot{\alpha} \in M$, so $q \Vdash \exists \beta \in M$ $\beta = \dot{\alpha}$. Hence $q \Vdash \dot{G} \cap D \cap M \neq \emptyset$. \square

Theorem 49. A partial order \mathbb{P} is proper iff $\forall \theta$ sufficient for \mathbb{P} $\{M : M \text{ countable}, M \prec H(\theta) \text{ and } \forall p \in M \text{ there is some } M\text{-generic condition } q \leq p\}$ contains a club.

models $M \models \exists$ a Suslin tree. But none of those models will be elementary submodels of any $H(\theta)$

Proof. \Rightarrow : Fix p and let $\{D_\alpha : \alpha < \kappa\}$ list all dense subsets of \mathbb{P} , each $D_\alpha = \{p_{\beta,\alpha} : \beta < \lambda_\alpha\}$ for some λ_α . By the contrapositive, we assume $\mathcal{N} = \{M \prec H(\theta) : M \text{ countable and there is no } M\text{-generic } q \leq p\}$ is stationary (so its complement contains no club).

Let $p \in G$ a \mathbb{P} -generic filter over V . There is $f \in H(\theta)^{V[G]}$, $f : \kappa \rightarrow G$ so each $p_{f(\alpha),\alpha} \in D_\alpha$ and each $p_{f(\alpha),\alpha} \leq p$.

Let $\mathcal{F} = \{M \text{ countable: } M \prec H(\theta)^{V[G]}, f \in M\}$. By model theory, \mathcal{F} is club in $[H(\theta)^{V[G]}]^\omega$, so $\mathcal{F}|_{H(\theta)}$ is club in $[H(\theta)]^\omega$. We show that $\mathcal{N} \cap \mathcal{F}|_{H(\theta)} = \emptyset$, hence $[H(\theta)^{V[G]}] \models \mathcal{N}$ is not stationary, so \mathcal{P} is not proper.

Suppose, by way of contradiction, that $M \in \mathcal{F}$, $M \cap H(\theta) \in \mathcal{N} \cap \mathcal{F}|_{H(\theta)}$. $[H(\theta)^{V[G]}] \models \forall \alpha \in M$ if $\alpha < \kappa$ then $p_{f(\alpha),\alpha} \in G$ and $f(\alpha) \in M$ (hence $p_{f(\alpha),\alpha} \in M \cap H(\theta)$) — $f(\alpha) \in M$ follows from $f \in M$. This is forced by some $r \in G$, i.e., some $r \Vdash \forall \alpha \in M$ if $\alpha < \kappa$ then $p_{\dot{f}(\alpha),\alpha} \in \dot{G}$ and $\dot{f}(\alpha) \in M$ (hence $p_{\dot{f}(\alpha),\alpha} \in M \cap H(\theta)$). Since G is a filter, there is $q \leq p, r$. So $q \Vdash \forall \alpha \in M$ if $\alpha < \kappa$ then $p_{\dot{f}(\alpha),\alpha} \in \dot{G}$ and $\dot{f}(\alpha) \in M$ (hence $p_{\dot{f}(\alpha),\alpha} \in M \cap H(\theta)$). I.e., $q \leq p$ and q is M -generic. But then $M \cap H(\theta) \notin \mathcal{N}$.

\Leftarrow : Suppose \mathbb{P} satisfies the right-hand side. Fix $p \in \mathbb{P}$. Define $\mathcal{M} = \{M \text{ countable: } p, \mathbb{P} \in M, M \prec H(\theta), \exists q \leq p \text{ } q \text{ is } M\text{-generic}\}$.

Subclaim 1. $\mathcal{N} = \{M[G] \cap \theta : M \in \mathcal{M}\}$ is club in $[\theta]^\omega$.

Proof. Let $a \in [\theta]^\omega \cap V[G]$. There is $f \in V[G]$, $f : \omega \rightarrow \theta$, f 1-1, $a = \text{range } f$. We construct $\{M_n : n < \omega\} \subset \mathcal{M}$ and $\{p_n : n \leq \omega\}$ where $M_n \prec M_{n+1}$, $p_n \in G \cap M_n$, $p_n \Vdash \dot{f}(n) \in M_n$. Let $M = \bigcup \{M_n : n < \omega\}$. $M \in \mathcal{M}$ by \mathcal{M} club. But by construction $a = \text{range } f \subset M[G]$ Hence \mathcal{N} is unbounded. The proof that \mathcal{N} is closed is similar. \square

By fact 41, if $M \in \mathcal{M}$ and q is M -generic, then $q \Vdash M[\dot{G}] \cap \theta = M \cap \theta$. By hypothesis, \mathcal{M} is club, so $\mathcal{M}|_\theta$ is club. Let S be stationary in $[\theta]^\omega$, and suppose C club in $[\theta]^\omega$, $C \in V[G]$. Without loss of generality, $C \subset \mathcal{M}|_\theta$. We want to show $S \cap C \neq \emptyset$.

