Math 409 Notes: Geometry

Jeremy Martin and Judith Roitman

January 10, 2014

1 Euclidean geometry

1.1 Straight-edge and compass constructions

A major problem for anyone who wants to build something is: how do you know you've built the thing you wanted to build? For example, how do you know the angle between walls is a right angle? How do you know you've cut two things of exactly the same length?

Thousands of years ago people couldn't go to the store to get rulers, templates, and table saws that guaranteed they were building what they thought they were. They could reliably make straight lines and circles — you can do this just with string, but they had other simple technologies as well. In particular, when discussing mathematics, the ancient Greeks focused on what you could build with a straight edge and compass[1]. But their compasses collapsed — once you finished one circle, you did not retain the exact radius for the next circle. Yet everywhere in the ancient world people were able to build things with great geometric precision.

In this section you will use straight-edges and compasses (the non-collapsing kind) to make some basic geometric constructions. You have almost certainly done this before, so I won't tell you how to do it. Instead, you'll figure it out working with a small group of classmates. It's crucial that you do *not* use your straight line segment as a ruler, i.e., you may not mark off lengths in any way or use such markings if they exist.

Tasks

1. Draw a straight line segment $\overline{AB}$. Draw a point p somewhere else on the page. Now draw a straight line segment of exactly the same length as $\overline{AB}$ with p as one end-point.

2. Draw a triangle ABC. Now copy it exactly somewhere else on the page.

3. Draw an equilateral triangle.

4. Draw an angle $\angle ABC$. Draw a point p somewhere else on the page. Copy $\angle ABC$ exactly, with p as its vertex.

5. Draw a straight line segment $\overline{AB}$. Construct a line perpendicular to $\overline{AB}$ that bisects its length.[2]

[1] not the thing that tells north from south, but the thing that makes circles

[2] known as the perpendicular bisector.

6. Draw a line ℓ and a point p that does not lie on ℓ. Construct a line through p that is perpendicular to ℓ. [Hint: #5 might help you.]

7. Draw a line ℓ and a point p that does not lie on ℓ. Construct a line through p that is parallel to ℓ.

8. Draw an angle $\angle ABC$. Bisect it.

9. Construct a square.

10. Construct a regular hexagon (six equal sides; six equal angles).

Question for discussion

How do you know that these constructions really work?

1.2 *Sketchpad* constructions

In the late 20th century people came up with computer environments that allow you to experiment with geometric constructions. The one we will be using is called *Geometer's Sketchpad*, and we will be using it a lot to make conjectures.

A *Sketchpad* construction is successful if, when we try to deform it by dragging vertices or sides, it retains the shape it's supposed to and does not retain any properties it's not supposed to have. For example, constructing a generic triangle means constructing a triangle which can be dragged to take on any triangular shape, but constructing an equilateral triangle means that no matter how you drag on the construction it remains equilateral, and constructing an isosceles triangle means that when you drag on it, it can take on the shape of any isosceles triangle (i.e., by dragging a suitable vertex, the angle between the equal sides can be anything between 0 and π).

Using *Sketchpad* we'll do the following in class:

1. Make a line segment, find its midpoint, and construct its perpendicular bisector.

2. Construct two parallel lines.

3. Make a circle and construct its diameter.

4. Make a generic triangle.

5. Construct an equilateral triangle.

6. Make a generic quadrilateral.

7. Construct a square.

We'll do a lot more for homework.

Question. Why do the constructions of the equilateral triangle and square work?

In the previous section, we mentioned that the ancient Greeks had a collapsible compass. How did they use it to exactly copy a straight line segment? Homework problem SA 12 uses a *Sketchpad* simulation in the homework to show how you can exactly copy a straight line segment with a *collapsing* compass. It uses the fact that the construction of an equilateral triangle does not require a fixed compass or a marked straight edge; you don't need to externally preserve any lengths in order to carry out the construction.

1.3 Plane geometry via axioms, Part I

We have kept asking: why do our geometric constructions work? That is, how do we know we are constructing the objects we want to construct?

So far, when we have tried to answer this question, we have appealed to various facts that seemed self-evident. But we can imagine someone challenging us: how do you know that this or that is true?

Why should we believe our mathematical conclusions? The answer to this question necessarily involves some notion of convincing argument. The ancient Greeks were perhaps the first to have the notion of proving things from a minimal set of assumptions, taking as little as possible for granted. The basic way they worked (and the basic way we work when we do mathematics) goes as follows.

First you have some *undefined notions*—you have to start somewhere.

Then you have some basic facts about these notions, called *axioms*. Again, you have to start somewhere.

Then you add *definitions*, which say what new words mean.

At some point you can prove *theorems*, which are statements that logically follow from what you've already done.

And you add more definitions and more theorems, and less important theorems called lemmas and corollaries, and so on until you've built up the enormous edifice we call mathematics. Which is still expanding, because every theorem you prove leads to more questions which lead to more definitions and theorems which lead to more questions, and so on. It's important to know the difference between undefined notions and axioms—things that don't have to be defined or proved—and theorems, which do have to be proved.

For example, we won't define the notion of distance. We will define a circle as the set of points equidistant from a given point (called the center). Note that this was really two definitions: circle, center.

We don't prove that a circle is the set of points equidistant from a given point. You can't prove it. It's a definition. But you can prove that if a line ℓ is tangent to a circle at a point p and if m is the diameter at p, then ℓ is perpendicular to m. In fact you *have to* prove it, otherwise you don't know whether it's true or not. Then it becomes a theorem and you can use it in proving subsequent theorems.

These distinctions were explicitly made by Euclid, who codified ancient Greek mathematical knowledge in his *Elements*, around the third century BCE. We tend to think of Euclid's *Elements* as being just about geometry, but in fact it included all of mathematics; geometry was only one part.

Euclid's idea was to have a very small set of undefined terms like *point* and *line* (i.e., "infinite straight line") — and a small set of axioms:

1. Any two distinct points determine a unique line.
2. Lines can be extended infinitely.
3. Every segment can be made into the radius of a circle.
4. All right angles are equal.
5. The parallel postulate: if two lines ℓ_1, ℓ_2 meet a third line p so that the sum of the inner angles on one side of p is less than π, then ℓ_1 and ℓ_2 can't be parallel.[3]

This notion of proving everything from a small set of basic principles had a powerful effect, ultimately shaping every area of mathematics. Indeed, the smaller the set of basic principles, the better a mathematical system is. The goal is to derive the most powerful rules possible from the simplest building blocks.

To the modern mind Euclid's axiom system is incomplete. He clearly was using a lot of physical intuition that was not made explicit in his axioms, and also suffered from quite a bit of vagueness. For example, he defined a point as "that which has no part", and an angle as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line."[4] This is circular—if you don't already know what an angle is you can't possibly understand it.

Still, Euclid's idea of stating your axioms explicitly and then deriving theorems from them was the birth of modern mathematics as we know it.

The fifth axiom is a famous story. It's clearly a much more complicated statement than the first four. This made a lot of geometers uncomfortable — after all, axioms are supposed to be simple principles that sensible people should all agree are obvious, and it takes a fair amount of effort even to figure out what that postulate is saying (let along whether it is "obvious"). Over the centuries, mathematicians spent a huge amount of effort trying to show that Euclid's fifth axiom could be proven from the first four. It turns out that it can't! More on this later.

It wasn't until about 100 years ago that the great mathematician David Hilbert definitively cleaned up Euclid's axioms.[5] We will take a middle path, using terms like "between," "interior," "right," "above," "opposite" without further explication, but also expanding Euclid's system of definitions and axioms to avoid both vagueness and complicated proofs of the obvious.

In this section we establish an axiom system that allows us to develop the geometry of the plane fairly rigorously. In the next section, we use this axiom system to prove some basic theorems. We will then have a short discussion of area. After that, we will continue to prove theorems in plane geometry using this system — including some fairly important theorems — as homework problems.[6] In accordance with contemporary practice, we will call such proofs Euclidean proofs, even though they will often not be the proofs given by Euclid.

Undefined terms

Any axiom system needs to be based on undefined terms. We will use five:

- point

[3]Note that we're using radian degrees here, as we will do throughout.
[4]From the translation at the Perseus digital library: http://www.perseus.tufts.edu/hopper/text?doc=Euc.+1.
[5]to see what they are, Google *Hilbert's axioms*
[6]Why not? Who says homework always has to be trivial imitations of what the teacher did in class?

- line (= infinite straight line)
- angle
- measure of a line segment
- measure of an angle

Here "measure" means "size," so the measure of a line segment is its length. In this chapter we will carefully distinguish between a measure (which is a number) and the thing being measured. Thus we write $\overline{AB}$ to name a line segment, and its measure, or length, is named by $m(\overline{AB})$; "$\overline{AB} = \overline{CD}$" means that the two lines segments actually are the same line segment, but "$m(\overline{AB}) = m(\overline{CD})$" means the two line segments have the same length. Similarly, we write $\angle ABC$ to name an angle, and its measure (which can be measured in degrees or radians) is written $m(\angle ABC)$. In the homework and in class we will eventually relax this notational distinction and write things like $\overline{AB} = \overline{CD}$ to mean that two line segments have the same length, and $\angle ABC = \angle DEF$ to mean that the two angles have the same measure. But not in this chapter.

When we discuss area, we will treat "area of a figure" as an undefined term as well.

In this class we assume (as the Greeks did) that everyone knows from experience what point, line, angle, length, and angle measure really mean. But an interesting exercise is to develop the theory axiomatically without appeal to intuition, and then see which structures fit different parts of the theory. In the 19th century people were surprised to discover that, in fact, there are alternative geometric structures. Some is this is discussed in Math 559.

Definitions

Undefined terms aren't enough. To make ourselves understood, we need more vocabulary, i.e., definitions based on the undefined terms. In this section we give some basic definitions. We will keep adding definitions as needed. That's what mathematicians do.

Definition 1. *Line segment and figure.*

1. Let A, B be on a line ℓ. The line segment $\overline{AB}$ is the set of all points on ℓ which lie between A and B.[7]

2. A figure is a collection of points which divide the plane into two parts, an inside (interior) and an outside (exterior).

3. If a figure consists of line segments, the sides of the figure are the maximal line segments.

4. A collection of points is collinear iff there is some line containing all of these points.[8]

(In a definition, "iff" just reminds you that the word being defined means *exactly* what the definition says it means.)

Definition 2. *Congruence and similarity:*

1. Two line segments are congruent iff they have the same measure.

[7] Remember: we are not defining "between". It is not hard to do this for collinear points (see definition 1.4 below, but the in more complicated situations the definition can be cumbersome.

[8] Axiom 1 will say that a collection with exactly two points is collinear.

2. Two angles are congruent iff they have the same measure.

3. Two figures are congruent iff one of them can be moved via a rigid motion so that it coincides with the other. In particular, if one of them consists of line segments, so does the other, corresponding sides have the same measure, and corresponding angles have the same measure. If figure $\mathcal{F}$ is congruent to figure $\mathcal{G}$ we write $\mathcal{F} \cong \mathcal{G}$.

4. Two figures are similar iff one of them is proportional to the other. In particular, if one of them consists of line segments so does the other, and corresponding sides are proportional. If figure $\mathcal{F}$ is similar to figure $\mathcal{G}$ we write $\mathcal{F} \approx \mathcal{G}$

Note that the definitions of congruence and similarity involve transformational geometry. This is actually how Euclid did it; his definition of congruence was: "Things that [can] coincide with one another are equal to one another."[9]

Definition 3. *Parallel lines: Two distinct lines are parallel iff they never meet.*

Definition 4. *Measures of angles.*

a. $m(\angle ABC) = \pi$ iff A, B, C are collinear and B is between A and C.

2. $\angle ABC$ bisects $\angle ABD$ iff $m(\angle ABC) = m(\angle CBD)$. Note that if $\alpha = m(\angle ABC)$ and $\beta = m(\angle ABD)$ then $\alpha = \beta/2$.

3. The line ℓ through $\overline{AB}$ is perpendicular to the line m through $\overline{BC}$ iff $m(\angle ABC) = \pi/2$. (We write $\overline{AB} \perp \overline{BC}$ or $\ell \perp m$.)

Definition 5. *Defining particular figures.*

1. A triangle is a figure with exactly three sides. It is isosceles iff it has at least two congruent sides. It is equilateral iff all of its sides are congruent.

2. A quadrilateral is a figure with exactly four sides. It is a parallelogram iff each pair of opposite sides is parallel. It is a rhombus iff it is a parallelogram with all four sides congruent. It is a rectangle iff it is a parallelogram in which all of its interior angles are $\pi/2$. It is a square iff it is a rectangle with all four sides congruent.

3. A circle is the set of points equidistant from a given point (called the center). The distance between each point on the circle and the center is called the radius.

From now on when we talk about measure we will relax and use "equal" instead of "congruent." E.g., we will say that two line segments or two angles are equal when they have the same measure. Note that this is not the usual meaning of "equal" — the line segments or angles need not occupy the same space.

A note of caution: We have defined certain things in this section. Later we will prove theorems about the things we've defined. Some of our theorems will precisely describe the objects we've just defined. For example: A triangle is equilateral iff all of its angles are equal. But "all of its angles are equal" is *not* the definition of an equilateral triangle. It is a property that is true of equilateral triangles and of no other triangles. Once we've proved this, we can use it. But until we've proved it, we can't use it.

[9]Stahl, *Geometry from Euclid to Knots*, p. 43

The axioms

At this point we have a bunch of things we can talk about, some of them undefined, some of them defined. Now it's time to say something about them. We start off with axioms. These are the statements we assume to be true without proof. In some sense they capture our basic intuitions. They are the formal ground for the theory we are developing: you have to start somewhere, and this is where we start.

An axiom about lines:

Axiom 1. *If A, B are distinct points, then there is exactly one line through A and B.*

Axiom 1 may seem obvious, but it fails on the sphere (spherical geometry) and also in hyperbolic geometry.

Five axioms about measure:[10]

Axiom 2. *If A, B, C are collinear then A is between B and C iff $m(\overline{BA}) + m(\overline{AC}) = m(\overline{BC})$.*[11]

The word "iff" is mathematician's jargon for "if and only if". That is, the axiom says that two different things are true. Assuming A, B, C are collinear: (i) if A is between B and C then $m(\overline{BA}) + m(\overline{AC}) = m(\overline{BC})$; (ii) if $m(\overline{BA}) + m(\overline{AC}) = m(\overline{BC})$ then A is between B and C. Logically, (i) and (ii) are two separate statements.

Axiom 3. *$m(\overline{AB}) = 0$ iff $A = B$.*

Similarly, this axiom says two things: (i) if $m(\overline{AB} = 0$ then $A = B$; (ii) if $A = B$ then $m(\overline{AB}) = 0$. And so on. There will be many uses of "iff" in this course.

Axiom 4. *$m(\overline{AB}) > 0$ iff $A \neq B$.*[12]

Axiom 5. *$m(\angle ABC) = 0$ iff all three points A, B, C are collinear and B is not between A and C.*

Axiom 6. *$m(\angle ABC) + m(\angle CBD) = m(\angle ABD)$ if the line segment $\overline{BC}$ is between the line segments $\overline{BA}$ and $\overline{BD}$.*

An axiom about logic:

Axiom 7. *Equals can be substituted for equals.*

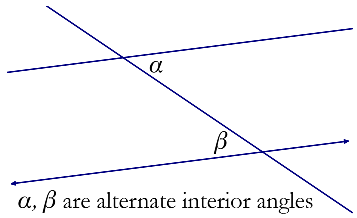
Two axioms about parallel lines:

Axiom 8. *Given a point P and a line ℓ there is exactly one line through P parallel to ℓ.*[13]

Axiom 9. *Given a transversal of two parallel lines, alternate interior angles are equal.*

[10] You might want to draw pictures to see what axioms 2, 5, and 6 are saying.
[11] You could use this as a definition of "between" in the case of collinear points...
[12] axioms 3 and 4 together tell us that measure of line segments is non-negative
[13] This is a very famous axiom, known as Playfair's axiom.

Here is a picture of axiom 9:



Axioms 8 and 9 distinguish Euclidean geometries from non-Euclidean geometries (e.g., spherical geometry and hyperbolic geometry). For thousands of years people tried to deduce them from the other axioms, but in the 19th century the Russian mathematicain Lobachevsky and the Hungarian mathematician Bolyai showed it couldn't be done.

Three axioms about triangles:

Axiom 10. *(SSS) Two triangles are congruent iff their corresponding sides are equal.*

Axiom 11. *(The triangle inequality) If A, B, C are distinct points, then $m(\overline{AB})+m(\overline{BC}) \geq m(\overline{AC})$*

Axiom 12. *Two triangles are similar iff their corresponding angles are equal.*

An axiom about circles:

Axiom 13. *The ratio between the circumference of any circle and its diameter is constant, i.e., there is a number π so that for any circle $\mathcal{C}$, (circumference of $\mathcal{C}$)/(diameter of $\mathcal{C}$) $= \pi$.*

Finally, two axioms that connect number and geometry:

Axiom 14. *For any positive whole number n, and distinct points A, B, there is some C between A, B so $n \times m(\overline{AC}) = m(\overline{AB})$.*

Axiom 15. *For any positive whole number n and angle $\angle ABC$, there is a point D so $n \times m(\angle ABD) = m(\angle ABC)$. Furthermore, $\overline{BD}$ is between the line segments $\overline{BA}$ and $\overline{BC}$.*

Axioms 14 and 15 tell us that any positive rational number can be a measure of either a line or an angle. So can any positive irrational number, but the axioms we've listed don't tell us that. This is because we are imitating Euclid, and the ancient Greeks were uncomfortable — in fact, downright disturbed — by irrational numbers. They were quite aware of π (see axiom 13) — although they did not use the symbol π — but did not know that π was not rational. They did know that $\sqrt{2}$ was irrational, but were not happy about this. According to one legend, the Pythagoreans killed any outsider who learned this secret. According to another legend, the Pythagoreans killed the person who proved that $\sqrt{2}$ is irrational. Both legends are probably false.

1.4 Plane geometry from axioms: Part II

At this point we've got both undefined and defined terms, and basic assumptions (the axioms). Now it's time to prove something. In this section we prove basic theorems using the axioms and definitions of the previous section. Some of these theorems may seem screamingly obvious, and that's part of the point — even the screamingly obvious needs to be proved.

The first three theorems say that certain things are unique: a line segment has exactly one midpoint, an angle has exactly one bisector, a line has exactly one perpendicular through a given point.

Theorem 1. *A line segment has exactly one midpoint.*

Proof. Let $\overline{AB}$ be a line segment. The first step is to show that there is at least one midpoint. By axiom 14 we can find a point C between A and B (that is, on $\overline{AB}$) such that

$$2 \cdot m(\overline{AC}) = m(\overline{AB}) \tag{1}$$

So we need to show that $m(\overline{AC}) = \overline{BC}$. By axiom 2, we know that $m(\overline{AC}) + m(\overline{CB}) = m(\overline{AB})$ so by axiom 7,

$$2 \cdot m(\overline{AC}) = m(\overline{AC}) + m(\overline{BC}) \tag{2}$$

hence, subtracting $m(\overline{AC})$ from both sides gives $m(\overline{AC}) = m(\overline{BC})$.

The second step is to show that $\overline{AB}$ doesn't have two distinct midpoints.

So suppose C, C' are midpoints. We will show that $C = C'$.

Either C' is between C and A, or C' is between C and B. Without loss of generality[14] C' is between C and B.

By definition of midpoint,

$$m(\overline{AC}) = m(\overline{CB}) = \frac{1}{2}m(\overline{AB}) \tag{3}$$

and

$$m(\overline{AC'}) = m(\overline{C'B}) = \frac{1}{2}m(\overline{AB}) \tag{4}$$

By axiom 2

$$m(\overline{AC}) + m(\overline{CC'}) + m(\overline{C'B}) = m(\overline{AB}) \tag{5}$$

By axiom 9

$$\frac{1}{2}(\overline{AB}) + m(\overline{CC'}) + \frac{1}{2}m(\overline{AB}) = m(\overline{AB}). \tag{6}$$

Hence $m(\overline{CC}) = 0$ and, by axiom 3, $C = C'$. □

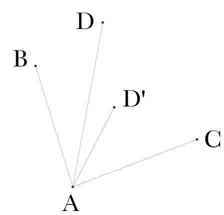
[14] this means that it doesn't matter which case we look at; the proof is essentially the same

(The little box at the end of a proof tells us that the proof is complete. In the old days, people would write QED for the Latin "quod est demonstrandum", i.e. "this is what had to be demonstrated.")

Theorem 2. *Every angle has exactly one bisector.*

Proof. Let $\alpha = \angle BAC$. In a proof similar to the first step of the proof of theorem 1, by axiom 15 we have at least one angle bisector; we have to show there aren't two.

Let D, D' be points so that $\overline{AD}$ and $\overline{AD'}$ bisect α. Since it doesn't matter which point we call D and which we call D', without loss of generality we have the following picture.