We construct the function $f \in V[G]$ as follows: $f : C \rightarrow \theta$, $f(c) = \sup c \in \theta$.

There is $M \in \mathcal{M}$, $\dot{f} \in M$ (hence $f \in M[G]$), $M \cap \theta \in S$. Let $q \leq p$, q is M -generic. Then $q \Vdash M \cap \theta = M[G] \cap \theta$ and $q \Vdash \dot{f}^{\leftarrow}[\theta]$ is club in $M[\dot{G}]$. By the latter, $q \Vdash$ if $\gamma \in M \cap \theta$ then there is $\dot{c} \in M[\dot{G}]$ $\gamma \in c$, $\dot{f}(\dot{c}) > \gamma$.²⁹

We only consider G with $q \in G$. In V let $\{\beta_n : n < \omega\} = M \cap \theta$. In $V[G]$ there is a sequence $\{\alpha_n : n < \omega\}$ cofinal in $M \cap \theta$ where each $\alpha_n \in M \cap \theta$, and there is $c_n \in M[G] \cap \dot{f}^{\leftarrow}(\alpha_n)$ (hence $c_n \subset M$) where $\beta_n \in c_n \subset c_{n+1}$, and $M \cap \theta = \bigcup_{n < \omega} \dot{f}^{\leftarrow}(\alpha_n)$. Each $c_n \in C$, so, since C is club, $M \cap \theta \in C$. Since $M \cap \theta \in S$, we are done. \square

Theorem 49 is the tool we use to prove partial orders proper, so let's do it.

Fact 32. *Countably closed partial orders are proper.*

Proof. Let \mathbb{P} be countably closed, $\mathbb{P} \in M \prec H(\theta)$, $\theta > |\mathbb{P}|$, M countable. Let $\{D_n : n < \omega\}$ list all dense subsets D of \mathbb{P} with $D \in M$. Let $p \in M \cap \mathbb{P}$. By elementarity we construct a descending sequence $\{p_n : n < \omega\}$ with each $p_n \leq p$, each $p_n \in D_n$. Let q be a lower bound for all p_n . Then if $r \leq q$ $p_n \leq r$ and $p_n \in D_n$, so q is M -generic. \square

²⁹because $M \prec H(\theta)$ and this is true if you substitute V for M (hence $H(\theta)$ for M)

Fact 33. *Ccc partial orders are proper.*

Proof. Let \mathbb{P} be ccc, $\mathbb{P} \in M \prec H(\theta), \theta > |\mathbb{P}|, M$ countable. Let $A \in M, A$ a maximal antichain in \mathbb{P} , hence A pre-dense.³⁰ Because \mathbb{P} is ccc, A is countable, so $\exists f : \omega \rightarrow A, f$ onto. Hence $\exists f \in M M \models f : \omega \rightarrow A, f$ onto. Since $\omega \subset M$, and enough of replacement holds in $H(\theta), A \subset M$. So every $q \in \mathbb{P}$ is M -generic, since every element of \mathbb{P} is compatible with some element of $A = A \cap M$. \square

An immediate corollary of the preceding two facts and fact 28 is that Mathias forcing is proper.

Fact 34. *Sacks forcing is proper.*

Proof. . Let $\mathbb{S} =$ Sacks forcing, $\mathbb{S} \in M \prec H(\theta), \theta > \mathfrak{c}, M$ countable. Let $\{D_n : n < \omega\}$ list $\{D \in M : D \text{ predense in } \mathbb{S}\}$. Let $p \in \mathbb{S}$. We construct an M -generic $q \leq p$.

Let $q_\emptyset \in D_0 \cap M. q_\emptyset \leq p$. Suppose we know $q_\sigma \in D_{|\sigma|}$. For $i < 2$ let $q_{\sigma \frown i} \leq q_\sigma, q_{\sigma \frown i} \in D_{|\sigma|+1} \cap M$ where $q_{\sigma \frown 0} \perp q_{\sigma \frown 1}$.

Let $q \leq \bigcap_{n < \omega} \bigcup_{|\sigma|=n} q_\sigma$. In the section on Sacks forcing we proved that $q \in \mathbb{S}$. If $r \leq q$ then for all $n < \omega$ there is σ with $|\sigma| = n$ and r, q_σ compatible. Since $q_\sigma \in D_n \cap M$, this completes the proof that q is M -generic. \square

³⁰recall that it suffices to consider predense sets which are maximal antichains

19 No S space — a case study in applied set theory

In this section we will give the proof of Todorcevic's theorem:

Theorem 50. *Cons(\exists no S space).*

19.1 Reduction to set theory

Definition 41. *An S space is a regular hereditarily separable Hausdorff space which is not hereditarily Lindelöf.*

The above definition has four technical topological terms: regular, Hausdorff, hereditarily separable, hereditarily Lindelöf. And, of course, underlying everything is a fifth term: the notion of a topology. So our first task is to remove the topology and turn the question into a question about sets, in this case, families of countable sets.

Definition 42. *(a) A topology on a set X is a family $\tau \subset \mathcal{P}(X)$ closed under finite intersection and arbitrary union, with $X, \emptyset \in \tau$. We call $u \subset X$ open iff $u \in \tau$; closed if $X \setminus u \in \tau$. (The most familiar example is the collection of open subsets of \mathbb{R}). Given a topology τ on X and $Y \subset X$, the subspace topology on Y consists of all $u \cap Y, u \in \tau$.*

(b) A topology is Hausdorff iff for any $x \neq y \in X$ there are open disjoint u, v with $x \in u, y \in v$.

(c) A topology is regular iff for any $x \notin H$ closed there are open disjoint u, v with $x \in u, H \subset v$. (d) A topology is separable iff it has a countable set D so that every open set has non-empty intersection with D — D is called dense. (\mathbb{R} is the most familiar example of a separable space.) A topology is hereditarily separable iff every subspace is separable. (Note that \mathbb{R} is hereditarily separable.

(e) A topology is Lindelöf iff every covering by open sets has a countable subcovering. It is hereditarily Lindelöf iff every subspace is Lindelöf. (Again, \mathbb{R} is Lindelöf, in fact hereditarily Lindelöf.)

Now let's reduce the problem of "no S space" to a problem in set theory. Suppose (X, τ) is an S space.

1. Since it is not hereditarily Lindelöf, we can assume (by possibly moving to a subspace) that it is not Lindelöf.
2. Hence we can find a sequence of points $\{x_\alpha : \alpha < \omega_1\}$ and a sequence of open sets $\{u_\alpha : \alpha < \omega_1\}$ where $x_\alpha \in u_\alpha$ and if $\beta < \alpha$ then $x_\alpha \notin u_\beta$.
3. Let's just consider the subspace $X^* = \{x_\alpha : \alpha < \omega_1\}$. It is Hausdorff, regular, and hereditarily separable because these properties are inherited by subspaces.
4. Each $u_\alpha \cap X^*$ is countable. This means that X^* has a base of countable open sets. Combined with regularity, this means that X^* has a base of countable clopen (= closed and open) sets.
5. In a standard move, we identify X with ω_1 under the map $x_\alpha \mapsto \alpha$.

Hence, if there is an S space there is one with the following properties:

1. It is a topology on ω_1

2. Each $\alpha \in u_\alpha$ clopen where $u_\alpha \subset \alpha + 1$

We call such a space a right-separated 0-dimensional topology on ω_1 . It translates into the following: a space generated by a Boolean algebra generated by a family $\{u_\alpha : \alpha < \omega_1\}$ satisfying (2).³¹ All we really care about is the Boolean algebra. A little more topology and we'll have our set-theoretic reduction.

Definition 43. *A subspace $Y \subset X$ is discrete iff for all $y \in Y$ there is open u_y with $\{y\} = Y \cap u_y$.*

For example, \mathbb{Z} is a discrete subspace of \mathbb{R} .

Fact 35. *A right-separated topology topology on ω_1 is hereditarily separable iff it has no uncountable discrete subspace.*

Proof. \Rightarrow : An uncountable discrete space is necessarily not hereditarily separable.