We have to show that either D lies on the line $\overline{AD'}$ or D' lies on the line segment $\overline{AD}$

Let $\beta' = \angle BAD', \beta = \angle BAD$. By definition, $m(\beta') = \frac{1}{2}m(\alpha) = m(\beta)$.

By axiom 6, $m(\beta) + m(\angle DAD') = m(\angle BAD') = m(\beta')$. So $m(\angle DAD') = 0$. By axiom 5, A, D, D' are collinear, and A does not lie between D, D'. Hence either D lies on the line segment $\overline{AD'}$ or D' lies on the line segment $\overline{AD}$ □

As an immediate corollary we have

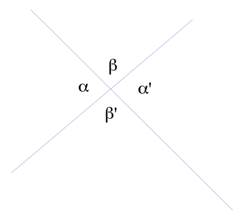
Theorem 3. *There is exactly one perpendicular to a given line through a given point on the line.*

Proof. Let C be a point on a line ℓ, and let A, B be points on ℓ with C between A, B. By definition 4, $m(\angle ACB) = \pi$. By theorem 2, $\angle ACB$ has a unique bisector. □

The next two theorems are about angles, and are not immediately obvious. Theorem 5 relies on the axioms about parallel lines, and fails when those axioms fail.

Theorem 4. *(Vertical angle theorem, or VAT) Vertical angles are congruent.*

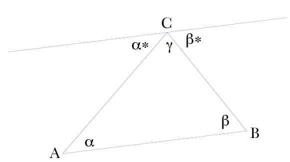
Here is a picture of theorem 4, where α, α' are vertical angles, as are β, β':



Proof. By definition 4, $m(\alpha)+m(\beta) = \pi = m(\beta)+m(\alpha')$, so $m(\alpha) = m(\alpha')$. A similar proof shows that $m(\beta) = m(\beta')$. □

Theorem 5. *The sum of the measures of the interior angles of a triangle is π.*

Proof. Consider the following diagram:



We want to show that $m(\alpha) + m(\gamma) + m(\beta) = \pi$.

Let ℓ be the line parallel to $\overline{AB}$ through C (which exists by axiom 8). By axiom 9, $m(\beta) = m(\beta^*)$ and $m(\alpha) = m(\alpha^*)$. By definition 4 and axiom 6, $m(\alpha^*) + m(\gamma) + m(\beta^*) = \pi$, so, by axiom 7, $m(\alpha) + m(\gamma) + m(\beta) = \pi$.

□

The next theorem may seem crashingly obvious, but in fact it is not. Since it depends on axiom 1, it fails in spherical and hyperbolic geometries.

Theorem 6. *Two distinct lines intersect in at most one point.*

The proof of theorem 6 turns out to be very brief, but it condenses some complicated logical reasoning. Before giving it, let's look carefully at how it works.

The method of proof is called the contrapositive. This means that to prove "if X is true then Y is true" we prove "if Y is false then X is false." Which is the same as saying "if not-Y is true then not-X is true."

The contrapositive is discussed in the appendix on proofs. Here's how it works in this particular instance:

We want to show that the hypothesis "two lines are distinct" implies the conclusion "they meet in at most one point." So we assume they do *not* meet in at most one point, i.e., they meet in at least two points, and prove that the lines are not distinct.

Now we're ready for the proof:

Proof. Suppose $p, q \in l_1 \cap l_2, q \neq p$, i.e., suppose ℓ_1, ℓ_2 meet in more than one point. By axiom 1, $\ell_1 = \ell_2$. □

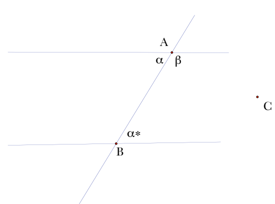
As we said, it's a very short proof.

The next theorem is proved by contradiction. In a proof by contradiction we assume the hypothesis, negate the conclusion, then derive a statement which contradicts something we know to be true given our assumptions. In symbols, to prove "if X then Y" we assume X and not-Y, and then prove not-Z, where we know that Z is true. We're committed to X. We're committed to Z. So not-Y must be false, i.e., Y is true. Hence if X is true, so is Y. Proofs by contradiction are discussed in the appendix on proofs.

This method is similar to proof by contrapositive, but is not the same.

Theorem 7. *Suppose two distinct lines are cut by a transversal. If a pair of alternate interior angles are congruent, the lines are parallel.*

Here's the picture:



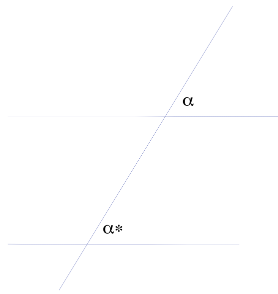
Proof. We assume that $\alpha = \alpha^*$ (the hypothesis) and that the lines are not parallel, that they meet in some point C (C is not shown). Then A, B, C form a triangle, i.e., they are not collinear. Let $\gamma = \angle(ACB)$. Then, by theorem 5, $\alpha^* + \beta + \gamma = \pi$. But $\alpha + \beta = \pi$. So $\gamma = 0$, i.e., A, B, C are collinear, a contradiction. So the theorem is proved. □

Let's see how this worked logically: We assumed $\alpha = \alpha^*$ (the hypothesis) and that the lines meet in point C (negating the conclusion). Then we went on to prove first that A, B, C form a triangle and then that A, B, C do not form a triangle. I.e., we proved the false statement "A, B, C form a triangle and A, B, C do not form a triangle." This proves the theorem.

An immediate corollary, left for homework, is:

Corollary 1. *Suppose two distinct lines are cut by a transversal. If a pair of corresponding angles is congruent, the lines are parallel.*

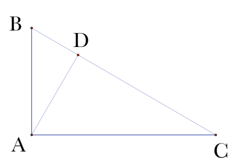
In the diagram below, α, α^* are corresponding angles.



The next basic theorem in our list is the Pythagorean theorem, one of the most profound theorems in early geometry, known to many ancient cultures. Here we give a proof which relies on similar triangles (the only proof in this chapter that does so). In the next section we'll give a proof which relies on area. And there are many, many more proofs. You can find 99 proofs, as well as generalizations, at http://www.cut-the-knot.org/pythagoras/index.shtml, the cut-the-knot website.[15]

Theorem 8. *(Pythagorean theorem) Let a, b, c be the lengths of the sides of a right triangle, where the side whose length is c is opposite the right angle. Then $a^2 + b^2 = c^2$.*

Proof. Let ΔABC be a right triangle, with $\angle CAB$ the right angle, and let D be on $\overline{BC}$ with $\overline{AD} \perp \overline{BC}$, as in the diagram:



I.e., $a = m(\overline{AB}), b = m(\overline{AC})$ and $c = m(\overline{BC})$.

Since $\angle BAC = \angle ADB = \angle CDA$, $\angle ABC = \angle ABD$, and $\angle ACB = \angle ACD$, by theorem 5, $\angle ACB = \angle BAD$ and $\angle CAD = \angle ABC$, so $\Delta ABD \approx \Delta DBA \approx \Delta DAC$.

Hence

$$\frac{a}{c} = \frac{m(\overline{BD})}{a} \text{ and } \frac{b}{c} = \frac{m(\overline{DC})}{b}$$

[15]In general, cut-the-knot is a fantastic resource.

so

$$a^2 = m(\overline{BD}) \cdot c \text{ and } b^2 = m(\overline{DC}) \cdot c.$$

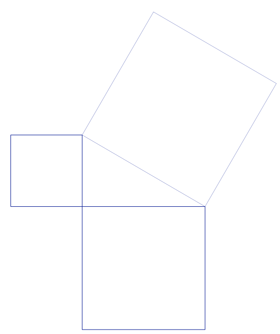
Puting this together we get

$$a^2 + b^2 = m(\overline{BD}) \cdot c + m(\overline{DC}) \cdot c = [m(\overline{BD}) + m(\overline{DC}] \cdot c = m(\overline{BC}) \cdot c = c^2$$

as desired.

□

This is in many ways an elegant proof, but by sidestepping area, it misses the point of the Pythagorean theorem. The point of the Pythagorean theorem is that if you put squares along the sides of the right triangle, their areas add up the way the theorem tells you:

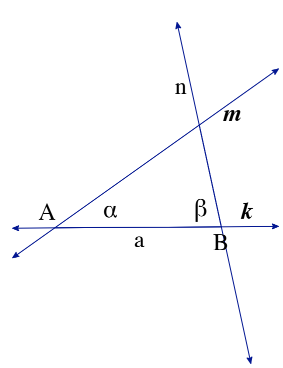


I.e.,what the Pythagorean theorem really says is that the big square in the diagram above has area equal to the sum of the areas of the two smaller squares. So a satisfactory proof needs to involve area. We'll do that in the next section.

The next two theorems prove two other criteria for triangle congruence, known as ASA and SAS. Some presentations of Euclidean geometry simply include them as axioms. Which sort of misses the point.

Theorem 9. *(ASA) Let $\alpha + \beta < \pi$. Let $a \in \mathbb{R}, a > 0$. Up to congruence there is only one triangle ΔABC where $m(AB) = a, m(\angle CAB) = \alpha$, and $m(\angle ABC) = \beta$.*

Proof. We may assume A, B are fixed. Our goal is to show that C is unique.



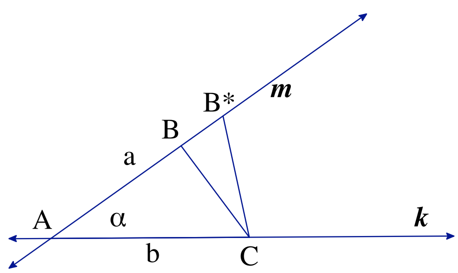
Let k be the line through $\overline{AB}$, m the line through A that makes angle α with k, n the line through B that makes angle β with k. By axiom 6, m, n are unique. By axiom 1, they meet in a unique point, C. □

Why is this the same as saying that if $m(\overline{AB}) = m(\overline{A'B'})$, $m(\angle CAB) = m(\angle C'A'B)'$ and $m(\angle CBA) = m(\angle C'B'A)'$ then $\Delta ABC \cong \Delta A'B'C'$? Given ΔABC and $\Delta A'B'C'$ satisfying these requirements, you can arrange them so that $A = A'$ and $B = B'$. But then you have the hypothesis of the theorem as stated.

As similar argument shows that the next theorem is the same as saying: if $m(\overline{AB}) = m(\overline{A'B'})$ and $m(\overline{CB}) = m(\overline{C'B'})$ and $m(\angle ABC) = m(\angle A'B'C')$ then $\Delta ABC \cong \Delta A'B'C'$.

Theorem 10. *(SAS) Let a, b be given lengths, α a given angle. Then, up to congruence, there is a unique triangle ΔABC with $m(\overline{AB}) = a, m(\overline{AC}) = b, \angle BAC = \alpha$.*

Proof. We may assume that A, C are fixed. Our goal is to show that B is unique.



So let k be the line through $\overline{AC}$, m be the line through A whose angle with k is α. By axiom 6, m is unique. Suppose there were two such triangles, $\Delta ABC, \Delta AB^*C$. Since m is unique, B, B^* both lie on m. Without loss of generality we may assume that B is between A and B^*. Then $a = m(\overline{AB^*}) = m(\overline{AB}) + m(\overline{BB^*}) = a + m(\overline{BB^*})$ so $m(\overline{BB^*}) = 0$ and $B = B^*$.

□

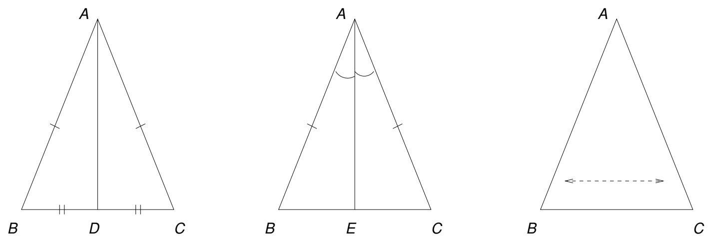
A natural question is: what about SSA? I.e., is there, up to congruence, a unique triangle ΔABC with $m(\overline{AB}) = a, m(\overline{BC}) = b$ and $m(\angle BCA) = \alpha$? If $\alpha = \pi/2$, the answer is "yes" by the Pythagorean theorem.

But in all other cases the answer is no, and it is a nice *Sketchpad* exercise given in the homework.

We apply axiom 10 and theorem 10:

Theorem 11. *(Thales' Theorem, a.k.a. the Isosceles Triangle Theorem (ITT)) The base angles of an iosceles triangle are equal. That is, if $AB = AC$ then $\angle ABC \cong \angle ACB$.*

We give three proofs. Here are the diagrams:



Proof 1 Proof 2 Proof 3

First proof. Let D be the midpoint of $\overline{BC}$. Then $AD = AD$; $AB = AC$ (by hypothesis); and $BD = CD$ (by definition of midpoint). So $\Delta ABD \cong \Delta ACD$ by SSS. By definition of congruence, $\angle ABC \cong \angle ACB$. □

Second proof. Let ℓ be the bisector of angle $\angle BAC$, and let E be the point where ℓ meets $\overline{BC}$. Then $AE = AE$; $AB = AC$ (by hypothesis); and $\angle BAE \cong \angle CAE$ (by definition of angle bisector So $\Delta ABD \cong \Delta ACD$ by SAS. Again, by definition of congruence, $\angle ABC \cong \angle ACB$. □

Third (and slickest) proof. Observe that $AB = AC$, $AC = AB$, and $BC = CB$. Therefore $\Delta ABC \cong \Delta BAC$ by SSS. In particular, $\angle ABC \cong \angle ACB$.

□

The real technique of the third proof is that the triangle is congruent to a reflected copy of itself. We didn't explicitly mention reflection in the proof, but that's what's really going on.

In a homework problem we'll show that the points D and E are actually the same point, but we can't assume that from the start. At this point you can't say "Let ℓ be the bisector of $\angle BAC$, and let ℓ meet $\overline{BC}$ at its midpoint", since we haven't proved that the bisector (of the angle where the equal sides meet) meets the base at its midpoint.

1.5 A few words about area

In this section we develop a few very basic facts about area.

First of all, the units are not the same units we use to measure length. If length is measured in feet, area is measured in square feet. If length is measured in meters, area is measured in square meters. If length is measured in light-years, area is measured in square light-years. The number we assign as an area tells us how many squares of a certain size fit into the shape — maybe they have to be chopped up and rearranged, but we are always talking about how many squares fit. You had a taste of this when you learned about the Riemann integral in calculus — there we talked about rectangles, but a rectangle is easily converted into a bunch of squares with maybe a part of a square left over.

Second, as you know from calculus, calculating areas is not trivial — that is what the theory of integration is about. In fact, deciding what objects have area is not always easy. In general, the interior of any simple closed curve has a well-defined positive area. What is a simple closed curve? A curve is simple if it doesn't cross itself, and closed if it ends up where it started. So circles, triangles, squares, rectangles, and so on are simple closed curves. The geometric figures we work with in this and the next chapter are all simple closed curves.

Not everything has a well-defined area. For example, if you look at all the points (x, y) so if x is rational then $-x < y < x$ but if x is irrational then $-x^2 < y < x^2$, how would you define its area?[16]

Here are the axioms for area:

Axiom 16. *Two congruent figures have the same area.*

Axiom 17. *If $P \subseteq Q$ and P, Q have area, then area$(P) \leq$ area(Q).*

Axiom 18. *If P, Q have area, then area$(P \cup Q) =$ area$(P)+$ area$(Q)-$ area$(P \cap Q)$.*[17]

Axiom 19. *The area of a rectangle is its length $\times$ height .*

An immediate corollary of axioms 19 is that the area of a square is the square of the length of one side, and that the area of a right triangle is $\frac{1}{2}ab$, where a, b are the lengths of the two sides forming the right angle. (Notice that we've dropped the units. This is common mathematical practice, but it drives physicists and engineers crazy. Even when you don't mention units, you should always know what they are. In particular, the units in these formulas are square units.)

Another corollary is that if one figure is similar to another, with constant of proportionality r, then the first figure's area is r^2 times the second figure's area. It's easy to see for rectangles: If rectangle P has length $r\ell$ and height rh, and rectangle Q has length rl and height rh then area$(P) = r^2 lh = r^2 \cdot$area(Q). In general, you look at the Riemann sum approach: estimate each area by rectangles, and notice that each rectangle in one Riemann sum has area r^2 times the corresponding rectangle in the other Riemann sum.

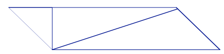
As an example, if one circle has radius 17 and another circle has radius 4, then the radius of the first circle is $(\frac{17}{4})$ times the radius of the second circle, i.e., the constant of proportionality is $\frac{17}{4}$. Hence the area of the first circle is $(\frac{17}{4})^2$ times the area of the second. And we would know that even if we didn't know the formula for the area of a circle — all we need to know is that all circles are similar to each other.

For a second example, suppose two figures are similar, and the perimeter of the first is $\frac{5}{12}$ times the perimeter of the second. Then the area of the first figure is $(\frac{5}{12})^2$ times the area of the second figure. (This example only works if the figures are similar: Take a square whose sides all have length 5, and a rectangle whose length is 2 and whose height is 8. Both have perimeters of 20, but the square has area 25 and the rectangle has area 16.)

We've got formulas for the area of a rectangle and the area of a right triangle. Here are diagrams showing how to derive the formulas for the area of a parallelogram and for the area of any triangle:

[16] The branch of mathematics which studies the area of complicated objects is called measure theory.

[17] And if this looks like the formula for the probability of the event "P or Q" — it should, because that's what it is.



In the top diagram, we took a parallelogram, chopped off a triangle on the left via a perpendicular line segment from a vertex, and moved the triangle to the right to get a rectangle. By axioms 16 and 18, moving things around like that doesn't change the total area. We know the area of the rectangle. So we know the area of the parallelogram: base $\times$ height (where the length of the perpendicular line segment is the height of the parallelogram). To turn this into a rigorous proof you have to show that when we move that triangle we indeed have a rectangle. You will give the details in a homework exercise.

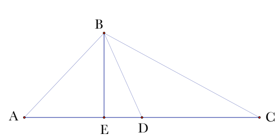
In the bottom diagram, we took a triangle, rotated it by π about the center of one side to get a parallelogram with area base $\times$ height, which doubles the area of the triangle by axioms 16 and 18. So the area of a triangle is $\frac{1}{2}$ base $\times$ height. To turn this into a proof you have to show that when we join the two triangles we get a parallelogram. Again, you'll give the details in a homework exercise.

The formula for the area of a triangle gives a very pretty theorem.

Theorem 12. *A median divides a triangle into two triangles of equal area.*

Here, a median is the line joining a vertex of a triangle to the midpoint of the opposite side.

Proof. Consider ΔABC.

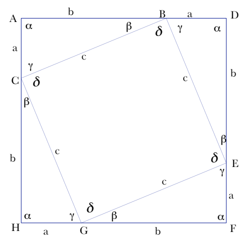


Let D be the midpoint of $\overline{AC}$, so BD is a median. Let $\overline{BE}$ be an altitude (i.e., a perpendicular line joining B to the opposite side.). The area of $\Delta ABD = \frac{1}{2}m(\overline{BE})m(\overline{AD})$. The area of $\Delta CBD = \frac{1}{2}m(\overline{BE})m(\overline{CD})$. But $m(\overline{CD}) = m(\overline{AD})$, so ΔABD and ΔCBD have the same area. $\square$

The basic area formulas also can be used to prove the Pythagorean theorem. The previous link to cut-the-knot gives many proofs, here is one.

Theorem 13. *(Pythagorean theorem) Let a, b, c be the lengths of the sides of a right triangle, where the side whose length is c is opposite the right angle. Then $a^2 + b^2 = c^2$.*

Proof. Let's begin with ΔABC, where $\alpha = \angle BAC = \pi/2$, $m(\overline{AC}) = a, m(\overline{AB}) = b$, and $m(\overline{BC}) = c$. We then duplicate the triangle and arrange our copies as in the diagram below, where A, B, D are collinear (so $\overline{A}D$ is a line segment), D, E, F are collinear (so $\overline{D}F$ is a line segment), F, G, H are collinear (so $\overline{F}H$ is a line segment), and H, C, A are collinear (so $\overline{H}A$ is a line segment):



We can do this because $\beta + \gamma < \pi$. And we know all the triangles are congruent by SSS.

Consider the quadrilateral $ADFH$. Its sides are all of length $a + b$, and its interior angles are right angles, so it's a square.

Now consider the quadrilateral $BEGC$. Its sides are all of length c, and its interior angles are all of size $\pi - (\beta + \gamma)$. But by theorem 5, $\alpha + \beta + \gamma = \pi$, i.e., $\pi - (\beta + \gamma) = \alpha = \pi/2$. So $BEGC$ is also a square, with area c^2.

The area of the square $ADFH$ is $(a + b)^2$. But the area of $ADFH$ is also 4(area of ABC)+ area of $BEGC = 4(ab/2) + c^2 = 2ab + c^2$.

I.e., $a^2 + 2ab + b^2 = 2ab + c^2$. So $a^2 + b^2 = c^2$ as desired.