\Leftarrow : Suppose (ω_1, τ) is right-separated, and Y is a non-separable subspace. Then for all $E \in [Y]^\omega$ there is u_E countable open, $u_E \cap Y \neq \emptyset, u_E \cap E = \emptyset$. We construct an increasing sequence of countable sets $\{E_\alpha : \alpha < \omega_1\}$ and points $x_\alpha \in u_{E_\alpha}$ where $E_{\alpha+1} = E_\alpha \cup u_{E_\alpha}$ and if α is a limit then $E_\alpha = \bigcup_{\beta < \alpha} E_\beta$. Let $Y = \{x_\alpha : \alpha < \omega_1\}$. But then each $u_{E_\alpha} \cap Y = \{x_\alpha\}$, so E is discrete. \square

So our task is the following: Given a right-separated 0-dimensional topology on ω_1 , force an uncountable discrete set. Do it with a proper forcing so we can iterate through all potential S spaces. And do it in a way that does not necessitate the use of large cardinals.

The following fact will prove useful:

Fact 36. *A right-separated 0-dimensional topology on ω_1 has an uncountable discrete subspace iff it has an uncountable subspace Y so every initial segment of Y is clopen in Y .*

Proof. \Rightarrow : Every subset, hence every initial segment, of a discrete space is clopen.

\Leftarrow : Suppose $Y \in [\omega_1]^{\omega_1}$ so that each $Y \cap \alpha = Y \cap u_\alpha$ where u_α is clopen. Let $Y = \{\beta_\alpha : \alpha < \omega_1\}$ in increasing order. Then each $\{\beta_{\alpha+1}\} = Y \cap (u_{\alpha+2} \setminus u_{\alpha+1})$, which is open in Y . \square

³¹the Boolean algebra generated by a family \mathcal{U} of subsets of X with $X \in \mathcal{U}$ is simply the closure of \mathcal{U} under finite union, finite intersection, and relative complement

19.2 Destroying one S space

Basic Hypothesis 1. τ is a right-separated 0-dimensional S topology on ω_1 and, for each α , u_α is clopen with $\alpha \in u_\alpha \subset \alpha + 1$.

Define \mathbb{P}_τ to be the set of all $p \in [\omega_1]^{<\omega}$ so that

1. if $\beta < \alpha \in p$ then $\beta \notin u_\alpha$
2. $\{\beta : p \cap u_\beta = \emptyset\}$ is uncountable

Forcings of the form \mathbb{P}_τ will be called canonical no-S forcings.

We want to show that if G is a \mathbb{P}_τ generic filter, then $\bigcup G$ is a cofinal discrete set in $(\omega_1)^V$.

To do this we need:

Claim 1. Fix $x \in [\omega_1]^{<\omega}$ and suppose $Z_x = \{\beta : x \cap u_\beta = \emptyset\}$ is uncountable. Define $\beta \in Z_x$ to be good iff $\{\gamma \in Z_x : (x \cup \{\beta\}) \cap u_\gamma = \emptyset\}$ is uncountable. Then there are uncountably many good points in Z_x .

Proof. Note that, by definition of Z_x , if $\beta, \gamma \in Z_x, \beta < \gamma$ and $(x \cup \{\beta\}) \cap u_\gamma \neq \emptyset$ then $\beta \in u_\gamma$.

So suppose Z_x is uncountable and, by contradiction, there are only countably many good β 's in Z_x . For each not good $\beta \in Z_x$ let $\gamma_\beta = \sup\{\gamma \in Z_x : (x \cup \{\beta\}) \cap u_\gamma = \emptyset\}$. $\gamma_\beta < \omega_1$. So there is $Y \in [Z_x]^{\omega_1}$ so that if $\beta < \delta, \beta, \delta \in Y$, then $\delta > \gamma_\beta$. Hence if $\beta < \delta, \beta, \delta \in Y$ then $\beta \in u_\delta$. So Y is an uncountable subset whose every initial segment is relatively clopen, a contradiction. □

Exercise: If G is a \mathbb{P}_τ -generic filter, then $\bigcup G$ is an uncountable discrete set in $(\omega_1)^V$. [Hint: For discrete, use (1); for uncountable, use (2) and the claim to show that each $D_\alpha = \{p : p \setminus \alpha \neq \emptyset\}$ is dense.]