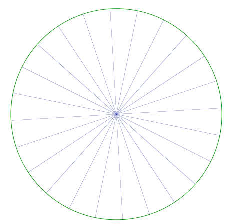
□

There are many proofs of the Pythagorean theorem, known in many cultures. The one above was known in ancient China.

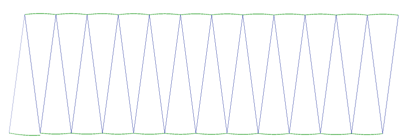
For the next theorem we'll leave our careful construction from axiomatics and leap ahead into the advanced notion of limit. While the formal definition of limit only arose in the 19th century, the kind of argument we'll give below was known in the ancient world.

Theorem 14. *Let r be the radius of a circle. Then the area of the circle is πr^2.*

Proof. In slightly different language, axiom 13 says that the circumference of a circle is $2\pi r$. So let's chop up the circle into congruent wedges (so they all have the same angle, which we'll call α). Let's use lots of them (in this case 24), as follows:



and then let's rearrange the wedges, as follows, into a figure we'll call W_α (because it depends on the angle α):



This figure W_α (we'll call it an almost parallelogram) is pretty close to a parallelogram P_α (which we haven't drawn). The width of P_α is pretty close to half the circumference, i.e., πr. The height of P_α is pretty close to r. The smaller the angle of the wedges, the closer the area of the almost parallelogram is to the area of the parallelogram. Also, the smaller the angle of the wedges, the closer the height of the parallelogram is to r and the closer the width of the parallelogram is to πr. So the area of P_α is pretty close to πr^2; in modern language, $\lim_{\alpha \to 0}$ (area of P_α) $= \pi r^2$. So the area of the almost parallelogram W_α is approximated by the area of the parallelogram P_α which approaches πr^2. But the area of the almost parallelogram is the sum of the areas of the wedges which is exactly the area of the circle.

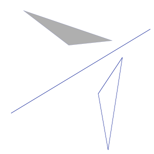
□

This proof is an example of a proof by continuity, that is, you have a continuous process (in this case, the angle of the wedges is getting smaller); the limit of that process gives you what you want. This case is particularly interesting, because the area of the almost parallelogram W_α = the area of the circle, no matter what α is. What changes is the area of the approximating parallelogram P_α.

2 Transformational geometry

2.1 The basic intuition

When we talk about transformations like reflection or rotation informally, we think of moving an object in unmoving space. For example, in the following diagram, when we say that the shaded triangle is the reflection of the unshaded triangle about the line we think about physically picking up the unshaded triangle and reflecting it about the line:



This is not how mathematicians think of transformations. To a mathematician, it is space itself (2D or 3D or...) that is being transformed. The shapes just go for the ride.

To understand how this works, let's focus on the following basic transformations of the plane: *translations* along a vector; *reflections* about a line; *rotations* by an angle about a point. To help us consider these as transformations of the plane itself, you've been given a transparency sheet. You'll keep a piece of paper fixed on your desk. You'll move the transparency. The transparency represents what happens when you move the entire plane. The paper that stays fixed tells you where you started from.

Project 1. Start by drawing **a dot** on your paper. Take a transparency sheet, put it over your paper, and trace the dot. What can you do to the transparency (i.e., plane) so that the dots will still coincide? I.e., which translations, reflections, and rotations leave the dot fixed?

Now draw **two dots** on the bottom sheet and trace them on the transparency. The dots should be two different colors, say red and blue. What can you do to the transparency so the dots still coincide, red on red, blue on blue? I.e., which translations, reflections, and rotations leave the two dots fixed? Which translations, reflections, and rotations put the blue dot on top of the red dot and the red dot on top of the blue dot?

Now try this with **three dots** (in three different colors, say red, blue and green) which are not collinear. Which translations, reflections, and rotations leave the three dots fixed? What about **three dots of the same color?** What about **two red dots and one blue dot?**

Now try this with a **straight line.** (Of course you can't draw an infinitely long line on the paper, but you can draw a line segment and pretend.) Which translations, reflections, and rotations leave the line fixed? Which translations, reflections, and rotations don't leave the line fixed but still leave it lying on top of itself?

The idea of transformational geometry is that by studying the behavior of individual transformations, and how different transformations interact with each other, we can understand the objects being transformed.

2.2 The basics

Let's formally define what a transformation is:

Definition 6. *A transformation of a space S is a map φ from the space S to itself which is 1-1 and onto.*

Transformational geometry has two aspects: it is the study of transformations of geometric space(s) *and* it studies geometry using transformations.

The first thing people realized when they started to get interested in transformations in their own right (in the 19th century) was that there was an algebra associated with them. Because of this, the development of the study of transformations was closely bound up with the development of abstract algebra.

In particular, people realized that transformations behaved a lot like numbers.

Closure under an operation. The notation for composition is $\varphi \circ \psi$, which means: first do ψ, then do φ. Composing two transformations results in a transformation, i.e., if φ and ψ are transformations, so is $\varphi \circ \psi$.

Existence of an inverse. Recall the definition of inverse: $\varphi^{\leftarrow}(x) = y$ iff $\varphi(y) = x$. Because a transformation φ is 1-1 onto, it has an inverse $\varphi^{\leftarrow}$.

Existence of an identity element. The identity transformation id is the transformation that leaves everything alone: $id(x) = x$ for all x.

Because we can compose transformations, because there is an identity transformation, because every transformation has an inverse, and because composition is associative,[18] the transformations of a given space form an algebraic structure called a group.[19] We will discuss this in more depth later. And we will see that there are other, smaller sets of transformations which have a group structure.

Here are some examples of transformations of $\mathbb{R}^2$ (the plane):

1. Reflecting the plane across a line ℓ: each point p is moved to a point p^* so that $\overline{pp^*} \perp \ell$ and the distance from p to ℓ equals the distance from p^* to ℓ.

2. Rotating the plane about a point c by a given angle θ: each point p is moved to a point p^* so $\overline{cp} = \overline{cp^*}$ and $\angle pcp^* = \theta$.

3. Translating a plane by a given vector $\vec{v}$: if $\vec{v}$ is the vector going from the origin to the point (a, b), each point (x, y) is moved to $(x + a, y + b)$.

4. Contracting or expanding the plane about a point c by a constant factor $k \neq 0$: each point p is moved to a point p^* on the line ℓ extending the line segment $\overline{cp}$; the distance from c to $p^* = |k| \times$ the distance from c to p; if $k > 0$ then p^* is on the same side of the line ℓ as p; if $k < 0$ then p^* is on the other side of ℓ from p.

5. Doing absolutely nothing (i.e., sending every point to itself). This is called the *identity transformation.* It might not look very exciting, but it's an extremely important transformation, and it's certainly 1-1 and onto.

[18] $\varphi \circ (\psi \circ \rho) = (\varphi \circ \psi) \circ \rho$.

[19] Groups are studied in Math 558.

All of these kinds of transformations can be applied to $\mathbb{R}^3$ (3-space) as well, with some modification. For example, reflection in $\mathbb{R}^3$ takes place across a plane, not across a line, and rotation occurs around a line, not a point. (Question for those who have had some linear algebra or vector calculus: How do these various transformations behave in $\mathbb{R}^n$?)

Here are some functions that are *not* transformations:

1. The function taking all points $(x, y) \in \mathbb{R}^2$ to the point $x \in \mathbb{R}$. It's not 1-1, and the space you start with isn't the space you end up with.[20]

2. The map taking all points $x \in \mathbb{R}$ to the point $(x, 0) \in \mathbb{R}^2$. It's not onto, and the space you start with isn't the space you end up with (even though $\mathbb{R}$ is geometrically isomorphic to its image).

3. Folding a plane across a line ℓ: this is 2-1 rather than 1-1 off ℓ, and it isn't onto the whole plane.

4. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. It's neither 1-1 nor onto. (On the other hand, the function $g(x) = x^3$ *is* a transformation.)

An important note. When we talk about transformations, we only care about where points end up, not how they get there. For example, the following three "recipes" all describe the same transformation:

- rotate the plane by $\frac{\pi}{2}$ about the origin.
- rotate the plane by $-\frac{3\pi}{2}$ about the origin.
- reflect the plane across the x-axis, then reflect across the line $y = x$.

To be precise: we consider two transformations $\varphi : S \to S$ and $\psi : S \to S$ to be the same iff $\varphi(p) = \psi(p)$ for all points p in S. It doesn't matter if φ and ψ are described by different recipes as long as they produce the same results.

Reflections, rotations and translations have a special property: they don't change the distance between any pair of points. That is, these transformations are *isometries*.[21] We'll come to back this idea later. For now, just notice that not every transformation is an isometry (for example, dilations are perfectly good transformations that are not isometries).

2.3 Notation for transformations

Here are the major types of transformations of the plane that we'll study, together with their notation:

[20] This is still an interesting map geometrically, even though it isn't a transformation. It's an example of *projection*; in this case, projecting a plane onto a line.

[21] From Greek: "iso" = same, "metry" = distance.

Transformation	Notation
Identity	id
Reflection across line ℓ	r_ℓ
Rotation about point x by angle θ	$\rho_{x,\theta}$
Translation by vector $\vec{v}$	$\tau\vec{v}$
"Glide reflection": first reflect across line ℓ, then translate by vector $\vec{v}$	$\gamma_{\ell,\vec{v}}$
Dilation about point x with constant factor k	$\delta_{x,k}$

Most of these are Greek letters which are mnemonics for the type of transformation they denote (ρ = rho = rotation; τ = tau = translation; γ = gamma = glide reflection; δ = delta = dilation). The exception is the Latin letter r for reflection. id is called the trivial symmetry (because it doesn't do anything).

In some sense, these are the "most interesting" kinds of transformations (though certainly not all possible transformations). The sense in which this is true will be stated clearly in several theorems.

This notation makes it easier to describe relations between transformations. For example, the fact that reflecting about a line twice ends up doing nothing can be expressed by the following equation: $r_\ell \circ r_\ell = id$. Instead of saying, "Rotating counterclockwise about a point x by angle θ is the inverse transformation of rotating clockwise about x by the same θ" — which is true, but extremely awkward — we can write the equation $(\rho_{x,\theta})^{-1} = \rho_{x,-\theta}$.

Observe that we are writing equations about transformations without reference to the points they are transforming. It is very convenient to be able to do this!

Here are some trivial facts:

Fact 1. *1. Every reflection is its own inverse:* $r_\ell \circ r_\ell = id$.

2. The inverse of rotation about a point by θ *is rotation about the same point by* $-\theta$*:* $\rho_{p,\theta} \circ \rho_{p,-\theta} = id$.

3. If first you rotate about a point by α *and then rotate about the same point by* β*, you've rotated about the point by* $\alpha + \beta$*:* $\rho_{p,\beta} \circ \rho_{p,\alpha} = \rho_{p,\alpha+\beta}$.

4. Dilating about a point by a factor of k *and then by a factor of* k^* *is the same as dilating about this point by a factor of* kk^**:* $\delta_{p,k^*} \circ \delta_{p,k} = \delta_{p,kk^*}$.

5. Translating by a vector $\vec{v}$ *followed by translating by a vector* $\vec{w}$ *is the same as translating by the vector* $\vec{v} + \vec{w}$*:* $\tau_{\vec{w}} \circ \tau_{\vec{v}} = \tau_{\vec{v}+\vec{w}}$.

Note how clumsy it is to say these things in regular language, and how elegant it is in mathematical notation.

Note that in general the commutative law does not hold: it is unusual for $\varphi \circ \psi = \psi \circ \varphi$.

2.4 Groups

We've already alluded to algebraic properties of transformations. In this section we look at this more closely.

Transformational geometry has two aspects: it is the study of transformations of geometric space(s) and it studies geometry using transformations.

While transformations were important in early geometry (Euclid's definition of congruence depends on the notion of isometry), the ancient Greeks did not study transformations in their own right. It wasn't until the 19th century that mathematicians began to focus on the study of transformation, and the first thing they noticed was that there was an algebra associated with them. Because of this, the development of the study of transformations was closely bound up with the development of abstract algebra.

In particular, people realized that transformations behaved a lot like numbers in the following ways.

- *Closure.* Since transformations are 1-1 and onto functions, you can compose any two transformations to get another transformation. Specifically, if φ and ψ are transformations of a space S, then so is $\varphi \circ \psi$. Remember, this means "first do ψ, then do φ", i.e.,

  $$(\varphi \circ \psi)(p) = \varphi(\psi(p)).$$

  It takes a little bit of checking to confirm that $\varphi \circ \psi$ is 1-1 and onto (this is left as an exercise).

- *Existence of an inverse.* Recall the definition of the inverse of a function: $\varphi^{-1}(p) = q$ if $\varphi(q) = p$. For φ to have an inverse, it needs to be 1-1, but that's not a problem because it's part of the definition of a transformation. Also, inverting a function switches its domain and range, but in this case both domain and range are just S. So φ^{-1} is also a transformation of S.

- *Existence of an identity element.* The *identity transformation*, denoted "id", is the transformation that leaves everything alone: $id(p) = p$ for all points $p \in S$. We've seen this before; it's certainly 1-1 and onto, so it's a transformation.

- *Associativity.* If φ, ψ and ω are three transformations of a space, then $\varphi \circ (\psi \circ \omega) = (\varphi \circ \psi) \circ \omega$. Indeed, for any point $x \in S$,

  $$(\varphi \circ (\psi \circ \omega))(x) = \varphi(\psi(\omega(x))) = ((\varphi \circ \psi) \circ \omega)(x).$$

Does this sound familiar? It should. It's an expansion of what we said after definition 13.

These four properties show up together in a lot of places.

- Consider the set $\mathbb{R}$ of real numbers and the operation of addition. If you add two real numbers, you get a real number. Every real number has an additive inverse, namely its negative. There's an additive identity, namely 0. And addition is associative: $(a + b) + c = a + (b + c)$. (One way to think about associativity is that it doesn't matter how you parenthesize an expression like $a + b + c$.)

- Every vector space[22] (under addition) has these four properties.

[22] studied in linear algebra

- Consider the set of *nonzero* real numbers and the operation of multiplication. Again, the operation is closed and associative. The number 1 is the identity element, and every real number r has the multiplicative inverse $1/r$.

Since we can compose two transformations to get a new transformation; since there is an identity transformation id; since every transformation has an inverse; and since composition is associative, the transformations of a given space, with the operation of composition, form a *group*.

One big difference between the group of real numbers under addition and the group of transformations under composition is that addition is commutative, but composition of transformations is not. That is, if r, s are real numbers, then $r + s = s + r$, but if φ, ψ are transformations, then it is rarely the case that $\varphi \circ \psi = \psi \circ \varphi$. That's okay — group operations don't have to be commutative.

The idea of a group is absolutely fundamental in mathematics.[23] As we'll see later on, groups come up all the time in geometry. In some sense, a lot of modern geometry is about groups just as much as it is about things like points and lines.

2.5 Transformations and geometry

In sections 2.2 and 2.3 we looked at transformations by themselves. Now we look at the interaction between transformations and sets of points. Until section 2.8, this chapter will focus on transformations of the plane $\mathbb{R}^2$.

First, some notation: if $\varphi : S \to S$ is a transformation, and $A \subset X$, then $\varphi[A]$ is the image of A, i.e., $\varphi[A] = \{\varphi(p) : p \in A\}$.

Definition 7. *A transformation φ fixes a point p iff $\varphi(p) = p$. It fixes a set A iff for all $p \in A, \varphi(p) = p$. It is a symmetry of a set A iff $\varphi[A] = A$. id is a symmetry of every set, called the trivial symmetry.*

Using this language, in section 2.1 we asked: which transformations leave a single point fixed? two points fixed? three points fixed? a line fixed? which transformations are symmetries of two points? of a line?

Examples

1. $\rho_{p,\theta}$ fixes p, no matter what θ is.
2. If $p \in \ell$ then $\rho_{p,\pi}$ is a symmetry of ℓ but does not fix ℓ.
3. r_ℓ fixes ℓ.
4. If $m \perp \ell$ then r_ℓ is a symmetry of m but does not fix m.
5. If $\vec{v} \neq \vec{0}$ then $\tau_{\vec{v}}$ has no fixed points.
6. If $\ell \parallel \vec{v}$ then $\tau_{\vec{v}}$ is a symmetry of ℓ.

Fact 2. *If φ fixes S then φ is a symmetry of S.*

[23] To learn more about groups, take Math 558.

The next step is to categorize transformations according to how much geometric structure they preserve.

For example, consider the transformation of the plane that takes the point (x, y) to the point (x, y^3); let's call this transformation φ. Note that φ is 1-1 and onto, as required. φ takes the line $y = x$ and turns it into the curve $y = x^3$. So it doesn't preserve straight lines. And this means it doesn't preserve the angle π, so it doesn't preserve angles. Which means that it doesn't preserve distance — by SSS, if it preserved distance it would have to preserve angles.

φ is an example of the kind of transformation we are not interested in.

Definition 8. *Let $\varphi : S \to S$ be a transformation.*

- *φ is rigid (also called an isometry) iff it preserves distances, that is, the distance between any two points p, q is the same as the distance between $\varphi(p)$ and $\varphi(q)$.*
- *φ is a similarity iff it preserves angles, that is, for any points p, q, s, $\angle pqs = \angle\varphi(p)\varphi(q)\varphi(s)$.*
- *φ is an affine map if it preserves straight lines, that is, if ℓ is a line, so is $\varphi[\ell]$.*[24]

In this course are interested in isometries and similarities. Affine maps are studied in more advanced geometry courses.

We've already observed that rotations, reflections, and translations are isometries. Dilations are similarities but not isometries. Here's a map which is affine: $\varphi((x, y)) = (x, 2y)$ — φ takes the line $y = mx + b$ to the line $y = 2mx + 2b$. It is not a similarity because φ leaves the x-axis fixed, and if $A = (1, 0), B = (0, 0), C = (1, 1)$ then $\angle ABC = \frac{\pi}{4}$ but, since $\varphi(A) = A, \varphi(B) = B, \varphi(C) = (1, 2)$, $\angle\varphi(A)\varphi(B)\varphi(C) > \frac{\pi}{2}$.

Fact 3. *1. Every isometry is a similarity.*

2. Every similarity is affine.

3. Not every affine map is a similarity and not every similarity is an isometry.

Proof. We've already given examples to show fact 3.3. We need to prove 3.1 and 3.2.

1. Suppose φ is an isometry. Let p, q, r be three non-collinear points and let $\varphi(p) = p'$, $\varphi(q) = q'$ and $\varphi(r) = r'$. We want to prove that $\angle pqr = \angle p'q'r'$. By definition of isometry, we know that $\overline{pq} = \overline{p'q'}'$, $\overline{pr} = \overline{p'r'}'$,, and $\overline{qr} = \overline{q'r'}$. But then $\Delta pqr = \Delta p'q'r'$ by SSS. Therefore $\angle pqr = \angle p'q'r'$,.

What if p, q, r are collinear, $\angle pqr = \pi$, i.e., q is between p, r? Suppose $\varphi(p) = p'$, $\varphi(q) = q'$ and $\varphi(r) = r'$. Then $\psi(r') = r, \psi(q') = q, \psi(r') = r$. By the previous paragraph, if $p', q'r'$ are not collinear, neither are p, q, r. So, by the contrapositive, p', q', r' are collinear. Since $\overline{pq} = \overline{p'q'}, \overline{qr} = \overline{q'r'}$, and $\overline{rp} = \overline{r'p'}$, if q is between p, r then, by axiom 2, q' is between p', r', i.e., $\angle p'q'r' = \pi$.

The case for $\angle pqr = 0$ is similar.

Hence φ preserves angles and is a similarity.

[24]You might think that "affine" should be called "linear," but that name is reserved for a special kind of affine map: linear maps are affine maps which fix the origin, that is, they are concerned with preserving algebraic structure as well as (in fact, more than) geometric structure.

2. If φ is a similarity, then it preserves angles, so in particular it preserves the angle π. That is, it preserves straight lines. (More precisely, if three points are collinear, then so are their images under φ, and if three points are not collinear, then neither are their images.)

□

Fact 4. *The isometries form a group, the similarities form a larger group, and the affine maps form a still larger group.*

Proof. We'll just consider the case of isometries — the proofs that the other two sets are groups work exactly the same way. To prove that the set of isometries forms a group, we show that it satisfies the four conditions listed in Section 2.4.

1. *Closure.* We need to show that the composition of two isometries is an isometry, i.e., that if φ and ψ preserve distance, then so does $\psi \circ \varphi$. Let p, q be any two points and let $p' = \varphi(p)$, $q' = \varphi(q)$, $p'' = \psi(p')$, $q'' = \psi(q')$. Then $\overline{pq} = \overline{p'q'}$ (because φ is an isometry) and $\overline{p'q'} = \overline{p''q''}$ (because ψ is an isometry), but that means that $\overline{pq} = \overline{p''q''}$. Note that $p'' = (\psi \circ \varphi)(p)$ and $q'' = (\psi \circ \varphi)(q)$. Therefore, $\psi \circ \varphi$ is an isometry.