Thus, if \mathbb{P}_τ is ccc, $V^{\mathbb{P}_\tau} \Vdash \tau$ is not a hereditarily separable topology on ω_1 . We could iterate all the \mathbb{P}_τ 's over, say, a model of $\text{CH} + 2^{\omega_1} = \omega_2$ (in which case we would only need to iterate ω_2 times) and be done.

The problem is that there are, under, say, CH , S space topologies τ on ω_1 for which \mathbb{P}_τ is not ccc. This is a result of Szentmiklóssy, the construction of something called a tight HFD. So we can't simply iterate ccc forcings.

And whatever we do, we can't use forcings which collapse ω_1 : then each S space we destroyed would be countable in the end; when we iterate, at the end we have a new ω_1 and have to start all over destroying its topologies...

There are several approaches to the existence of non ccc canonical no-S forcings which preserve ω_1 .

The easiest to describe is the following: first force a generic club \dot{C} by the countably closed forcing \mathbb{P}_{club} as we did before. Then let $\dot{Q} = \{p \in \mathbb{P}_\tau : \text{if } \alpha < \beta \in p \text{ then } \exists \gamma \in \dot{C} \alpha < \gamma < \beta\}$. It is easy to show that $\mathbb{P}_{\text{club}} * \dot{Q}$ forces a discrete subspace which is cofinal in $(\omega_1)^V$, but not so easy to show that $V^{\mathbb{P}_{\text{club}}} \Vdash \dot{Q}$ is ccc, hence $\mathbb{P}_{\text{club}} * \dot{Q}$ is proper, hence ω_1 is not collapsed.

A second way involves the combinatorial principle TOP, which is fairly complicated to state, and which implies that if τ is S, then \mathbb{P}_τ is ccc. TOP can be proven to be consistent via proper

forcing without using full PFA, that is, the consistency of TOP can be proven without recourse to large cardinals, in an iteration which alternates with ccc instances of canonical no-S forcings. Thus, one way or another, every S space is destroyed.

A third way is to use finite sequences of models to separate elements of the potential discrete subset.³² This is actually not so different from the first method described, where elements are separated by a club, but because it is not an iteration the proof that it is proper is easier. So this is the approach we will take.

This will suffice for a proof under PFA. In the next section we will turn our attention to the question of iteration: how do we know we can iterate and take care of everything without appealing to large cardinals?

So let $\tau, \{u_\alpha : \alpha < \omega_1\}$ be as in the basic hypothesis. Define \mathbb{P}^τ to be the set of all pairs (x_p, \mathcal{M}_p) where

1. $x_p \in \mathbb{P}_\tau$
2. \mathcal{M}_p is a finite \in -chain of countable elementary submodels of $H(\omega_2)$, i.e., if $N, M \in \mathcal{M}_p$ then either $N \in M$ or $M \in N$.³³
3. $\tau, \{u_\alpha : \alpha < \omega_1\} \in M$ for all $M \in \mathcal{M}_p$ ³⁴
4. $\forall \alpha < \beta \in x_p \exists M \in \mathcal{M}_p \alpha < \omega_1 \cap M < \beta$

The order is: $p \leq q$ iff $x_p \supset x_q$ and $\mathcal{M}_p \supset \mathcal{M}_q$.

Exercise If G is a \mathbb{P}^τ -generic filter, then $\bigcup_{p \in \dot{G}} x_p$ is discrete, cofinal in $(\omega_1)^V$.

By the exercise, it suffices to show that \mathbb{P}^τ is proper.

Let θ be sufficient for \mathbb{P}^τ , M a countable elementary submodel of θ with $\tau, \{u_\alpha : \alpha < \omega_1\}, \mathbb{P}^\tau \in M$. Fix $p \in M$. Define $q = (x_p, \mathcal{M}_p \cup \{M \cap H(\omega_2)\})$.

Claim 2. q is \mathcal{M} -generic.

Before proving the claim, some definitions: Fix $m < \omega$.

- If $s \in [\omega_1]^m$ we write $s = \{\alpha_i : i < m\}$ in increasing order. For $k \leq m, s|_k = \{\alpha_i : i < k\}$.
- If $S \in [\omega_1]^m$ and $k < m$ we write $S|_k = \{s|_k : s \in S\}$
- If $S \in [\omega_1]^m$ we say S is ω_1 -branching iff $\forall k < m \forall s \in S|_k \{\beta : s \cup \{\beta\} \in S_{k+1}\}$ is uncountable.