2. *Inverses.* Suppose that φ is an isometry. In particular φ is a transformation, so it has an inverse transformation φ^{-1}, which we want to show is an isometry. So, let p, q be any two points and let $p^* = \varphi^{-1}(p)$, $q^* = \varphi^{-1}(q)$. Then $\varphi(p^*) = p$ and $\varphi(q^*) = q$. Since φ is an isometry, $\overline{pq} = \overline{p^*q^*}$. Hence φ^{-1} is an isometry.

3. *Identity element.* The identity transformation is an isometry, because clearly $\overline{pq} = \overline{id(p)id(q)}$.

4. *Associativity.* Isometries are functions, so their composition satisfies the associative law. □

2.6 The structure of isometries

In this section we focus on isometries.

There are three major theorems about isometries. Two of their proofs are fairly complicated, so we won't give them. But we will give applications.

Theorem 15. *(The three-point theorem) An isometry of the plane is determined by any three non-collinear points. I.e., if φ, ψ are isometries, p, q, s are non-collinear points, $\varphi(p) = \psi(p), \varphi(q) = \psi(q)$, and $\varphi(s) = \psi(s)$, then $\varphi = \psi$.*

When you did the 3-dot example in project 1 you experienced the truth of this theorem, but a formal proof is quite technical.

The three-point theorem is useful for checking whether two isometries are equal: all you have to do is check that they agree on each of three non-collinear points. (While formally any three non-collinear points will do, in practice, to produce a clear explanation you may have to use some ingenuity in choosing those points appropriately.)

Here are three applications of the 3-point theorem.

Example 1. *The composition of perpendicular reflections.*

Suppose $\ell \perp m$ and p is the intersection of ℓ, m. We want to show that $\rho_{p,\pi/2} = r_m \circ r_\ell$. The trick is to pick the three points cleverly. So we set $q \neq p$ be on ℓ and $s \neq p$ be on m What is $\rho_{p,\pi/2}(p)?r_m \circ r_\ell(p)$? What is $\rho_{p,\pi/2}(q)?r_m \circ r_\ell(q)$? What is $\rho_{p,\pi/2}(s)?r_m \circ r_\ell(s)$? For all three points, the results were the same. So the transformations are the same.

Example 2. *The composition of parallel reflections.*

Let n, m be distinct parallel lines. We want to show that $r_m \circ r_n$ is a reflection, and we want to know exactly which reflection it is. Again, we want to pick our points cleverly. Let's try putting p, q on m and r on n. What is $r_m \circ r_n(p)?r_m \circ r_n(q)?r_m \circ r_n(r)$?

Example 3. *The inverse of a reflection.*

Show that a reflection is its own inverse: for any ℓ, $r_\ell \circ r_\ell = id$. What three points would be good to choose? Remember, they cannot all lie on ℓ.

The number three crops up again in the next theorem:

Theorem 16. *(The three reflection theorem) Every isometry is the composition of at most three reflections.*

Proof. We are going to prove this using *Sketchpad* as our worksheet, so that all diagrams are made in *Sketchpad*.

On your screen you have three non-collinear points labeled A, B, C, and three other points A^*, B^*, C^* where $\Delta ABC \cong \Delta A^*B^*C^*$. So there is an isometry that takes A to A^*, B to B^*, and C to C^*. Call this isometry ψ.

Now we have to come up with a way to get from A, B, C to A^*, B^*, C^* in at most three steps, all of them reflections, and do it in a way so that it clearly generalizes to any isometry.

The first reflection is r_ℓ, where ℓ is the perpendicular bisector of $\overline{AA^*}$. r_ℓ necessarily moves A to A^*. If $\psi = r_\ell$ you're done. In the example on your screen, it doesn't so you're not.

Okay, now we've gotten A moved to A^*. Let $B^{**} = r_\ell(B), C^{**} = r_\ell(C)$.

The second reflection is r_m, where m is the perpendicular bisector of $\overline{B^*B^{**}}$ r_m moves B^{**} to B^*. This leaves A^* fixed, i.e., $r_m \circ r_\ell(A) = A^*$. Why? Because of isosceles triangles: $\overline{A^*B^{**}} = \overline{A^*B^*}$, so A^* is on the perpendicular bisector of $\overline{B^*B^{**}}$. And, by definition, $r_m \circ r_\ell(B) = B^*$.

If $\psi = r_m \circ r_\ell$ you're done. In the example on your screen it doesn't so you're not.

At this point we've moved A to A^*, B to B^*, but C has moved to a point C^{***} which need not equal C^*.

The third reflection is r_n, where n is the perpendicular bisector of $\overline{C^{***}C^*}$. Since $\overline{A^*C^{***}} = \overline{A^*C^*}$ and $\overline{B^*C^{***}} = \overline{B^*C^*}$, we have that both A^*, B^* lie on this perpendicular bisector, so r_n leaves them fixed.

Hence $r_n \circ r_m \circ r_\ell(A) = A^*, r_n \circ r_m \circ r_\ell(B) = B^*, r_n \circ r_m \circ r_\ell(C) = C^*$. I.e., by the 3-point theorem, $\psi = r_n \circ r_m \circ r_\ell$.

□

The three reflection theorem is a beautiful theorem — it tells us that reflections generate all the isometries of the plane, and that you need at most three of them. Its main application is the proof of the next theorem, which, in case you thought everything was about the number 3, is about the number 4:

Theorem 17. *(Isometry classification theorem)There are only four types of isometries of the plane: reflection, rotation, translation, glide reflection.*

Note that the isometry classification theorem applies to the identity, since the identity is translation by the vector of length 0 and also rotation about any point by 2π.

There are many ways to prove the isometry classification theorem, all of them tedious, so we won't give a proof. But all of the proofs rely to some extent on the three reflection theorem.

The isometry classification theorem has a nice corollary:

Corollary 2. *Let φ be an isometry. Either φ is a symmetry of some line or it fixes a point.*

Proof. By the isometry classification theorem, there are only four cases to consider.

- If φ is a reflection r_ℓ, then φ fixes every point on ℓ, so it certainly is a symmetry of ℓ.
- If φ is a rotation $\rho_{p,\theta}$, then it fixes the point p.
- If φ is a translation $\tau_{\vec{v}}$, then it is a symmetry of any line parallel to the translation vector $\vec{v}$.
- If φ is a glide reflection $\gamma_{\ell,\vec{v}}$, then it is a symmetry of the line ℓ. □

2.7 Symmetries of bounded figures: two dimensions

What can we say about the set of symmetries of a figure?

First we have to be clear about our terms: a *figure* is built out of finitely many curves and line segments, and is bounded, i.e., contained inside some circle — if you can draw it on a finite page, it's bounded. This includes things like triangles, pentagons and circles, but not things like *filled-in* circles or infinite strips in the plane.[25]

Second, let's agree that when we talk about a symmetry of a figure, we restrict ourselves to isometries. This captures our intuition. There are many transformations that look like isometries in a small region of space but then do strange things outside it, and it complicates our discussion too much to talk about those.

The symmetries of a figure $\mathcal{F}$ are closed under composition; the identity transformation of the plane is a symmetry of $\mathcal{F}$; and each symmetry of $\mathcal{F}$ has an inverse which is also a symmetry of $\mathcal{F}$. So they form a group, which we'll call $Sym(\mathcal{F})$.

Finally, in this subsection, we are interested only in bounded convex figures.

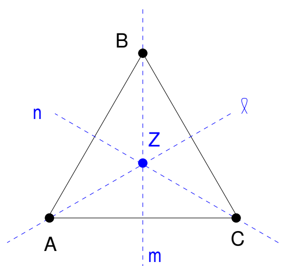
[25]More formally, a figure is a 0-dimensional bounded union of curves — in mathematics, straight lines and straight line segments are exactly the curves with curvature = 0.

Definition 9. *A figure $\mathcal{F}$ is convex iff it is a closed curve (that is, it divides the plane into an inside and an outside) and, if the points p, q are inside $\mathcal{F}$ then the line segment $\overline{pq}$ is also inside $\mathcal{F}$.*

For example, the san serif letters S, T, A are not closed curves, B is a closed curve but not convex, and O is convex.

Example 4. *The symmetries of an equilateral triangle.*

Suppose that $\mathcal{F}$ is an equilateral triangle ΔABC. In addition to the identity, the group $Sym(\Delta ABC)$ contains two nontrivial rotations — namely $\rho_{Z,2\pi/3}$ and $\rho_{Z,4\pi/3}$, where Z is the center of the triangle — and three reflections: r_ℓ, r_m, and r_n, where ℓ, m, n are the perpendicular bisectors of the three sides of the triangle.



That is,

$$Sym(\Delta ABC) = \{id,\ \rho_{Z,2\pi/3},\ \rho_{Z,4\pi/3},\ r_\ell,\ r_m,\ r_n\}.$$

Here's how we know that this is the complete list of symmetries. Every symmetry φ of ΔABC takes vertices to vertices; that is, φ is a permutation of the set $\{A, B, C\}$. (For example, r_m fixes B and swaps A with C, while $\rho_{Z,2\pi/3}$ maps A to C, B to A, and C to B. The identity, of course, fixes each of the three vertices.)

A triangle has three vertices, and there are at most $6 = 3! = 3 \times 2 \times 1$ possible ways to rearrange a set with three elements. So any ΔABC has cannot have more than 6 symmetries. Therefore, we know that the symmetries we've listed are **all** the symmetries of an equilateral triangle.

(In general, any isometry φ of a figure made up of straight line segments is determined by what it does to the vertices — it must take vertices to vertices. This always gives an upper limit for the number of isometries.)

Example 5. *The symmetries of convex quadrilaterals.*

A parallelogram is a convex quadrilateral whose only non-trivial symmetry is $\rho_{p,\pi}$ for some p (which we call the center of the parallelogram — it's where the diagonals meet).

A rectangle is a convex quadrilateral with one non-trivial rotational symmetry and two reflection symmetries. Knowing this, you can prove that the rotational symmetry must be $\rho_{p,\pi}$, and the reflection symmetries must be r_ℓ, r_m where $\ell \perp m$ and p is the intersection of ℓ, m.

Similarly, a square is a convex quadrilateral with three non-trivial rotational symmetries and four reflection symmetries.

Example 6. *The definition of a circle.*

Let p be a point in the plane. A figure $\mathcal{F}$ is a circle with center p iff every rotation through p is a symmetry of $\mathcal{F}$ and every reflection across a line containing p is also a symmetry of $\mathcal{F}$.

Example 7. *Subgroups of symmetry groups.*

Let's look at the rotational symmetries of an equilateral triangle: $\rho_{Z,2\pi/3}, \rho_{Z,4\pi/3}, id.$[26] They form a subgroup:[27] closed under composition, under inverse, including the identity, and associative. But the reflection symmetries are not a subgroup: $r_\ell \circ r_m$ is a rotation, not a reflection.

The rotational symmetries of a rectangle are also a subgroup (because any $\rho_{p,\pi}$ is its own inverse) but the reflection symmetries are not ($r_\ell \circ r_m$ is a rotation).

Similarly, the rotational symmetries of a square are a subgroup, but the reflection symmetries are not (because the composition of two distinct non-parallel reflections is a rotation).

Symmetry groups are very important in algebra. But they are also important in geometry. To a modern geometer — this means any geometer since the late 19th century — what characterizes a geometric figure isn't the number or characteristics of its sides and/or angles, but its symmetry group.

This may sound strange, but that's how modern geometers think.

Theorem 18. *Rotations and reflections may be symmetries of a bounded set; translations and glide reflections are not.*

We already know from the examples that rotations and reflections may be symmetries of a bounded set. Why can't translations and glide reflections be?

Proof. The proof depends on the following theorem: For any bounded set of points $\mathcal{F}$ there is a unique smallest circle $\mathcal{C}_\mathcal{F}$ that $\mathcal{F}$ fits inside — $\mathcal{C}_\mathcal{F}$ is called the *circumscribed circle* about $\mathcal{F}$.

Because $\mathcal{C}_\mathcal{F}$ is unique, any symmetry of $\mathcal{F}$ is a symmetry of $\mathcal{C}_\mathcal{F}$. And by the transformational geometry definition of a circle (example 6) only rotations and reflections are symmetries of a circle. □

The center of $\mathcal{C}_\mathcal{F}$ is called the *circumcenter* of $\mathcal{F}$.

Definition 10. *A polygon is a closed bounded figure whose sides are straight lines.*

For example, triangles, rectangles, pentagons, hexagons... are polygons. An n-gon is a polygon with n sides. In a polygon, the number of sides equals the number of angles.

In general, symmetries of polygons are not that interesting, in fact many polygons have no symmetry other than the identity. For example, if a triangle has no two angles equal then it has no non-trivial symmetries.

But one class of polygons has highly non-trivial symmetry groups, and that is the class of regular polygons.

[26] Recall that $id = \rho_{Z,0} = \rho_{Z,2\pi}$.

[27] A subgroup is exactly what you think it should be: a smaller group inside a larger one.

Definition 11. *A polygon is regular iff all of its sides have the same length and all of its angles are the same.*

Regular polygons intuitively look symmetric. Now we have a mathematical way of saying what that means: a regular polygon has a lot of symmetries.

The circumcenter of a regular polygon is easy to describe: it is the point of intersection of all of its angle bisectors. Which is also the point of intersection of all the perpendicular bisectors of its sides. How convenient! We call this point the *center* of the polygon.[28]

Suppose P is a regular n-sided polygon (or "n-gon"). How big is the set $Sym(P)$? Let c be the center of P. If $\varphi \in Sym(P)$ then $\varphi(c) = c$. Also, φ takes vertices to vertices, and it preserves adjacency among vertices; if v and w are vertices of P that are adjacent to each other, then so are $\varphi(v)$ and $\varphi(w)$. Fix v, w adjacent vertices. The points c, v, w are noncollinear, so by the three-point theorem, φ is determined by what it does to c, v, w. There are n possibilities for $\varphi(v)$ — any vertex of P could be $\varphi(v)$, and, under rotations, for any vertex z of P there is some rotation $\rho \in Sym(P)$ with $z = \rho(P)$. Once we've determined $\varphi(v)$, there are two possibilities for $\varphi(w)$, namely, the two vertices adjacent to $\varphi(v)$), and by reflection we see that if s, t are the vertices adjacent to $\varphi(v)$ then there is some $\psi \in Sym(P)$ with $\psi(v) = s$; and another ψ' with $\psi'(v) = t$. So we see that P has exactly $2n$ symmetries. In fact, we can say exactly what the symmetries are.

Theorem 19. *Let P be a regular polygon with n sides. The non-trivial symmetries of P are as follows:*

- *all reflections across its angle bisectors;*
- *all reflections across the perpendicular bisectors of its sides; and*
- *all rotations about its center by $\frac{2k\pi}{n}$, for $0 < k < n$.*

Proof. All of these transformations are certainly symmetries of P. On the other hand, if $\varphi \in Sym(P)$, then $\varphi(c) = c$, where c is the center of P (as defined above), so φ must either be a rotation about p or a reflection across a line containing c. Any rotation or not reflection that is not one of those listed above does not take vertices to vertices. □

Before talking about non-regular polygons, we introduce the "multiplication" table for $Sym(\mathcal{F})$. This is often useful in looking for patterns among symmetries.

Table 1 is the "multiplication" table for the symmetries of an equilateral triangle, where the rows and columns are labeled by the symmetries and the entry in column φ and row ψ is $\varphi \circ \psi$.

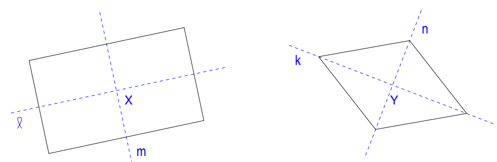
Notice that this table is not commutative, for example $r_\ell \circ r_m = \rho_{Z,4\pi/3}$, but $r_m \circ r_\ell = \rho_{Z,2\pi/3}$. Notice that he composition of two rotations, or of two reflections, is a rotation, while the composition of a rotation and a reflection (in either order) is a reflection. (Analogy: the product of two positive numbers or of two negative numbers is positive, while the product of a positive number with a negative number is negative.) Finally, notice that the grid has the "Sudoku property": each label occurs exactly once in each row and exactly once in each column. Every group table has this "Sudoku property."

[28] In general, there are several possible candidates for center — another is the *centroid*, or the center of balance — but for regular polygons, as for circles, all the definitions coincide, so we are safe in talking about *the* center.

Table 1: Symmetries of an equilateral triangle

	id	$\rho_{Z,2\pi/3}$	$\rho_{Z,4\pi/3}$	r_ℓ	r_m	r_n
id	id	$\rho_{Z,2\pi/3}$	$\rho_{Z,4\pi/3}$	r_ℓ	r_m	r_n
$\rho_{Z,2\pi/3}$	$\rho_{Z,2\pi/3}$	$\rho_{Z,4\pi/3}$	id	r_m	r_n	r_ℓ
$\rho_{Z,4\pi/3}$	$\rho_{Z,4\pi/3}$	id	$\rho_{Z,2\pi/3}$	r_n	r_ℓ	r_m
r_ℓ	r_ℓ	r_m	r_n	id	$\rho_{Z,2\pi/3}$	$\rho_{Z,4\pi/3}$
r_m	r_m	r_n	r_ℓ	$\rho_{Z,4\pi/3}$	id	$\rho_{Z,2\pi/3}$
r_n	r_n	r_ℓ	r_m	$\rho_{Z,2\pi/3}$	$\rho_{Z,4\pi/3}$	id

Now let's discuss polygons that are not regular, but have non-trivial symmetries. For example, let's discuss the $Sym(\mathcal{R})$ where $\mathcal{R}$ is a rectangle, and $Sym(\mathcal{S})$ where $\mathcal{S}$ is a rhombus.



In example 5 we said that a rectangle's non-trivial symmetries are: one rotational symmetry (about the intersection of the diagonals) and two reflection symmetries (across the perpendicular bisectors of each pair of opposite sides). And we said that the non-trivial symmetries of a rhombus are: one rotational symmetry (across the intersection of the diagonals) and two reflection symmetries (across the diagonals).

That's strange — it looks like the symmetry groups look the same algebraically. This becomes clear when we look at the actual multiplication tables:

Table 2 gives the multiplication table for the symmetry group of a rectangle, and table 3 gives the multiplication table for the symmetry group of a rhombus.

Table 2: Symmetries of a rectangle

	id	$\rho_{X,\pi}$	r_ℓ	r_m
id	id	$\rho_{X,\pi}$	r_ℓ	r_m
$\rho_{X,\pi}$	$\rho_{X,\pi}$	id	r_m	r_ℓ
r_ℓ	r_ℓ	r_m	id	$\rho_{X,\pi}$
r_m	r_m	r_ℓ	$\rho_{X,\pi}$	id

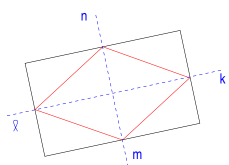
These two multiplication tables are essentially the same: if you take the first table and replace X with Y, ℓ with k, and m with n, you get the second table. Algebraically, we say that the symmetry groups of the rectangle and the rhombus are *isomorphic*. There's a good reason for this: the two figures can be superimposed so that their symmetry groups consist of exactly the same sets of transformations.

This is an example of how modern mathematics uses groups to study geometric objects. The

Table 3: Symmetries of a rhombus

	id	$\rho_{Y,\pi}$	r_k	r_n
id	id	$\rho_{Y,\pi}$	r_k	r_n
$\rho_{Y,\pi}$	$\rho_{Y,\pi}$	id	r_n	r_k
r_k	r_k	r_n	id	$\rho_{Y,\pi}$
r_n	r_n	r_k	$\rho_{Y,\pi}$	id

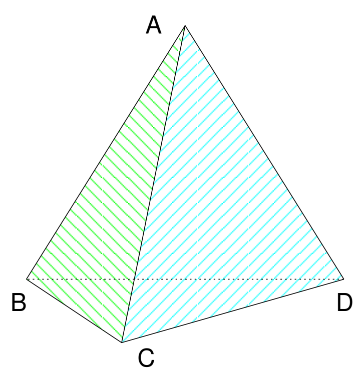
fact that the symmetry groups of the rhombus and rectangle are the same indicates that there's some relationship between the two figures. And you can see the relationship in the diagram below:



Of course you don't need groups to realize that you can form a rhombus by joining the midpoints of a rectangle, and that every rhombus can be formed that way, but the same technique can be applied to more complicated figures where the relationship is not so obvious.

2.8 A taste of symmetry in 3D

Example 8. *Let $\mathcal{T}$ be a regular tetrahedron (i.e., a triangular pyramid in which every side is an equilateral triangle). Call the vertices A, B, C, D.*



How many symmetries does $\mathcal{T}$ have? If φ is a symmetry, then clearly $\varphi(A)$ can be any of $\varphi(A)$, $\varphi(B)$, $\varphi(C)$ or $\varphi(D)$. Four choices there.

Having chosen $\varphi(A)$, there are three choices for $\varphi(B)$ (any of the other three vertices).