Suppose $\tau \in M$ a countable elementary submodel of $H(\omega_2)$, $S \in M$ where S is ω_1 -branching, and $s \in M \cap S|_k$ for some $k < m$. What can we conclude?

- $M \models \{\beta : s \cup \{\beta\} \in S|_{k+1}\}$ is uncountable.
- Hence $M \models \exists Y$ a countable dense subset of $\{\beta : s \cup \{\beta\} \in S|_{k+1}\}$.

³²this turns out to be highly suggestive, that is, using finite sequences of models as part of the forcing condition is useful elsewhere

³³by definition of \prec , if $M \in N$ and $M, N \prec H(\theta)$ for some θ then $M \prec N$

³⁴note that we can't require $\mathbb{P}^\tau \in M$ — that would be too self-referential

- So $\exists Y \in M$ $M \models Y$ a countable dense subset of $\{\beta : s \cup \{\beta\} \in S|_{k+1}\}$.
- So $H(\omega_2) \models Y$ a countable dense subset of $\{\beta : s \cup \{\beta\} \in S|_{k+1}\}$.
- So Y really is a countable dense subset of $\{\beta : s \cup \{\beta\} \in S_{k+1}\}$.
- Since Y is countable, $Y \subset M$.

Hence if $s \in M \cap S|_k$ and u countable clopen then there is infinite countable $Y \in M, M \supset Y, Y \setminus u \neq \emptyset, s \cup \{\beta\} \in S_{k+1}$ for all $\beta \in Y$. The important fact here is that u need not be in M .

This is the essential combinatorics (in fact a bit stronger than necessary) we need.

Subclaim 2. *Let $r \leq q, r^* = r \cap M$ where M is a countable elementary submodel, and $x_r = x_{r^*}$. If $s \leq r^*$ with $s \in M$, then s, r are compatible.*

Proof. Let $w = (x_s, \mathcal{M}_s \cup \mathcal{M}_r)$. By hypothesis, $x_w \supset x_r$; by definition $\mathcal{M}_w \supset \mathcal{M}_r$, so if $w \in \mathbb{P}^\tau$ then $w \leq r$.

$x_s \in \mathbb{P}_\tau$ since $s \in \mathbb{P}^\tau$. And x_s is separated by \mathcal{M}_s , so by \mathcal{M}_w . We need to check that if $N \in \mathcal{M}_s$ and $N^* \in \mathcal{M}_r \setminus \mathcal{M}_s$ then $N \in N^*$.

$M \cap H(\omega_2) \in \mathcal{M}_q$, hence \mathcal{M}_r . Since $s \in M \cap H(\omega_2)$, \mathcal{M}_s is a finite subset of $M \cap H(\omega_2)$, and if $N^* \in \mathcal{M}_r \setminus \mathcal{M}_s$ then either $N^* = M \cap H(\omega_2)$ or $M \cap H(\omega_2) \in N^*$. In either case, by definability, $N \in N^*$. \square

We are ready to prove the claim.

Proof. Let $r \leq q, D$ dense open in $\mathbb{P}^\tau, D \in M$. By density, we may assume $r \in D$. Define $r^* = r \cap M, x = x_r \setminus M, n = |x|$. If $n = 0, x_r = x_{r^*}$, so by extending r^* to some $r_s \in D \cap M$, by the subclaim we have s compatible with r and are done.

So assume $n \geq 1$ and define $T = \{a \in [\omega_1]^n : \exists s \in D, s \leq r^* \ x_s \setminus x_{r^*} = a\}$. $T \in M$.

Subclaim 3. *T is ω_1 -branching.*

Proof. Fix $a \in T, k < n$. $a = \{\alpha_i : i < n\}$ and, since $a = x_s \setminus x_{r^*}$ for some s , there is an elementary chain of models $\{M_i : i < n\}$ where each $\alpha_i \in M_i$ and each $\alpha_{i+1} \notin M_i$. Suppose $E_{a,k} = \{\beta : a|_k \cup \{\beta\} \in T|_{k+1}\}$ is countable. Then $E_{a,k} \in M_k$ so $\sup E_{a,k} < \omega_1 \cap M_k$. But $\alpha_{k+1} \in E_{a,k}$ and $\alpha_{k+1} \geq \omega_1 \cap M_k$, a contradiction. \square

Define $u = \bigcup_{\gamma \in x} u_\gamma$. Let $\hat{T} = \{a|_k : a \in T, k \leq n\}$. Define $X_0 = \{\alpha : \{\alpha\} \in \hat{T}\}$. $X_0 \in M$. X_0 is uncountable. There is $Y_0 \in [X_0]^\omega \cap M$ with Y_0 dense in X_0 , hence there is $\alpha_0 \in Y_0 \setminus u$ (because u is countable clopen).

Let $X_1 = \{\alpha : \{\alpha_0, \alpha\} \in \hat{T}\}$. $X_1 \in M$. There is $Y_1 \in [X_1]^\omega \cap M$ with Y_1 dense in X_1 , hence there is $\alpha_1 \in Y_1 \setminus u$.

Etc. For each $k < n$ we construct $\alpha_k \notin u$ with $\{\alpha_i : i \leq k\} \in \hat{T}, \alpha_k \in M$.

Let $a = \{\alpha_i : i < n\}$. $a \in T \cap M$. So there is $s \in D \cap M, s \leq r^*, a = x_s \setminus x_{r^*}$.

Since $a \cap u = \emptyset$, s, r are compatible. \square

By the claim, \mathbb{P}^τ is proper. Hence it adds an uncountable discrete subspace. So τ is no longer an S space.

I.e., given an S space we can destroy it with proper forcing. What about destroying all S spaces?

Under PFA we are done. But PFA involves large cardinals, and we want a consistency result with just ZFC.

So we turn our attention to iteration. Suppose $\text{CH} + 2^{\omega_1} = \omega_2$ holds in the ground model. Suppose forcing with a countable iteration of \mathbb{P}^τ 's of length α with $|\alpha| \leq \omega_1$ preserves $\text{CH} + 2^{\omega_1} = \omega_2$. Then we can enumerate all names of potential S spaces in intermediate models in a sequence of length ω_2 , and destroy each in a countable support iteration of length ω_2 . Which will complete the proof of "there are no S spaces."

But unfortunately the second hypothesis fails: $V^{\mathbb{P}^\tau} \Vdash |(\omega_2)^V| = \omega_1$. The next section shows what goes wrong and how to fix it.

19.3 Iterating without large cardinals

First, let's see what goes wrong.

Claim 3. *Let $N \in M, N \prec M \prec H(\omega_2)$, N, M countable. Then $\omega_1 \cap N \in M$.*

Proof. $N \in M, \omega_1 \in M$, so $\omega_1 \cap N \in M$. □

Under the hypothesis of the claim, $\omega_1 \cap N < \omega_1 \cap M$

Fact 37. *Let \mathbb{P}^τ be as in the previous section, where τ is an S space topology. $V^{\mathbb{P}^\tau} \Vdash |(\omega_2)^V| = \omega_1$.*

Proof. Fix $\delta < \omega_1$ and let $\mathcal{N}_\delta = \{N \prec H(\omega_2) : \tau, \{u_\alpha : \alpha < \omega_1\} \in N \text{ and } \omega_1 \cap N = \delta\}$. By the preceding claim, each $|\mathcal{M}_p \cap \mathcal{N}_\delta| \leq 1$, and if $\emptyset \neq \mathcal{M}_p \cap \mathcal{N}_\delta \neq \mathcal{M}_q \cap \mathcal{N}_\delta \neq \emptyset$, then p, q are incompatible.

For $\alpha < \omega_2$ define $D_\alpha = \{p : \sup \bigcup \mathcal{M}_p > \alpha\}$. Each D_α is dense.

Let G be a \mathbb{P}_τ -generic filter. Then $G \cap D_\alpha \neq \emptyset$ for each $\alpha < \omega_2$. Define $\mathcal{M} = \bigcup_{p \in G} \bigcup \mathcal{M}_p$. Then $M, N \in \mathcal{M} \Rightarrow \omega_1 \cap N \neq \omega_1 \cap M$. Define the function $f \in V[G]$ as follows: for $M \in \mathcal{M}$, $f(\omega_1 \cap M) = \sup(\omega_2 \cap M)$. f is a partial function from a cofinal subset of ω_1 to a cofinal subset of ω_2 . (Exercise: why?) Hence f collapses ω_2 to ω_1 . □

So we have to change the partial order.