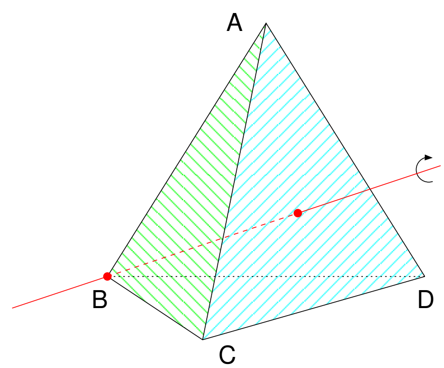
Having chosen $\varphi(A)$ and $\varphi(B)$, there are two choices for $\varphi(C)$. Then, once we choose $\varphi(C)$, there is only one possibility left for $\varphi(D)$.

In total, there are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ possible symmetries of $\mathcal{T}$.

In the tetrahedron, as in the equilateral triangle, each vertex is adjacent to each other vertex. Contrast this with a square, where each vertex is only adjacent to two of the other three vertices. If a square's vertices are, in clockwise order, A, B, C, D, and you rewrite them as B, A, C, D, you've changed something important: your old B was not adjacent to D, but your new B is. This is why there are only 8 symmetries of the square. But in the tetrahedron, this doesn't happen. Anything that works in one configuration of vertices works in all of them. So all possible symmetries of $\mathcal{T}$ actually *are* symmetries.

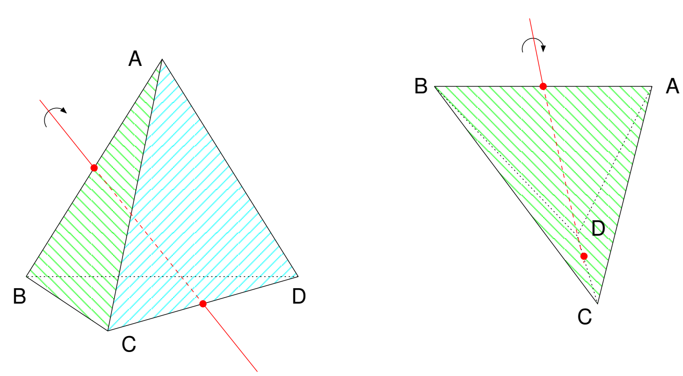
How can we describe the symmetries of a tetrahedron geometrically?

You can draw a line connecting a vertex with the center of the opposite triangle and rotate $\mathcal{T}$ by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ around this line, as in the following figure.



There are four ways to choose that vertex-triangle pair, so we get a total of eight rotations this way.

Another way to find a line of rotation is to connect the midpoints of two opposite edges of $\mathcal{T}$, as in the following figure. (It's probably easiest to visualize if you dangle $\mathcal{T}$ from one of its edges — the right-hand figure is an attempt at illustrating this.)



There are three such pairs of opposite edges, giving three more permutations.

There are twelve more symmetries of the tetrahedron: reflecting across the plane containing an edge and the midpoint of the opposite edge (for example, the edge $\overline{AC}$ and the midpoint of $\overline{BD}$) —there are six of those; and, finally, six compositions of reflections and rotations that haven't already been listed.

So far, this gives us 23 permutations. Throw in the identity, and you've got 24.

2.9 More on similarity

What do we mean when we say two figures are similar? We defined similarity without reference to transformations in definition 2, but that definition rested on the notion of proportionality.[29] Transformations allow a short, precise definition:

Definition 12. *Two figures are similar iff there is a similarity taking one to the other.*

Recall axiom 12: two triangles are similar iff their corresponding angles are equal. This may lead us to think that "corresponding angles are equal" is a good criterion for similarity of two figures, but it isn't. For example, all rectangles have all angles $= \pi/2$, so no matter how you set up a correspondence between angles of two rectangles, their corresponding angles are congruent. But not all rectangles are congruent to each other, e.g., a square is congruent only to another square. So knowing angles of particular figures isn't enough to know about similarity: *all* angles on the plane must be preserved.

A convenient Euclidean theorem (you will find it useful in the homework) is the following:

Theorem 20. *(SAS for similar triangles). Suppose ΔABC and ΔDEF have the property that, for some fixed $r > 0$, $\overline{DE} = r\overline{AB}, \overline{EF} = r\overline{BC}$ and $\angle ABC = \angle DEF$. Then $\Delta ABC \sim \Delta DEF$.[30] Hence $\overline{DF} = r\overline{AC}$.*

Proof. The last sentence follows immediately from $\Delta ABC \sim \Delta DEF$ and $\overline{DE} = r\overline{AB}$, so it suffices to prove $\Delta ABC \sim \Delta DEF$.

Let $\Delta D'E'F'$ be a triangle with $\overline{D'E'} = r\overline{AB}, \overline{E'F'} = r\overline{BC}$ and $\overline{F'E'} = r\overline{CA}$. Then $\Delta D'E'F' \sim \Delta ABC$. So $\angle D'E'F' = \angle ABC = \angle DEF$. Hence, by the usual SAS, $\Delta DEF \cong \Delta D'E'F'$. So $\Delta DEF \sim \Delta ABC$. □

An important theorem we won't prove is that, just as with isometries, a similarity is characterized completely by what it does to any three non-collinear points..

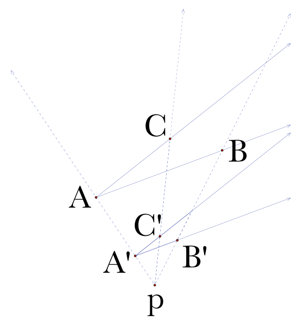
Theorem 21. *Suppose φ, ψ are similarities. Let p, q, r be three non-collinear points. Then $\varphi = \psi$ iff $\varphi(p) = \psi(p), \varphi(q) = \psi(q)$, and $\varphi(r) = \psi(r)$*

Theorem 22. *Dilations are similarities.*

Proof. Let φ be dilation about p by a factor of r, and let $\angle BAC$ be any angle. We'll consider the case where p is not on the angle. Let $A' = \varphi(A), B' = \varphi(B), C' = \varphi(C)$. We consider the case where p does not lie on the angle and $1 > r > 0$, as in the following diagram:

[29] Middle school mathematics spends a lot of time on ratio and proportion, hence on similarity (although the term doesn't always appear in middle school texts).

[30] Recall that $\sim$ is the symbol for "is similar to".



We want to show that $\angle CAB = \angle C'A'B'$.

By SAS for similar triangles, we know that $\Delta A'pB' \sim \Delta ApB$. Similarly, $\Delta A'pC' \sim \Delta ApC$. So $\angle pAC = \angle pA'C'$ and $\angle pAB = \angle pA'B'$. $\angle CAB = \angle pAC - \angle pAB = \angle pA'C' - \angle pA'B' = \angle C'A'B'$ and we are done.

Note that r doesn't matter: while the diagram would look different if $r < 0$ or $r > 1$, the proof works word for word in all three cases..

The case where p lies on the angle is left to the reader.. □

There are certain classes of figures in which all the figures are similar.

- Any two circles are similar: move the center of the first to the center of the second by a translation; then use a dilation that makes the radii coincide.
- All squares are similar.
- All parabolas are similar (hint: look at the polar coordinate equation for a parabola, and note that such equations differ only by a constant of proportionality).

Dilations composed with isometries are similarities: all of these transformations preserve angles. In fact, this is the only way to get a similarity.

Fact 5. *Every similarity is a dilation composed with isometry.*

Proof. Let φ be a similarity, and let Δ be a triangle. Let $\Delta^* = \varphi[\Delta]$. Then Δ^* is a triangle whose angles are the same as the angles in Δ. Let δ be a dilation that shrinks (or expands) Δ to a triangle congruent to Δ^*. Let ψ be an isometry moving $\delta[\Delta]$ onto Δ^*. Then $\varphi = \psi \circ \delta =$ a dilation followed by an isometry. □

The converse of fact 5 is the reason why the two definitions of similarity — the one via proportionality and the one via transformations — are the same. If $\mathcal{F}$ and $\mathcal{G}$ are similar via proportionality, then there is a dilation that shrinks (or expands) $\mathcal{F}$ to a figure $\mathcal{F}^*$ congruent to $\mathcal{G}$, and then there is an isometry so that the image of $\mathcal{F}^*$ is $\mathcal{G}$.

Another major theorem which we will not prove is:

Theorem 23. *A similarity which has either no fixed points or at least two fixed points is an isometry.*

We will prove:

Corollary 3. *A similarity which is not an isometry has exactly one fixed point.*

Proof. By the contrapositive: Suppose φ is a similarity which does not have exactly one fixed point. Then it has no fixed points or at least two. So by theorem 23 φ, is an isometry. □

Another corollary to theorem 23 is that a similarity which has no fixed point is either a translation or a glide reflection, and that a similarity which fixes at least two points is a reflection.

2.10 Transformational geometry proofs

Transformational geometry can be used to prove theorems, just as Euclidean axioms can. The axioms are basic facts about transformations. We will not list all of these basic facts, but, as a sample, here are some basic facts about the reflection r_ℓ:

- r_ℓ fixes ℓ.
- If $m \not\parallel l$ and the angle between m and ℓ is α,[31] then the angle between $r_\ell[m]$ and ℓ is α.
- If m is parallel to ℓ then so is $r_\ell[m]$, and the distance between $r_\ell[m]$ and ℓ is the distance between m and ℓ.
- If $m \perp \ell$ then $r_\ell[m] = m$, i.e., r_ℓ is a symmetry of m (but it doesn't fix m).

And so on for the other similarities.

We also define various figures transformationally:

Definition 13. *(a) A circle $\mathcal{C}$ is a convex figure so that there is a point p (called the center) so that if ℓ is a line through p then r_ℓ is a symmetry of $\mathcal{C}$.*[32]

(b) A parallelogram $\mathcal{P}$ is a convex quadrilateral so that if p is the intersection of the diagonals then $\rho_{p,\pi}$ is a symmetry of $\mathcal{P}$.

(c) A rectangle is a parallelogram $\mathcal{R}$ so that if ℓ connects the midpoints of opposite sides, then r_ℓ is a symmetry of $\mathcal{R}$.

(d) A kite is a convex quadrilateral $\mathcal{K}$ with reflection symmetry about at least one of its diagonals.

(e) A rhombus is a convex quadrilateral $\mathcal{S}$ with reflection symmetry about each of its diagonals.

Note that if $\mathcal{C}$ is a circle by definition 13, and α is any angle, then $\rho_{p,\alpha}$ is a symmetry of $\mathcal{C}$. Why is that? Because any rotation through p is the composition of two reflections $r_\ell \circ r_m$, where p is the intersection of ℓ, m.

[31] "the angle between m and ℓ" means: the angle between m and ℓ which is $< \pi$

[32] We already defined a circle transformationally, but the definition contained redundancies.

In the homework you'll be asked to go from some of these definitions to the usual ones. For example, if $\mathcal{K}$ is a kite via definition 13, then $\mathcal{K}$ is a kite in the usual sense. In the Euclidean homework we proved (or will prove) that the diagonals of a kite are perpendicular. So by composition, if $\mathcal{S}$ is a rhombus then $\rho_{p,\pi}$ is a symmetry. And since a square is both a rhombus and a rectangle, we immediately know its symmetries.

Theorem 24. *The base angles of an isosceles triangle are equal.*

Proof. Let $\overline{AB} = \overline{BC}$ and let ℓ be the angle bisector of $\angle ABC$. Let D the intersection of $\overline{AC}$ and ℓ. Then $r_\ell(B) = B$. By definition of ℓ, $\angle ABD = \angle CBD$. By hypothesis, $\overline{AB} = \overline{BC}$. So $r_\ell(A) = C$ and $r_\ell(C) = A$. Hence $\angle ABC = \angle r_\ell(A) r_\ell(B) r_\ell(C) = \angle CBA$. $\square$

Theorem 25. *If $\mathcal{T}$ is a triangle and $\mathcal{M}_\mathcal{T}$ is the triangle whose vertices are the midpoints of the sides of $\mathcal{T}$, then $\mathcal{T} \approx \mathcal{M}_\mathcal{T}$.*

Proof. Let $\mathcal{T} = \Delta ABC$ and let E be the midpoint of $\overline{AB}, F$ the midpoint of $\overline{BC}$, and G the midpoint of $\overline{AC}$. Consider $\delta_{A,1/2}$. Then $\delta_{A,1/2}(A) = A, \delta_{A,1/2}(B) = E$, and $\delta_{A,1/2}(C) = G$. So (by the transformational definition of similarity) $\Delta ABC \approx \Delta AEG$ and $\overline{EG} = \frac{1}{2}\overline{BC}$.

Similarly, $\overline{EF} = \frac{1}{2}\overline{AC}$, $\overline{FG} = \frac{1}{2}\overline{AB}$.

Since two triangles are similar iff corresponding sides have the same proportion, $\Delta ABC \approx \Delta FEG$. $\square$

Let's give a transformational geometry proof of a familiar theorem about isosceles triangles (proved in the homework by Euclidean means).

Theorem 26. *If $\overline{AB} = \overline{CB}$ then the perpendicular bisector of $\overline{AC}$ is the angle bisector of $\angle ABC$.*

Proof. Let ℓ be the angle bisector of $\angle ABC$. As in theorem 24, $A = r_\ell(C)$. Let D be the intersection of ℓ and $\overline{AC}$. Then $\angle ADB = \angle r_\ell(A)DB = \angle CDB$. Since $\angle ADC = \pi$, $\angle ADB = \frac{\pi}{2}$. Also, $\overline{AD} = \overline{r_\ell(A)D} = \overline{CD}$. Hence ℓ is the perpendicular bisector of $\overline{AC}$.

$\square$

We give a transformational geometry proof of SAS for similar triangles (theorem 20):

Proof. Suppose $\overline{AB} = r\overline{DE}, \overline{BC} = r\overline{EF}$ and $\angle ABC = \angle DEF$. Let φ be an isometry with $\varphi(B) = E$ and $\varphi(A), D, E$ collinear, $\varphi(A), D$ are on the same side of E and $\Delta ABC \cong \Delta\varphi(A)E\varphi(C)$. By angle preservation, $\varphi(C), F, E$ are collinear and $\varphi(C), F, E$ are on the same side of E. Consider $\delta = \delta_{E,1/r}$. $\delta(\varphi(A)) = D, \delta(\varphi(C)) = F$, so $\Delta ABC \cong \Delta\varphi(A)E\varphi(C) \sim DEF$, as desired. $\square$

Finally we sketch a proof using both Euclidean and transformational geometry (as well as other methods) of a very important theorem:

Theorem 27. *A circle maximizes area for a fixed perimeter. Stated more precisely: Fix a length k. Among all the figures with perimeter k, the circle will have the largest area, and every other figure with perimeter k will have smaller area.*

Proof. In some sense this proof is only a sketch of a proof, since it relies on a fact and a definition that are beyond the scope of this class:

(1) There is a shape which maximizes area for a fixed perimeter.

(2) The centroid of a figure is defined to be the balance point, that is, a lamina of that shape will balance on a fulcrum placed at the centroid.

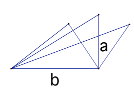
As a corollary of (2),

(3) any line through the centroid splits the figure into two equal areas.

We will use the following fact

(4) among all triangles with two sides of length a and b respectively, the one with a right angle between those sides maximizes the area.

The proof of (4) is by picture:



Here we have three triangles with sides b and a. If we take b as the base, then the area is $\frac{1}{2}bh$ where h is the height of the triangle. If the angle between the side of length b and the sides of length a is $\frac{\pi}{2}$, then $h = a$. For any other angle, $h < a$.

Now we're ready to focus on the theorem.

Suppose $\mathcal{F}$ is a figure which maximizes area for a fixed perimeter.

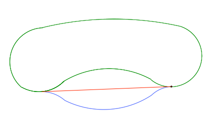
The corollary to (2) immediately gives us that there is a point p (namely the centroid) so that if ℓ is a line through p then r_ℓ splits $\mathcal{F}$ into two equal areas.

By definition 7 there are two things left to show:

(a) $\mathcal{F}$ is convex, and

(b) if ℓ goes through p then r_ℓ is a symmetry.

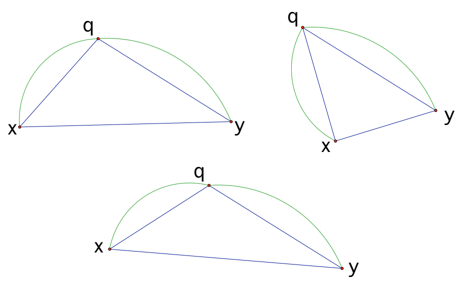
To show (a): Suppose $\mathcal{F}$ is not convex. Then we have a picture something like this, where the green lines are the original shape:



We can reflect part of the perimeter about some line (the red line in the picture) to get a figure with a larger area and the same perimeter, as in the picture. Which contradicts $\mathcal{F}$ having maximum area.

To show (b): . Let q be a point on $\mathcal{F}$, suppose ℓ splits $\mathcal{F}$ into two equal areas, and let x, y be the intersections of ℓ and $\mathcal{F}$, q not on ℓ. The claim is that $\angle xqy = \frac{\pi}{2}$.

We can see this by considering q a hinge, as below, where the diagram shows us three possibilities for half of $\mathcal{F}$.



Half the area of $\mathcal{F}$ = the area of the triangle + the areas of the two curved segments. The areas of the two curved segments stays constant as the triangle changes. The sides of the triangle inside $\mathcal{F}$ keep their lengths, i.e., $\overline{xq}$ has constant length, and $\overline{yq}$ has constant length. So the area of $\mathcal{F}$ is maximized when the area of the triangle is maximized, i.e., by (4), when $\angle xqy = \frac{\pi}{2}$.

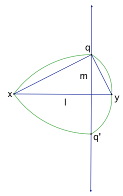
Since ℓ splits the area of $\mathcal{F}$ in half, p is on ℓ. (This is a converse of (3), so it needs a small proof, left to the reader.) (b) will be proved if we can show the following:

(5) $r_\ell(q)$ is on $\mathcal{F}$.

Why does (5) complete the proof? Because if (5) holds for any ℓ through p and any q on the half of $\mathcal{F}$ we're looking at, then r_ℓ is a symmetry of $\mathcal{F}$, so $\mathcal{F}$ is a circle.

To prove (5):

Given q on $\mathcal{F}$ and ℓ a line through p we may assume that q not in ℓ (because if q on ℓ then $r_\ell(q) = q$ and we're done). Let m be the line through q perpendicular to ℓ. Let q' be the intersection of m and the other half of $\mathcal{F}$.



Since q' is on $\mathcal{F}, \angle xq'y = \frac{\pi}{2}$. We need to show that $q' = r_\ell(q)$.

Since reflecting the triangle preserves the right angle, $\angle xr_\ell(q)y$ is also a right angle. The function $f : l \to \pi$ given by $f(s) = \angle xsy$ (where s is a point on ℓ) is a continuous function. By a continuity argument, there are exactly two points s, s^* with $f(s) = \frac{\pi}{2} = f(s^*)$, one on each side of ℓ. We know that q is one of those points, and q' is the other. We know that $q \neq r_\ell(q)$. Thus $q' = r_\ell(q)$.

$\square$

You have more transformational geometry proofs on the homework. These proofs all work the same way: find the right transformation and let it do the work for you.

2.11 Tesselations

In this and the next section we are interested in symmetries of unbounded objects.

A *tesselation* is usually defined to be a tiling of the plane where there are no gaps or overlaps.

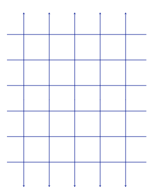
This is ambiguous. Do we have a finite number of types of tiles? An infinite number? Must each tile be bounded? Can we use unbounded tiles (e.g., an infinite strip)? Different definitions are used in different places, depending on what point the author is trying to make.

We will define a tesselation to be a tiling of the plane by only a finite number of types of bounded figures.

In the next few projects, we will explore tilings using pattern blocks. There are six types of pattern blocks: the large regular hexagon, the trapezoid (= half the large hexagon), the skinny parallelogram, the fat parallelogram (= 1/3 the large hexagon), the equilateral triangle (= 1/6 the large hexagon), and the square.

All of these pieces fit together in a nice way: except for the long side of the trapezoid, all the sides have exactly the same length (and the long side of the trapezoid is two of those lengths).

As with figures, a symmetry of a tiling is an isometry that takes the tiling into itself. For example, consider the following tiling:[33]



It has the following symmetries:

- any translation that takes the center of one square to the center of another
- any symmetry of any square[34]
- any reflection across the side of any square
- rotation by $k\pi/2$ about the vertex of any square

Project 6. Construct a complex tiling of the plane using pattern blocks which has many symmetries. Find all of these symmetries.

Project 7. Construct a tiling of the plane using pattern blocks whose only non-trivial symmetry is a rotation by π about a single point.

[33] Of course our picture is finite. We have to imagine the tiling repeating itself forever in all directions. When you create tesselations, you have to make it clear how the pattern continues.

[34] carrying the whole tiling along with it

Project 8. Construct a tiling of the plane using pattern blocks whose only non-trivial symmetry is a reflection across a single line.

Project 9. Construct a tiling of the plane using pattern blocks whose only non-trivial symmetries are gotten by repeated translation using a single vector.

Projects 7, 8, 9 are somewhat surprising: most of the time when we create a tesselation using pattern blocks, it has many symmetries. One could easily conjecture that every pattern block tesselation has many symmetries. Projects 5, 6, and 7 are counterexamples to this conjecture. Counterexamples are as important to mathematics as theorems.

Now let's focus on tesselations that use only one shape. If you can tile the plane using only one shape, we say, not surprisingly, that this shape tiles the plane. For example, a square tiles the plane. And every pattern block piece tiles the plane.

Theorem 28. *If P is a regular polygon that tiles the plane, then each angle of P is some $2\pi/n$ for n a positive integer.*

Proof. All the angles of P are the same, and when we fit the vertices of the tiles together, they must add up to 2π. □

Corollary 4. *The only regular polygons that can tile the plane are: the equilateral triangle, the square, and the regular hexagon.*

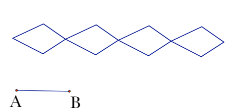
Theorem 29. *(a) Every triangle tiles the plane.*

(b) Every quadrilateral tiles the plane.

Project 10. We will "prove" theorem 29 on *Sketchpad*: On *Sketchpad*, begin with one tile — for (a) an arbitrary triangle, for (b) an arbitrary quadrilateral (be sure to consider both convex and non-convex cases) — and find a transformation of that tile that, repeated over and over, produces a tessellation. Deform your original tile (you might want to make it recognizable by coloring it in some way) and check that, indeed you still have a tessellation.

2.12 Friezes and their symmetries

A *frieze* is a pattern gotten by repeatedly translating a finite pattern (called the generating pattern) by a fixed vector infinitely many times. For example, the frieze below is gotten by repeatedly translating the quadrilateral by the vector $\vec{AB}$.



Since friezes are produced by repeatedly translating a finite pattern by parallel vectors, they are always contained within infinite two parallel lines (called borders). To make the exposition easier, we will assume, without loss of generality, that these parallel lines are horizontal, and that the top border is as low as possible, the bottom border as high as possible (i.e., there are no infinite vertical or horizontal gaps). Note that the borders do not actually have to appear in the frieze — for example, no borders appears in the example above.

A trivial frieze consists of a set of infinite parallel horizontal lines bounded between two infinite horizontal lines. From now on we consider that our friezes are not trivial, that is, there is some non-horizontal line or curve or shape. So by "frieze pattern" or "frieze" we will always mean "non-trivial frieze pattern" or "non-trivial frieze."[35]

Theorem 30. *(Niggli, 1926) There are exactly seven symmetry groups of frieze patterns.*

The rest of this section is the proof of Niggli's theorem.

As already mentioned, a symmetry group is just the set of symmetries of a set $\mathcal{S}$. A symmetry group is determined by its set of *generators*, i.e., if you have certain transformations in the group you necessarily have others.

For example, suppose a frieze has one symmetry which is a horizontal reflection across a line ℓ, and another symmetry which is a vertical reflection across a line m. Then, since a horizontal reflection followed by a vertical reflection is the same as a rotation about the point of intersection by π, the frieze also has to have a symmetry which consists of rotation by π about the point of intersection.

Here is a general fact:

Fact 6. *If h is a symmetry of a set $\mathcal{S}$, so is $h^{\leftarrow}$, and so is any transformation which consists of repeated compositions of h or $h^{\leftarrow}$.*

Niggli's theorem is proved by carefully setting out all the possibilities of "if a frieze has these kinds of symmetries it must have those," and it turns out there are only seven possibilities.

We set out these possibilities in a series of lemmas.

First, let's ask about translations of a frieze.

Lemma 1. *For every frieze there is a horizontal translation τ which is a symmetry of the frieze; $\tau^{\leftarrow}$ is also a symmetry of the frieze; and every other translation symmetry of the frieze is a composition of finitely many copies of τ or $\tau^{\leftarrow}$.*

Proof. Let $\mathcal{F}$ be a frieze. Because the borders are horizontal, any translation symmetry of $\mathcal{F}$ must be horizontal. By the definition of frieze, there is at least one translation symmetry. Since $\mathcal{F}$ is non-trivial, there is a non-horizontal line or curve or shape in the frieze. Let p be a point in this non-horizontal line or curve or shape. Let q be the closest translation of p under a symmetry of $\mathcal{F}$ which is to the right of p. Then the translation τ which moves p to q is the smallest translation symmetry of $\mathcal{F}$ to the right, i.e., if τ' is also a symmetry of the frieze and $\tau'(p)$ is to the right of p,

[35] This is for the technical reason that Niggli's theorem, the main theorem of this section, doesn't apply to trivial friezes.

then $\overline{p\tau'(p)} \geq \overline{p\tau(p)}$. Most of the rest of the lemma follows from fact 6. The only question is: can there be other translation symmetries?

Let d be the distance between p and q. Suppose τ^* is a translation symmetry which is not a finite composition of either τ or $\tau^{\leftarrow}$ and let $q^* = \tau^*(p)$. Then $d^* =$ (the distance between p and q^*) is not a multiple of d. Because d is minimal, $d^* > d$. So there is some k a positive integer with $kd < d^* < (k+1)d$. Without loss of generality, assume q^* is to the right of p. Then $q' = (\tau^{\leftarrow})^k(\tau^*(p)) = (\tau^{\leftarrow})^k(q^*)$ is a translation of p under a symmetry of $\mathcal{F}$ which is to the right of p, but the distance between p and q' is less then d. This contradicts the minimality of d. □

(If you didn't understand the last paragraph of the proof of the lemma, try letting $k = 1, k = 2$, etc.).

So the symmetries of a frieze which are translations are all the same: they are repetitions of a single horizontal translation or of its inverse.

Lemma 2. *The only possible rotational symmetry of a frieze is rotation by π about a point on the mid-line between the borders.*

Proof. A rotational symmetry has to carry one border into the other. And the only rotation which carries a line into another line parallel to itself is a rotation by π about a point on the mid-line. □

This tells us that if a frieze has rotational symmetry, it has only one kind of rotational symmetry. Going back to the definition of symmetry tells us more: the center of rotation of a frieze is (a) either at the center of the generating pattern or (b) at the right (hence left) edge of the generating pattern (or of course at the center or edge of any repetition of the pattern).

Lemma 3. *Let c be the center of the generating pattern, and let p be the center of one vertical edge of the pattern. If a frieze has rotational symmetry, then both $\rho_{c,\pi}$ and $\rho_{p,\pi}$ are symmetries.*

Proof. Let $\vec{v}$ be the vector of translation, i.e., translation symmetries of the frieze are integer multiples of $\tau_{\vec{v}}$. Depending on whether p is to the right or left of c, $\rho_{p,\pi} = \pm\tau_{\vec{v}}\rho_{c,\pi}$, hence $\rho_{c,\pi} = \mp\tau_{\vec{v}}\rho_{p,\pi}$ □

Lemma 4. *The only possible reflection symmetries of a frieze are reflection across a horizontal or vertical axis. If it is about a horizontal line, the line must be the midline between the borders.*

Proof. As with lemma 2, this is because a reflection symmetry has to carry one border into the other. □

Lemma 5. *(a) If a frieze has vertical reflection symmetry, then the generating pattern embeds in a rectangle (whose edges may or may not be visible), and the generating translation vector is the length of this rectangle.*

(b) If a frieze has vertical reflection symmetry, then the lines of vertical symmetry are the vertical line through the center of the pattern and the vertical lines through the edges of the pattern.

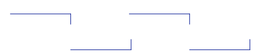
Proof. (a) is immediate. We prove (b): Let $\vec{v}$ be the generating translation vector. As in lemma 3, if k is the vertical line through the center of the pattern and m is the vertical line through the edge, then $r_k = \pm\tau_{\vec{v}}r_m$, $r_m = \mp\tau_{\vec{v}}r_k$ □

Lemma 6. *Any glide reflection symmetry of a frieze involves reflection across the midline.*

Proof. Again, this is because any symmetry of a frieze has to carry one border into another. □

Given a frieze $\mathcal{F}$, we say that a glide reflection symmetry of $\mathcal{F}$ is trivial iff it is the composition of a horizontal reflection symmetry with a translation symmetry. Every frieze with a horizontal reflection symmetry has such trivial glide reflection symmetries. A glide reflection symmetry is not trivial if it is the composition of a horizontal reflection which is *not* symmetry of the frieze and a translation symmetry which is *not* a translation symmetry of the frieze.

Here is an example of a frieze with a non-trivial glide reflection symmetry:



It has no reflection symmetry, but if we reflect it about the midline and then translate by half the generating translation, we have a symmetry of the frieze.

So our possible frieze symmetries other than the mandatory translation symmetries are:

- ρ_p = rotation about a point p on the midline h by π [36](and by lemma 2, we know exactly which centers of rotation work).
- r_h = reflection across the mid-line h
- r_v = reflection across a vertical line (and by lemma 4 we know exactly which vertical lines work)
- γ = glide reflection, where the reflection is about h.

There are four of these altogether, leading to $2^4 = 16$ potential symmetry groups. How do we winnow it down to seven?

The answer is: the presence of some symmetries automatically implies the presence of others, which ends up ruling out 9 of the possible combinations. Here's how we do it.

Lemma 7. *Let v be a vertical line, and c the point of intersection of v and h. Then $r_h r_v = \rho_c$.*

Proof. Draw a small triangle and see what $r_h r_v$ do to it. See what ρ_c does to it. They do the same thing, so we are done since three points determine an isometry. □

[36]Since the only possible angle is π, for notational convenience we're not mentioning the angle.

Lemma 8. *If c is the point of intersection of h and a vertical line v, then $\rho_c r_v = r_h$.*

Proof. As in lemma 7. □

Lemma 9. *If c is the point of intersection of h and a vertical line v, then $\rho_c r_h = r_v$.*

Proof. As in lemma 7. □

Lemma 10. *If v is a vertical line and c is some point not on v, then $\rho_c r_v$ is a glide reflection.*

Proof. As in lemma 7. □

Lemma 11. *Let γ be a glide reflection where the reflection is through h. If v is a vertical line, then $\gamma r_v = \rho_c$, for c a point on h but not on v.*

Proof. As in lemma 7. □

Lemma 12. *If γ is a glide reflection about h, $\mathcal{F}$ is any frieze, and r_h is a symmetry of $\mathcal{F}$, then $r_h\gamma$ is a symmetry of $\mathcal{F}$ iff γ is a trivial glide reflection.*

To prove lemma 12, compare what happens to one point under $r_h\gamma$ to what would happen if γ were non-trivial.

We summarize these lemmas in table 4. In this chart, "glide reflection" means "non-trivial glide reflection."

Table 4: Frieze symmetries

Conclusion:	If a frieze has these symmetries	then it has those symmetries	reason
1	vertical and horizontal	rotation	lemma 7
2	vertical and rotational	horizontal	lemmas 8 and 12
3	horizontal and rotational	vertical	lemma 9
4	glide reflection and vertical	rotation	lemma 11

All friezes have translation symmetry. Here are the seven cases describing what combinations of symmetries are possible:

Case I. The frieze has rotational symmetry...

- 1. and horizontal symmetry. By table 4 it has vertical symmetry. By lemma 12 it has no glide reflection symmetry.
- 2. and glide reflection symmetry but no horizontal symmetry. By table 4 it has no vertical symmetry.
- 3. no horizontal symmetry and no glide reflection symmetry. By table 4 it has no vertical symmetry.

Case II. The frieze has no rotational symmetry...

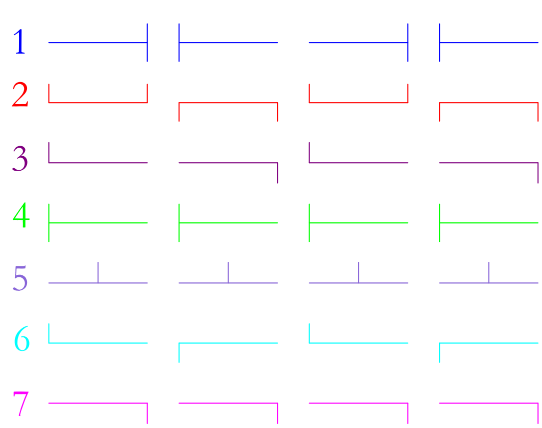
- 4. and horizontal symmetry. By table 4 it has no vertical symmetry and, by lemma 12, no glide reflection symmetry.
- 5. and vertical symmetry and no horizontal symmetry. By table 4 it has no glide reflection symmetry.
- 6. and glide reflection and no vertical or horizontal symmetry.
- 7. and no other symmetries except translation.

To summarize in yet another chart, table 5 lists the possibilities:

Table 5: frieze symmetries

case	rotational	horizontal	vertical	non-trivial glide
1	yes	yes	yes	no
2	yes	no	no	yes
3	yes	no	no	no
4	no	yes	no	no
5	no	no	yes	no
6	no	no	no	yes
7	no	no	no	no

There are seven cases, so Niggli's theorem is almost proved. What remains is to show that there is a frieze for every case. This is done in the illustration below:



We have shown that there are seven potential symmetry groups. We have shown that there is a frieze for each of these symmetry groups. So we have proved Niggli's theorem.

3 Polyhedra

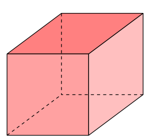
3.1 The basics

In the previous chapter we looked briefly at the symmetries of a tetrahedron. Tetrahedra are an example of a particular kind of solid known as a *polyhedron* (plural: *polyhedra*). Its characteristics are:

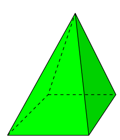
- it is made up of polygons glued together along their edges
- it separates $\mathbb{R}^3$ into itself, the space inside, and the space outside
- the polygons it is made of are called *faces*
- the edges of the faces are called the edges of the polyhedron
- the vertices of the faces are called the vertices of the polyhedron.

The most familiar example of a polyhedron is a cube. Its faces are squares, and it has 6 of them. It also has 12 edges and 8 vertices.

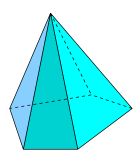
Another familiar example is a pyramid. A pyramid has a bottom face, which can be any polygon (you are probably most familiar with pyramids that have square bottoms), and the rest of its faces meet in one point.



Cube



Square pyramid

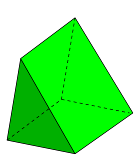


Pentagonal pyramid

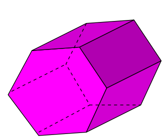
Project 1 If the bottom face of a pyramid has n sides, how many faces, edges, and vertices will it have?

Another familiar example is a prism, which is a polyhedron with two congruent parallel faces in which the other faces are rectangles. The two congruent faces can be triangles, quadrilaterals, or anything else.

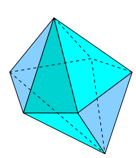
Another, perhaps slightly less familiar example is a bipyramid, which is built by taking two pyramids with congruent bases and gluing the bases together, so that only the triangular faces are left.



Triangular prism



Hexagonal prism



Pentagonal bipyramid

Project 2 If one of the parallel faces of a prism has n sides, how many faces, edges, and vertices will the prism have?

Project 3 If the base (or "equator") of a bipyramid has n sides, how many faces, edges, and vertices will the bipyramid have?

While there are lots of different polyhedra, they all have some common features just by virtue of being polyhedra.

Theorem 31. *In any polyhedron*

- *every vertex must lie in at least three faces. (Otherwise, the polyhedron collapses to have no volume.)*
- *every face has at least three vertices. (It's a polygon, so it better have at least three sides.)*
- *every edge must lie in exactly two faces. (Otherwise, the polyhedron wouldn't have an inside and an outside.)*
- *the sum of the angles of the faces meeting at a vertex must be $< 2\pi$ (If the sum were exactly 2π the faces would lie in a plane; if it were $> 2\pi$ the faces would overlap.)*

As usual, you can learn a lot by constructing —- or proving you can't construct — non-standard examples. For example:

Project 4 Can you construct a polyhedron with two parallel faces, one a triangle, the other a rectangle?

Project 5 Can you construct a polyhedron so that exactly one face is not a rectangle?

Project 6 Can you construct a polyhedron in which every face is a regular hexagon?

An interesting thing to do with solids is to cut them along a plane and see what cross-section you get.

Project 5 What happens when you cut a cube along a plane (a) parallel to one of its faces; (b) perpendicular to one of its faces and through a diagonal of the face; (c) through an opposite pair of vertices that divides the cube into two equal volumes.

You may remember that the circle, ellipse, parabola and hyperbola are gotten in similar fashion: by cutting a double cone in various ways.

3.2 The Platonic solids

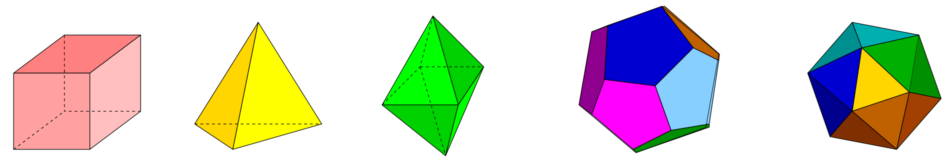
Definition 14. *A polyhedron is called regular (or: a Platonic solid) iff (a) all of its faces are congruent; (b) all of its faces are regular polygons; and (c) each of its vertices meets the same number of edges as every other vertex.*

So cubes are Platonic, but most prisms and pyramids are not.

You may have heard of the following Platonic solids besides the cube:

- tetrahedron (four equilateral triangles)
- octahedron (8 equilateral triangles)

- icosahedron (20 equilateral triangles)
- dodecahedron (12 regular pentagons)



Cube Tetrahedron Octahedron Dodecahedron Icosahedron

You haven't heard of any others because

Theorem 32. *There are exactly five Platonic solids.*

This is a surprising theorem. Generally when we define a nice class of objects in mathematics it is large, in fact usually infinite (think of the set of primes, the collection of all regular polygons...). But this class is not only finite it has only five things in it.

On the other hand, maybe it is not so surprising. We have run into something like this before, in the proof that the only regular polygons that tile the plane are equilateral triangles, squares, and regular hexagons.

One part of the proof — that there are at least five Platonic solids — is already done. We know what they are. In fact you've built them from patterns. So the interesting part is proving that there aren't any more. In order to do this you need to examine carefully how you fit polygons together to make polyhedra — we already did this a little in theorem 31. What more can you say about the number of faces that touch at the same vertex? What can you say about the angles at which the faces meet? And so on.

Project 6. Fold a piece of paper. Cut so that you have two congruent polygons joined at the fold. Let p be a point at the edge of the fold, and ℓ, m the sides through p which do not coincide with the fold. Fold and unfold so that you narrow and expand the angle between ℓ and m at p. When is this angle the biggest possible? If the two faces were part of a polyhedron, would you need at least one more face at p? Could you have more than one more face at p?

Project 7 Fit some polygons together at a vertex p to start a polyhedron. What is the sum of the angles touching p? Now flatten the part touching p (you will have to cut at least one edge to do this). No matter what your polyhedron is, can the sum of the angles touching p sum up to more than 2π? Can they sum up to exactly 2π? Explain briefly.

Now that we've done some general exploration of fitting polygons together to make polyhedra, let's prove theorem 1.

First, we note that Platonic solids are very special, because each vertex must belong to the same number of faces, say n, and each face must be a polygon with the same number of sides, say s. What can we say about these numbers?

By theorem 31

Lemma 13. $n \geq 3$ *and* $s \geq 3$.

Now let's talk about angles.

Suppose $\mathcal{P}$ is a Platonic solid. Each face F is a regular polygon with s sides and s angles. The sum of the interior angles of F is $\pi(s-2)$, so each angle of F is $\pi - \frac{2\pi}{s}$. We use this to prove

Corollary 5. *A Platonic solid cannot have faces with six or more sides.*

Proof. Suppose p is a vertex of a polyhedron meeting at least three faces of a Platonic solid where each face has s sides. Each angle of each face is $\pi - \frac{2\pi}{s}$. So the angle for the vertex of a third face meeting p must be $< 2\pi - 2(\pi - \frac{2\pi}{s}) = \frac{4\pi}{s}$. Hence that $\pi - \frac{2\pi}{s} < \frac{4\pi}{s}$. So $n < 6$. □

Conclusion 1 The faces of a Platonic solid must be equilateral triangles, squares, or regular pentagons.

By theorem 31

Lemma 14. *A Platonic solid whose faces are squares cannot have more than three faces meeting at a vertex.*

Lemma 15. *A Platonic solid whose faces are pentagons cannot have more than 3 faces meeting at a vertex.*

Lemma 16. *A Platonic solid whose faces are triangles cannot have 6 or more faces meeting at a vertex.*

Conclusion 2 We are left with the following possibilities for Platonic solids:

- faces are equilateral triangles; the number of faces meeting at a vertex may be 3 (tetrahedron), 4 (octahedron), or 5 (icosahedron).
- faces are squares; the number of faces meeting at a vertex must be 3 (cube).
- faces are regular pentagons; the number of faces meeting at a vertex is 3 (dodecahedron).

Which almost proves the theorem. We also need

Project 8. Suppose you have a bunch of equilateral triangles, squares, or regular pentagons. Suppose you are told how many of such faces each vertex of a regular polyhedron must meet. Suppose your constraints are consistent with conclusion 2. Then there is exactly one way to construct the regular polyhedron, and it must be one of those listed in conclusion 2.