Given $\tau, \{u_\alpha : \alpha < \omega_1\}$ as in the basic hypothesis, define $\mathcal{N} = \bigcup_{\alpha < \omega_1} \mathcal{N}_\delta$, and define \mathbb{Q}_τ to be the set of all $p = (x_p, e_p)$ where

1. $x_p \in \mathbb{P}_\tau$
2. each element of e_p is $N \cap \omega_1$ for some $N \in \mathcal{N}$.
3. $\tau, \{u_\alpha : \alpha < \omega_1\} \in N$ for all $N \in \mathcal{N}$
4. $\forall \alpha < \beta \in x_p \exists \gamma \in e_p \alpha < \gamma < \beta$

Note that the proof of subclaim 3 goes through, since it never uses $M_i \in M_{i+1}$, so T is ω_1 -branching. Since $a, r^* \in N \Rightarrow T \in N$, we don't use ϵ -chains in our proof of proper. So, similar to the proof that \mathbb{P}^τ is proper, we have

Exercise: \mathbb{Q}_τ is proper.

$|\mathbb{Q}_\tau| = \omega_1$, so clearly has the ω_2 -cc, hence doesn't collapse cardinals $\geq \omega_2$. Being proper, it doesn't collapse ω_1 . So it doesn't collapse cardinals.

Let's see what happens when we iterate.

An element of \mathbb{Q}_τ is a subset of $([\omega_1]^{<\omega})^2$, so all of the elements of the next $\mathbb{Q}_{\dot{\tau}}$ in the iteration are still named by ω_1 objects in the ground model.³⁵ So an element of the countable support iteration of length γ will look like a function $p : \gamma \rightarrow ([\omega_1]^{<\omega})^2$ where only countably many $p(\alpha) \neq (\emptyset, \emptyset)$.

Assume $V \models \text{CH} + 2^{\omega_1} = \omega_2$. By CH, an iteration \mathbb{P} of length ω_2 will thus have size $||[\omega_2]^\omega|^{\omega_1} = \omega_2$, and for $\gamma < \omega_1$, $|\mathbb{P}_\gamma| = \omega_1$.

By the generalized Δ -system lemma, ω_2 many conditions in \mathbb{P} cannot be an antichain. (The proof is a good exercise in using the generalized Δ -systems lemma.) So \mathbb{P} has the ω_2 -cc. We already know it is proper. Hence it does not collapse cardinals.

Since $|\mathbb{P}_\gamma| = \omega_1$, then $V^{\mathbb{P}_\gamma} \models \text{CH} + 2^{\omega_1} = \omega_2$

And $2^{\omega_1} = \omega_2 \Rightarrow |\{\tau : \tau \text{ is an S space topology on } \omega_1\}| \leq \omega_2$.

So at each stage $\gamma < \omega_2$, $\exists\{\dot{\tau}_{\gamma,\alpha} : \alpha < \omega_2\}$ $V^{\mathbb{P}_\gamma} \models$ if $\dot{\tau}$ is an S space topology on ω_1 then there is α with $\dot{\tau} = \dot{\tau}_{\gamma,\alpha}$.

Let $\varphi : \omega_2 \rightarrow (\omega_2)^2$ where if $\varphi(\gamma) = (\beta, \alpha)$ then $\beta \leq \gamma$. At stage $\gamma + 1$ we consider the topology $\dot{\tau}_{\varphi(\gamma)}$. If $\mathbb{P}_\gamma \Vdash \dot{\tau}_{\varphi(\gamma)}$ is S, we force with $\mathbb{Q}_{\dot{\tau}_{\varphi(\gamma)}}$. Otherwise we do nothing.

In this way, we consider every possible S space topology that arises, and destroy each one. We do it without collapsing cardinals. The proof of "Cons(\exists no S space)" is complete.

Final note on proof: In other forcings using finite \in -chains of models, the move to simply consider finite sequences of $\omega_1 \cap M$'s won't work, since more about the models is used than simply the first missing ordinal. The fix instead moves from considering a chain $M_0 \in M_1 \in \dots \in M_k$ to considering finite sets of countable elementary submodels $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_k$ where every element in \mathcal{M}_i has the same transitive collapse (including the same $\omega_1 \cap M$), and we can find chains $M_0 \in M_1 \in \dots \in M_k$ where each $M_i \in \mathcal{M}_i$. When we extend a condition, we are allowed to add finitely many models to each \mathcal{M}_i . Details are, as they say, left to the reader.

³⁵of course from the standpoint of the ground model we don't know exactly which ones, since we add objects to $H(\omega_2)$ as we iterate