3.3 Euler's formula

Project 9 For each of the Platonic solids, investigate $v =$ the number of vertices, $f =$ the number of faces, $e =$ the number of edges. Do this with at least one pyramid, at least one prism, and at least one bipyramid. See if you can come up with a formula relating these three variables.

Do not read further until you have completed project 9. Otherwise you'll have missed your chance to discover one of the most important formulas in mathematics — once somebody tells you there is a formula, it's really not so hard to figure out what it is. In fact this is so important that I'm putting a page break here to reduce temptation.

Theorem 33. *(Euler's formula) The formula you came up with in project 9 holds for all convex polyhedra.*[37]

Euler's genius was realizing that there *is* a formula. This formula has obvious applications in geometry and topology, but also in other areas – if you go to the website http://www.cut-the-knot.org/do_you_know/polyhedra.shtml you can see a proof that there are only five Platonic solids using Euler's formula. The essence of Euler's formula is to look at an object made up of parts with smaller dimension — for a three-dimensional polyhedron these parts are the vertices (dimension 0), edges (dimension 1), and faces (dimension 2) —- and find the relationship among the number of these parts. This idea generalizes to higher dimensions and to different sorts of shapes (for example, "polyhedra" with holes, and non-convex polyhedra.)

There are many proofs of Euler's formula, and George Lakatos has written a charming (and profound) book *Proofs and Refutations*, which uses proofs of this formula to delve deeply into the philosophy of mathematics. My favorite proof is from George E. Martin's *Transformation Geometry: An Introduction to Symmetry*[38]

"To prove the famous formula, imagine that all the edges of a convex polyhedron are dikes, exactly one face contains the raging sea, and all other faces are dry. We break dikes one at a time until all the faces are inundated, following the rule that a dike is broken only if this results in the flooding of a face. Now, after this rampage, we have flooded $f-1$ faces and so destroyed exactly $f-1$ dikes. Noticing that we can walk with dry feet along the remaining dikes from any vertex p to any other vertex along exactly one path, we conclude there is a one-to-one correspondence between the remaining dikes and the vertices excluding p. Hence there remain exactly $v-1$ unbroken dikes. So $e=(f-1)+(v-1)$ and we have proved *Euler's formula*."

I always imagine Godzilla on this rampage — maybe you like King Kong better. It's a vivid, dramatic proof. But is it correct?

If we take away the colorful imagery, the proof boils down to four lemmas about what happens after all those dikes are broken:

Lemma 17. *At the end of this procedure, we have flooded $f-1$ faces.*

Lemma 18. *At the end of this procedure, we can walk with dry feet along the unbroken edges from any vertex to any other vertex.*

Lemma 19. *At the end of this procedure, there is a unique path consisting of unbroken edges connecting any two vertices.*

Lemma 20. *At the end of this procedure, there are exactly $v-1$ unbroken edges.*

Our task is to show:

I. Lemmas 17, 18, 19, 20 are correct.

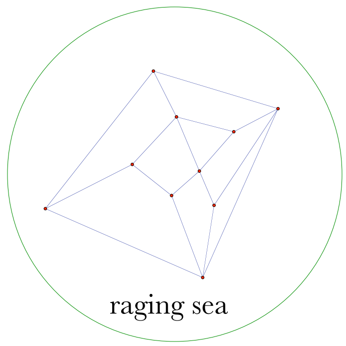
II. Lemmas 17, 18, 19, 20 prove theorem 33.

Proof. For lemma 17: The only face we have not flooded is the one which originally contained the raging sea.

[37] To make things confusing, there are two Euler's formulas — the other is $e^{\pi i}=-1$.

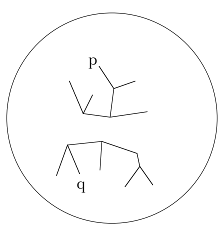
[38] Springer-Verlag, 1982, New York. The quote is from p. 198.

For the rest of the proof, let's imagine that we punctured the face that originally contained the raging sea and flattened the polyhedron out by stretching so that the original polyhedron looks something like this:

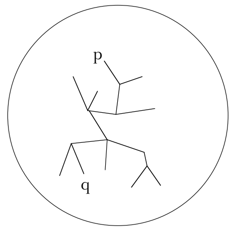


We will prove lemmas 18, 19, 20 twice: first with an informal argument accompanied with a picture; then with a more formal argument, unaccompanied by a picture.

For lemma 18: Suppose, by way of contradiction, that at the end of this procedure there were two vertices p, q so you couldn't walk with dry feet from p to q. Then you'd have a picture that looked something like this:



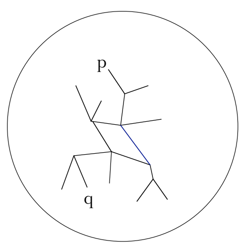
But then some edge would have been cut needlessly, since a picture like this



would also have all faces flooded, and we would be able to connect p, q.

More formally: what if there is some pair of vertices that are cut off from each other, i.e., you can't walk from one to the other? Then there must have been some edge fB which acted as a bridge: the dike network was connected just before B was broken, and disconnected just after B was broken. But that could only happen if the raging sea had already flooded *both* sides of B — and in that case we wouldn't have broken B in the first place. So it's impossible for there to be two mutually un-walkable-between vertices.

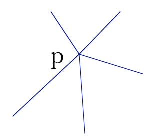
For lemma 19: Suppose, by way of contradiction, there were two paths between vertices p, q:



Then there would be a face that was not flooded.

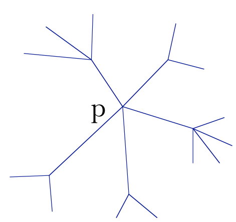
More formally: suppose, by way of contradiction, that there were two different paths between some pair of vertices. But then those paths would enclose some unflooded region, and that means that we didn't break enough edges.

For lemma 20: Pick a vertex p. There are exactly $v - 1$ vertices different from p. We let V_1 be the set of vertices that you can reach from p by only one unbroken edge, and let E_1 be the set of unbroken edges between p and the elements in V_1:



Note that $\#V_1 = \#E_1$, where $\#X$ is the size of X.

Let V_2 be the set of vertices that you can reach from p by exactly two unbroken edges, and let E_2 be the set of new unbroken edges we need to travel from p to a vertex in V_2:



Since a path from p to a vertex in V_2 starts with an edge in E_1 and adds an edge in E_2, we have $\#V_2 = \#E_2$.

And so on. $V_n =$ the set of vertices that you can reach from p by exactly n unbroken edges; $E_n =$ the set of new unbroken edges we need to travel from p to a vertex in V_n; $\#V_n = \#E_n$.

Eventually (because there are only finitely many vertices and edges to start with) we stop.

By lemma 6, each vertex other than p sits in some V_n. So we have at least $v - 1$ unbroken edges.

Are there any other unbroken edges?

Pick any edge $\overline{qr}$. p connects with q, so $q \in V_n$ for some n. Since r connects directly to q, either $r \in V_{n-1}$ (and $\overline{qr} \in E_n$) or $r \in V_{n+1}$ (and $\overline{qr} \in E_{n+1}$).

So when we count all the edges in all the E_n's, we've counted all the edges.

More compactly, we can work backwards: pick a vertex p. There are exactly $v - 1$ vertices other than p. By lemma 18, you can get from any other vertex q to p, and by lemma 19, for each other vertex q, there is a unique edge $f(q)$ that is the first step from q to p. Similarly by lemma 19, if q and q' are different vertices, then it cannot be the case that $f(q) = f(q')$ (otherwise one or both of the paths to p involves doubling back). On the other hand, every edge is $f(q)$ for *some* q (because the network is connected, so it's possible to walk from p towards that edge and eventually cross it). That is, the function $f : \{\text{vertices other than } p\} \to \{\text{edges}\}$ is a bijection, and so $e = v - 1$.

□

The proof of theorem 33 follows quickly from these lemmas:

Proof. . A given edge is either cut or not cut, and no edge can be both. By lemma 20, $v-1$ is the number of edges not cut. Since every time we cut an edge we flood a face, $f-1$ is the number of edges cut. So $e=(v-1)+(f-1)$, i.e., using a little algebra, $v+f-e=2$, as desired. □

3.4 An application of Euler's formula

When combined with other observations about the numbers e v, e and f, Euler's formula has other consequences. Remember that the *degree* of a vertex is the number of edges attached to it. For instance, all vertices in a cube have degree 3, while all vertices in an octahedron have degree 4. The degrees don't have to be the same in an arbitrary polyhedron: In a pentagonal pyramid (see p. 1), the apex of the pyramid has degree 5, while each of the base vertices has degree 3.

If you add up all the degrees of vertices in a polyhedron (in fact, in any graph), each edge will be counted twice. That is,

$$\text{(degree of vertex \#1)} + \text{(degree of vertex \#2)} + \cdots + \text{(degree of vertex \#}v\text{)} = 2e. \qquad (7)$$

If you add up the numbers of edges in all the faces of a polyhedra, you will again count each edge twice (because each edge lies in exactly two adjacent faces). That is,

$$\text{(number of edges inface \#1)} + \text{(number of edges inface \#2)} + \cdots$$
$$\cdots + \text{(number of edges inface \#}f\text{)} = 2e. \qquad (8)$$

For example, a cube has 8 vertices of degree 3 each (so the sum of all degrees is 24), and 6 quadrilateral faces (so the sum in equation 8 is also 24), and 12 (= 24/2) edges.

Formula equation 7 goes by the name of the *Handshaking Theorem* in graph theory — if you think of the vertices as people and each edge as a handshake between two people, then the degree of v is the number of people with whom v shakes hands, so the theorem says that adding up those numbers for all people, then dividing by 2, gives the total number of handshakes.

You can verify that these general rules are true for your favorite polyhedron.

Now every face has to have at least 3 sides. So the sum in equation 8 has to be *at least* $3f$. this observation together with equation 8, it tells us that $3f \leq 2e$ or equivalently $f \leq \frac{2e}{3}$. Now, substitute this inequality into Euler's formula to get rid of the f:

$$2 = v - e + f \leq v - e + \frac{2e}{3} = v - \frac{e}{3}$$

i.e.,

$$6 \leq 3v - e$$

i.e.,

$$e \leq 3v - 6.$$

Also, every vertex has to have degree at least 3, so the same calculation says that

$$e \leq 3f - 6$$

or equivalently

$$\frac{2e}{f} \leq 6 - \frac{12}{f} < 6.$$

So what? Well, $2e/f$ is the average number of edges in a face (just because there are f faces in total, and the sum of their numbers of edges is $2e$). Therefore, we have proved:

Theorem 34. *In every polyhedron, the average number of sides in a face is less than 6.*

Corollary 6. *There is no polyhedron all of whose faces are hexagons.*

4 Appendix: Proofs

4.1 How to prove something

A proof is simply a convincing argument. It is the way we convince ourselves and others that a mathematical statement is true. "Convincing" here means "logically convincing." We are not charming people, browbeating people, using rhetoric or psychological tricks. We are instead setting out a logical train of thinking so that the conclusion follows inescapably from the hypotheses.

Constructing a proof has five steps. If a proof is uncomplicated some of these steps can run into each other, but you should be aware of all five.

1. Be clear on your conclusion. That is: what are you trying to prove?

2. Be clear on your hypotheses. That is, what are you assuming? What information does that give you?

3. What general approach are you taking?

4. What steps are you taking and how do you justify them?

5. Now tighten this up into the final proof.

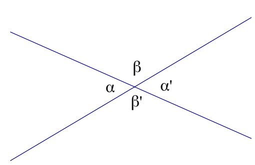
Step 5 is what you will hand in on homework and exams. Steps 1 through 4 are what get you to step 5. When you're trying to prove something, please don't skip steps 1 through 4. As you become more experienced you will find that steps 1 through 4 come more quickly, but they can never be skipped.

Example

We will carefully examine the proof of theorem 4, the vertical angle theorem (VAT). Since this is theorem 4, we can use theorems 1, 2 and 3, and also any of our definitions and axioms.

Step 1. What are you trying to prove?

a. In the following diagram, I am trying to prove that $\alpha = \alpha'$ and $\beta = \beta'$.



Step 2. What are you assuming, and what information does that immediately give you?

a. I am assuming I have two lines that cross at a point, with angles labelled as in the diagram above.

b. Because the lines cross at a point, they are not parallel.

Note: 1.a is just setting up the situation. At this point we don't know if 1.b will be helpful or not.

Step 3. What general approach are you taking?

a. Definition 4.1 tells us that $\alpha + \beta = \pi, \alpha' + \beta = \pi, \alpha + \beta' = \pi$ and $\alpha' + \beta' = \pi$.

b. I will try to use basic algebra to work with the equations in 3.a.

Note: Step 3 is always somewhat tentative: this is what you are going to try. You don't know if it will work yet. If it doesn't work, you try to find another general approach.

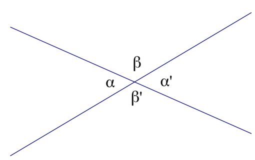
Step 4. What are your steps and how are you justifying them?

From definition 4.1, $\alpha + \beta = \pi = \alpha' + \beta$. By basic algebra, $\alpha = \pi - \beta = \alpha'$. Hence $\alpha = \alpha'$. Similarly, by definition 4.1, $\alpha + \beta = \pi = \alpha + \beta'$. By basic algebra, $\beta = \pi - \alpha = \beta'$, so $\beta = \beta'$.

Step 5. Now tighten this up into the final proof.

Theorem. Vertical angles are equal.

Proof. Consider two lines meeting at a point as in the following diagram (where we've labelled the angles):



We want to show that $\alpha = \alpha'$ and $\beta = \beta'$. By definition 4, $\alpha + \beta = \pi = \beta + \alpha'$, so $\alpha = \alpha'$. A similar proof shows that $\beta = \beta'$.

Comment: Note how condensed the final proof is compared with steps 3 and 4. Also note that while 2.b was part of our preliminary thinking, we didn't have to use it.

4.2 A little logic

All proofs take the form $P \Rightarrow Q$ where P is the hypothesis and Q is the conclusion.[39] We start with P and go down a logical path ending with Q. (A maze is a good analogy.) Often there is more than one way to do it. (Think of a maze with several paths.) Usually it's fairly easy to figure out what the hypothesis is and what the conclusion is, but sometimes it's tricky. To figure out which is the hypothesis and which the conclusion, you don't know have to know what the words mean, you just have to follow the logical form.

Here are some examples.

1. Dilations are similarities.[40]

Hypothesis: φ is a dilation. *Conclusion*: φ is a similarity.

2. Slithy toves wabe whenever it's brillig.[41]

[39] "$P \Rightarrow Q$" means "If P then Q."

[40] This is theorem 20. At this point in the course you might not know what a dilation is or what a similarity is. That doesn't matter.

[41] This is adapted from the poem *Jabberwocky* by the mathematician and logician Charles Dodgson, better known

Hypothesis: It's brillig. *Conclusion*: Slithy toves wabe.

3. There is exactly one perpendicular to a given line through a given point on the line.[42]

There are many ways to parse this out. Here are two of them: **Version 1**: *Hypothesis*: A point p is on the line m, and p is also on the lines k, k', where $k \perp m$ and $k' \perp m$. *Conclusion*: $k = k'$. **Version 2** *Hypothesis*: A, B, C are collinear with B between A, C. *Conclusion*: There is only one line through B which is perpendicular to $\overline{AC}$.

4. $\sqrt{2}$ is not rational.

Hypothesis: $x = \sqrt{2}$. *Conclusion*: x is not rational.

4.3 Mathematical language

Mathematical language is different from ordinary language in a number of ways.[43]

The only tense is the present tense. When we prove that the sum of the interior angles of a triangle is π, we don't mean that maybe sometime in the past it was different, or it might change sometime in the future. Nor do we mean that a triangle developed over time so that its angles eventually summed up to π...

There is no ambiguity — this is where definitions come in. You don't have to argue about what a circle is: it is a set of points equidistant from a given point (the center). As for undefined terms — think of "line" or "point" — you don't have to argue about what they really are, you just limit your talk about them to what your definitions and axioms and theorems allow you to say. "What is a line, really?" is a question of philosophy, not mathematics.[44]

Because there is no ambiguity, *every statement is either true and false.*[45]

"Or" means "or/and." E.g. when a mathematician says "Sam Brownback is a Republican or a Kansan" she is telling the truth.

"If... then..." has nothing to do with causality. Causality takes place in time, and since in mathematics everything is in the present tense, nothing can be caused by anything else. This is a radical change from ordinary language. Consider the sentence "If you get a college degree you will get a better job." Your getting a college degree supposedly *causes* you to get a better job. First you get a college degree. Because you did this, later you get a better job than you would have if you hadn't gotten a college degree.[46] "If you get a college degree you will get a better job" cannot be a theorem of mathematics because it involves time and causality.

"If... then", "iff", "or", "and", "not" are called logical connectives.

as Lewis Carroll.

[42] This is theorem 3.

[43] In this section we will use ordinary language at times to illustrate various points, but when we do we will be using ordinary language in mathematical ways.

[44] "Does this axiom system capture what I want to capture about lines and points?" is a fair question, but it too is not mathematical, it is what's called meta-mathematical.

[45] We might not *know* whether a mathematical statement is true or false, but not knowing its status says something about us, not about the statement.

[46] Steve Jobs and Bill Gates, who never graduated from college, might disagree.

4.4 Truth values

When is a mathematical statement true and when is it false? Remember that there are no intermediate values besides true or false.[47] So if P is true, then $\neg P$ is false.[48] If P is true and Q is true then "P and Q " is true; otherwise "P and Q " is false. If either P is true or Q is true (or both are true) then "P or Q " is true; otherwise it is false.

The first interesting question to consider is: when do we know that $P \Rightarrow Q$ is true? The answer might strike you as strange: $P \Rightarrow Q$ is *false* exactly when P is true and Q is false; otherwise it is true. In particular, if P is false, then $P \Rightarrow Q$ is true.

Why is this? $P \Rightarrow Q$ is true means: "if P then Q" is true, i.e., if P is true then Q is true. But this statement says nothing about when what happens when P is false.

Why should this be? Let's look at a simple example. Consider the statement "If $x = \sqrt{2}$ then x is not rational." This statement is only about those x's which equal $\sqrt{2}$. It doesn't say anything about any other numbers. So whether 3 is rational or irrational, whether π is rational or irrational, doesn't affect the truth of the statement "If $x = \sqrt{2}$ then x is rational."

Similarly, let's consider the statement "Every odd number bigger than 1 is a prime."Why is it false? Because 9 is odd and 9 is not a prime, i.e., 9 satisfies the hypothesis but not the conclusion.[49]. Whether 2 is prime (it is) is irrelevant, since 2 is not odd; similarly, whether 78 is prime (it's not) is irrelevant.

If this seems too abstract, remember what it's like to be a kid and have your mom or dad or grandmother or babysitter say: "If you finish your dinner you'll get dessert." The only way you'll think you've been lied to is if you finish your dinner (the hypothesis is true) but you didn't get dessert (the conclusion is false). If you *don't* finish your dinner but still get dessert, you won't think you've been lied to, you'll think you're lucky. And in mathematicsland, if you haven't been lied to, you were told the truth. Because every statement is either true or false.

Repeating with emphasis: *The only way $P \Rightarrow Q$ can be false is if P is true and Q is false. In any other situation, $P \Rightarrow Q$ is true.*

An important abbreviation which we don't use in ordinary language is "iff" or "$\Leftrightarrow$".[50] "$P \Leftrightarrow Q$" means "$P \Rightarrow Q$ and $Q \Rightarrow P$." $P \Leftrightarrow Q$ is true exactly when (P is true exactly when Q is true).[51] For example: "$ABCD$ is a square iff $\overline{AC} = \overline{DB}$ and $\overline{AC} \perp \overline{DB}$" is true because if a quadrilateral is a square then the diagonals are equal and perpendicular to each other, and if the diagonals of a quadrilateral are equal and perpendicular to each other then the quadrilateral is a square. If you have one (a square, *or* a quadrilateral-whose-diagonals-are-equal-and-perpendicular-to-each-other) — either one — you have the other. If you don't have one — either one —you don't have the other.

When you're asked to prove a statement of the form $P \Leftrightarrow Q$ you have to give two proofs, one of $P \Rightarrow Q$ and one of $Q \Rightarrow P$.

[47]This raises all kinds of sophisticated issues in logic — Google *Gödel's theorem* if you're interested — but we won't discuss these issues here.

[48]$\neg$ is the handy mathematical symbol for "not."

[49]There's a joke about this: the physicist says "3 is a prime, 5 is a prime, 7 is a prime, 9 is an experimental error..." while the engineer says "3 is a prime, 5 is a prime, 7 is a prime, 9 is a prime..."

[50]"iff" drives spellcheck crazy.

[51]Which is the same as: P is false exactly when Q is false.

In ordinary discourse we often conflate $\Rightarrow$ and $\Leftrightarrow$. For example, when you're told "If you eat your dinner you'll get dessert" you hear an implicit threat: "If you don't eat your dinner you won't get dessert." The two statements together give you "Eat your dinner iff you get dessert." Which is a different statement from the original.

4.5 Converse and contrapositive

Because of this break from causality, contrapositives (explained in the next paragraph) and converses (ditto) act a little differently than they do in our standard, informal, ambiguous way of speaking and writing.

Let's take a standard mathematical implication of the form $P \Rightarrow Q$. Its converse is $Q \Rightarrow P$. Its contrapositive is $\neg Q \Rightarrow \neg P$. If we know $P \Rightarrow Q$ is true, what do we know about its converse and its contrapositive?

Let's consider the converse, $Q \Rightarrow P$. It is false exactly when Q is true and P is false. But when P is false and Q is true, $P \Rightarrow Q$ is true, not false. So $Q \Rightarrow P$ is not the same as $P \Rightarrow Q$. For example, the implication "If a number is divisible by 6 then it is divisible by 3" is true, but its converse "If a number if divisible by 3 then it is divisible by 6" is not true. Sometimes both an implication and its converse are true; sometimes both are false; sometimes one of them is true and the other is false.

Note: A very common error is to try to prove the converse instead of the original statement. Don't do this.

On the other hand, consider the contrapositive, $\neg Q \Rightarrow \neg P$. It is false exactly when $\neg Q$ is true and $\neg P$ is false. Which is exactly when Q is false and P is true. Which is exactly when $P \Rightarrow Q$ is false. I.e., $P \Rightarrow Q$ is true exactly when $\neg Q \Rightarrow \neg P$ is true; $P \Rightarrow Q$ is false exactly when $\neg Q \Rightarrow \neg P$ is false.

If you want to prove $P \Rightarrow Q$ you can, instead, prove the contrapositive $\neg Q \Rightarrow \neg P$.

As a simple examples, let's once again analyze the proof of theorem 6: two distinct lines intersect in at most one point.

First we put the statement in ... $\Rightarrow$ ... form: Suppose ℓ, m are lines. If $\ell \neq m$ then there is at most one point in both ℓ and m.

The contrapositive is: Suppose ℓ, m are lines. If there are two or more points in both ℓ and m then $\ell = m$. [52]

Proof of contrapositive: Suppose $p, q \in l \cap m, p \neq q$. By axiom 1, $\ell = m$.

By proving the contrapositive, we proved the original statement.

[52]Note that the statement "Suppose ℓ, m are lines" goes before the hypotheses of both the statement and the contrapositive. That's because it is an overarching statement of context. Technically, we are defining the universe of objects we are talking about, and since this universe applies to both the hypothesis and the conclusion, we define it before the hypothesis.

4.6 Proof by contradiction

Proof by contradiction is similar to proof by contrapositive, but instead of simply assuming the negation of the conclusion we assume the negation of the conclusion *and* we assume the hypothesis. Then something happens which can't happen. Oops.

More formally, if we want to prove $P \Rightarrow Q$ by contradiction, we first assume P and $\neg Q$. From this we derive something which either contradicts something earlier in the proof (so we'd have a statement which is both true and false) or which we already know is false (say, $0 = 1$). So that means there was something wrong with our premise: you can't have both P and $\neg Q$. I.e. either P is false, or both P and Q are true. But these are exactly the circumstances in which $P \Rightarrow Q$ is true.

For an example of a proof by contradiction using only basic facts of arithmetic, consider

Theorem 35. *$\sqrt{2}$ is irrational.*[53]

Proof. Suppose $\sqrt{2}$ is rational. (I.e., suppose $x = \sqrt{2}$ — this was our hypothesis — and x is rational — this negates our conclusion.) Then

$$\sqrt{2} = \frac{n}{m} \tag{9}$$

where (†) n, m are positive integers with no common divisors other than 1. Since $1 < \sqrt{2} < 2$, $m \neq 1$. By equation 9,

$$2 = \frac{n^2}{m^2} \tag{10}$$

I.e., $\frac{n^2}{m^2}$ is an integer, so m, n have common divisors, which contradicts (†). [Hence (†) is both true and false, which can't happen. The premise of our argument must be false.] So $\sqrt{2}$ is irrational. □

Note: The part of the proof within the square brackets is usually left out of proofs by contradiction, but I put it in to make the reasoning crystal clear.

4.7 Quantifiers

In mathematics we are very careful about distinguishing "there exists" (written $\exists$) from "for every" (written $\forall$). For example: $\exists x\ x^2 = 2$ is true in the real numbers (because there is a real number whose square is 2), false in the integers (because there is no integer whose square is 2), while $\forall x\ x^2 = 2$ is false in any reasonable number system (what kind of number system could have the square of *every* number = 2?).

$\exists x\ P(x)$ is true if you can find at least one x for which P holds. $\exists x\ P(x)$ is false if P fails for all x.

[53] There are many proofs of this theorem, including some lovely geometric ones, at the cut-the-knot website. To access them, go to http://www.cut-the-knot.org/proofs/sq_root.shtml.

$\forall x\ P(x)$ is true if P is true for all x. $\forall x\ P(x)$ is false if you can find at least one x for which P does not hold.

Things get more complicated when we alternate quantifiers. For example, let $P(x, y)$ abbreviate "$x \neq y$"

$\forall x\ \exists y\ P(x, y)$ means: for every x there is at least one $y \neq x$, i.e.: there are at least two things. This statement is true in most contexts.

$\exists x\ \forall y\ P(x, y)$ means: there's some x which doesn't equal *any* y, including itself. This statement is false in all contexts, since every $x = x$.

Now let's go backwards, putting in quantifiers and logical connectives explicitly.

"Some birds are blue and other birds are red" becomes: $\exists x$ (x is a bird and x is blue) and $\exists x$ (x is a bird and x is red).

"Not every bird can fly" becomes: $\neg\forall x$(x is a bird $\Rightarrow$ x can fly). "No bird can fly" becomes: "$\forall x$ x is a bird $\Rightarrow \neg$(x can fly)". These statements are not equivalent. "Not every bird can fly" is true: penguins can't fly. "No bird can fly" is false: robins can fly.

"Every number is bigger than another number" becomes: $\forall x$(x is a number $\Rightarrow \exists y$ y is a number and $x > y$). This statement is true for the integers, but false for the positive integers, since no positive integer is smaller than 1.

5 Homework problems

5.1 Euclidean proofs

EP 1. (a) Explain why the construction of an angle bisector works.

(b) Explain why your the construction of a perpendicular bisector works.

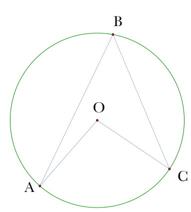
EP 2. (Converse of ITT) Let ΔABC be a triangle with $\angle BAC \cong \angle BCA$. Prove that $\overline{AB} = \overline{BC}$. **Honors**: do this in two ways.

EP 3. Prove that if two distinct lines are cut by a transversal and a pair of corresponding angles is congruent, then the lines are parallel. (This is corollary 1 in chapter 1.)

EP 4. Suppose $\overline{AB} \cong \overline{BC}$. Prove that the angle bisector of $\angle ABC$ is the perpendicular bisector of $\overline{AC}$. [Hint: in the diagrams for the proofs of theorem 11, show that $D = E$.]

EP 5. Take an equilateral triangle. Connect the midpoints of each side. Prove that this inner triangle is also equilateral.

EP 6. Prove the conjecture you developed in exercise SA 3 for the picture below:



Here O is the center of the circle, A, B, C are points on the circle.

Honors Prove your conjecture in SA 3 when either $\overline{AB}$ crosses the line $\overline{OC}$, or $AOCB$ form a convex quadrilateral?

EP 7. State AAS clearly. Prove it.

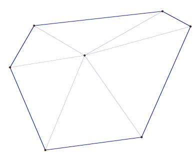
EP 8. State SSA clearly. Give an example of two triangles where the hypothesis SSA holds but the triangles are not congruent.

EP 9. Prove that the perpendicular bisectors of the sides of any triangle meet in a point. (Hint: Let p be a point where two of the perpendicular bisectors meet. Which points is p equidistant from?)

EP 10. We define the distance of a point p from a line ℓ to be the length of the line segment from p perpendicular to ℓ. Prove that a point is equidistant from the sides of an angle iff it is on the angle bisector.

EP 11. Use EP 10 to show that the three angle bisectors of a triangle meet in one point.

EP 12. What is the sum of the interior angles of a convex polygon with n sides? Prove it. Use a diagram like the one below (the diagram is for $n = 6$, but your argument should be completely general).



EP 13. Carefully write up proofs justifying the area formulas for a parallelogram and for a triangle.

EP 14. (a) Prove that a convex quadrilateral is a parallelogram iff its opposite sides are congruent and its opposite angles are congruent.

(b) Prove that a convex quadrilateral is a parallelogram iff its diagonals bisect each other.

(c) Let $ABCD$ be a parallelogram and let E be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$. Prove that the area of ΔAEB is $\frac{1}{4}$ the area of the parallelogram.

EP 15. Let $A \neq B$ be the points of intersections of two circles with centers P, Q. Prove that $\overline{AB} \perp \overline{PQ}$.

EP 16. We say that a convex figure S is *inscribed* in a polygon T iff (a) S is inside T, (b) S touches every side of T, and (c) each side of T meets S at only one point. Prove that if a circle is inscribed in a rectangle, the rectangle is a square.

Honors. Let T be a polygon. Prove that if a convex figure S satisfies (a) and (b), then it satisfies (c).

EP 17. Take a bunch of congruent line segments and join them together so all the angles where consecutive lines meet are congruent. Prove that the angle bisectors meet in one point. [Note: the lines don't necessarily form a polygon — they might cross each other or the figure might not be closed.]

EP 18. Using definition 3 and axiom 12 in the section of Euclidean geometry, prove the following:

(a) Let r be any positive real number. Given ΔABC, a point D on $\overline{AB}$ and a point E on $\overline{BC}$ so $\frac{\overline{BD}}{\overline{AB}} = \frac{\overline{BE}}{\overline{CB}} = r$, then $\overline{DE} \parallel \overline{AC}$ and $\overline{DE} = r\overline{AC}$.

(b) Use (a) to prove that the line segment joining the midpoints of two sides of a triangle is parallel to and has half the length of the third side.

(c) Prove that, given ΔABC, points D on $\overline{AB}$ and E on $\overline{BC}$, with $\overline{DE}||\overline{AC}$ then $\Delta ABC, \Delta DBE$ are similar triangles.

EP 19. Consider a triangle ΔABC. Let E be the midpoint of $\overline{AB}$, F the midpoint of $\overline{CB}$, G the midpoint of $\overline{AC}$. By EP 18, $\Delta ABC, \Delta EBF$ are similar.

(a) Let P be the point of intersection of $\overline{BG}, \overline{EF}$. Prove that ΔBFP is similar to ΔBCG.

(b) Using (a), show that $\overline{FP} = \frac{1}{2}\overline{EF}$.

(c) Prove that triangles $\Delta ABC, \Delta FGE$, and ΔAEG are all similar.

(d) The area of $\Delta GEF = r\cdot$ the area of ΔABC. What's r?

EP 20. (a) Prove that the diagonals of a parallelogram meet in a right angle iff the parallelogram is a rhombus.

(b) Prove that the diagonals of a parallelogram are congruent iff the parallelogram is a rectangle.

EP 21. Prove that the angle bisectors of a convex quadrilateral meet in one point iff a circle can be inscribed in the quadrilateral.

Honors Does this generalize for arbitrary convex polygons? Give brief reasons.

EP 22. Prove that the vertices of a convex quadrilateral lie on a circle iff each pair of opposite vertices sums to π.

EP 23. Prove the conjecture you developed in SA 8 for a convex quadrilateral.

Honors What if the quadrilateral is not convex? From *Sketchpad* you have a conjecture. Can you prove it?

EP 24. Prove that in a regular polygon the point of intersection of the angle bisectors is also the point of intersection of the perpendicular bisectors of the sides. (Note that you will need to show that these points of intersection exist before showing that they are the same.)

EP 25. Why does the construction in SA 10 work?

EP 26. Suppose you are given a circle without being given the center. How can you construct the center?

EP 27. Let $\mathcal{T}$ be a triangle, and let $\mathcal{M}_\mathcal{T}$ be the triangle formed by its midpoints.

(a) Prove that $\mathcal{M}_\mathcal{T} \approx \mathcal{T}$.

(b) Prove that the midpoints of $\mathcal{M}_\mathcal{T}$ lie on the medians of $\mathcal{T}$.

(c) Use (b) to show that the medians of $\mathcal{M}_\mathcal{T}$ are the medians of $\mathcal{T}$.

(d) Now let $\mathcal{T}_0 = \mathcal{T}, \mathcal{T}_1 = \mathcal{M}_{\mathcal{T}_0}, \mathcal{T}_2 = \mathcal{M}_{\mathcal{T}_1}$, and so on, i.e., each $\mathcal{T}_{n+1} = \mathcal{M}_{\mathcal{T}_n}$. Using (b) and a proof by continuity, show that the medians of $\mathcal{T}$ meet in one point.

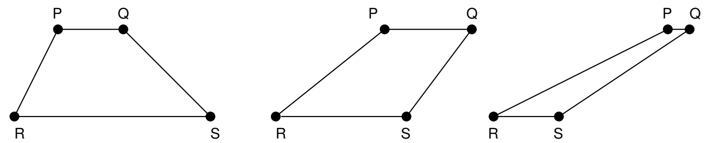
[Hint: You developed the intuition behind this proof in SA 9.]

EP 28. Prove that a rhombus inscribed in a circle is a square.

EP 29. Suppose three circles in the plane, all of the same radius r, pass through a common point O. Show that their other points of intersection A, B, C lie on a circle of radius r. [Hint: a *Sketchpad* construction might give you a sense of what this is about.]

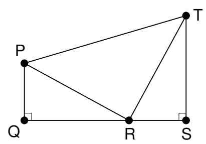
EP 30. A trapezoid is a convex quadrilateral with at least two parallel sides. Here are some examples:[54]

[54] So parallelograms are examples of trapezoids. The trapezoids in pattern blocks are called *isosceles trapezoids* because the non-parallel sides have the same length, but trapezoids which are not parallelograms do not have to be isosceles.



Find a formula for the area of a trapezoid and prove that it works for all trapezoids.

EP 31. The following proof of the Pythagorean theorem was discovered in 1876 by future U.S. President James A. Garfield. Construct a right triangle ΔPQR with sides PQ, QR of lengths a and b and hypotenuse PR of length c. Construct another right triangle ΔRST such that $\Delta PQR \cong \Delta RST$ and R is between Q and S, as shown below. With this setup, prove that $c^2 = a^2 + b^2$. [Hint: Use EP 30.]



5.2 *Sketchpad* assignments

Unless otherwise suggested, send these assignments to me as e-mail attachments. I grade them mostly by dragging stuff and seeing if the construction does what it's supposed to. Be sure to name each sketch exactly as suggested, and be sure your e-mail has your name and the problem(s) in the subject line. For example, if I were handing in # 83 and was told to label it TRAP, the subject line would read: Roitman TRAP.[55] If you hand a *Sketchpad* assignment in early and I discover that it's not correct, I'll send it back with comments so you can correct the problem. So it's a good idea to hand them in early.

SA 1. Warm-ups: (a) Construct a regular hexagon using *Sketchpad* and send your construction in a sketch named HEX. Within your sketch write: this is a regular hexagon.[56]

(b) Construct a square using *Sketchpad* and send your construction in a sketch named SQUARE.

SA 2. Construct a triangle whose base is the diameter $\overline{AB}$ of a circle. Place a point C on the circle. Measure $\angle ACB$. Send your construction in a sketch named SEMI. Within your sketch, tell me what you found.

SA 3. Construct a figure as in EP 6. Move points A, B, C around. Send me your construction in a sketch named DIAMTRI. Within your sketch, tell me the relationship between $m(\angle AOC)$ and $m(\angle ABC)$.

Explain why the conclusion of SA 2 is a corollary of the conclusion in SA3.

Honors Tell me what happens when either $\overline{AB}$ crosses the line $\overline{OC}$, or $AOCB$ form a convex quadrilateral.

SA 4. Construct circle with diameter $\overline{AC}$. Place a point D anywhere. Send me your construction in a sketch named INOUT. Within your sketch tell me: what can you say about $\angle ADC$ when D is inside (not on) the circle? when D is outside the circle?

[55] Don't worry, there are a lot fewer than 83 problems in this section.

[56] I know that's kind of dumb, but this is so you learn how to write in a sketch.

SA 5. The following exercise has three parts. We begin by constructing an arbitrary triangle.

(a) In blue, construct the perpendicular bisectors of the sides of the triangle. Deform the triangle. On your sketch, tell me whether these three lines meet in 1, 2, or 3 points.

(b) In red, construct the three angle bisectors of the triangle. Deform the triangle. On your sketch, tell me whether these three lines meet in 1, 2, or 3 points.

(c) In green, construct the three lines that connect a vertex to the midpoint of the opposite side. Deform the triangle. On your sketch, tell me whether these three lines meet in 1, 2, or 3 points.

Send me the completed sketch (all three parts) under the name TRILINES.

SA 6. (a) Use EP 9 to circumscribe a circle about an arbitrary triangle ΔABC, starting with a sketch of the triangle. (Hint: once you have the triangle, you need to figure out where the center of the circle must be — that's where you'll use the EP exercise. Use "circle by center + point" to construct the circle.) As you move the vertices of the triangle, the circle should change with the triangle — this is what you'll be graded on. Be sure to hide any extra lines: all I should see is the triangle, the circle, and the vertices of the triangle. Send me the sketch under the name CIRCUMCIRCLE. *Do not* simply place three points on the circle and connect them —- I can tell by "show all hidden" whether you've done this sketch correctly.

(b) Measure $\angle ABC$ and, in the sketch of (a), say what happens when (i) the center of the circle p is inside the triangle; (ii) the center of the circle p is on the triangle; (iii) the center of the circle p is outside the triangle.

SA 7. Similarly (I won't tell you which EP problem(s) to cite), use *Sketchpad* to inscribe an arbitrary circle in a triangle (that is, the sides of the triangle are tangent to the circle), and send me the sketch under the name INCIRCLE.

SA 8. Create an arbitrary quadrilateral $\mathcal{Q}$ (i.e., one that can be deformed to any quadrilateral shape). Construct the quadrilateral $\mathcal{R}$ whose vertices are the midpoints of the sides of $\mathcal{Q}$. Send me the sketch under the name QUAD. In the sketch, tell me what kind of quadrilateral $\mathcal{R}$ is.

SA 9. (a) Create an arbitrary triangle $\mathcal{T}$ and construct the triangle $\mathcal{M}_{\mathcal{T}}$ whose vertices are the midpoints of the sides of $\mathcal{T}$. Construct the medians of $\mathcal{T}$ and the midpoints of the sides of $\mathcal{M}_{\mathcal{T}}$. Send me this sketch under the name MID. Answer all of the following questions within the sketch:

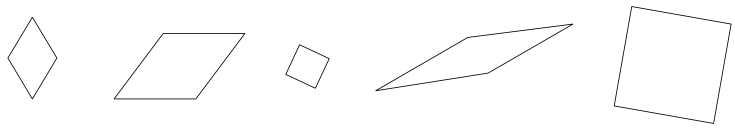
(b) What is the relation of the midpoints of $\mathcal{M}_{\mathcal{T}}$ and the medians of $\mathcal{T}$? What is the relation of the medians of $\mathcal{M}_{\mathcal{T}}$ and the medians of $\mathcal{T}$?

(c) Compare the area of $\mathcal{M}_{\mathcal{T}}$ and the area of $\mathcal{T}$.

(d) Now do this 5 times: $\mathcal{T}_0 = \mathcal{T}; \mathcal{T}_1 = \mathcal{M}_{\mathcal{T}_0}; \mathcal{T}_2 = \mathcal{M}_{\mathcal{T}_1}; \mathcal{T}_3 = \mathcal{M}_{\mathcal{T}_2}; \mathcal{T}_4 = \mathcal{M}_{\mathcal{T}_3}, \mathcal{T}_5 = \mathcal{M}_{\mathcal{T}_4}$. What can you say about the medians of all of these triangles? Compare the area of $\mathcal{M}_{\mathcal{T}_i}$ and the area of $\mathcal{T}$. [Write your conclusion to (d) in your sketch but do *not* send me a sketch of part (d).]

SA 10. Draw a circle. Put four points on it and connect them to form the quadrilateral $ABCD$. Calculate $\angle ABC + \angle ADC$. Send me this sketch under the name CIRCQUAD. In the sketch, tell me what happens to this sum as you move the points around on the circle.

SA 11. Construct a rhombus $ABCD$. To earn full credit, you should be able to drag the vertices around so that the lengths and angles can vary freely. For example, you should be able to make your figure look like any possible rhombus (such as all of the following):



Send your construction to me in a sketch named RHOMBUS. Before you send it, hide everything except the four vertices and four sides of the rhombus.

SA 12. Simulate the construction with a straight-edge and *collapsing* compass of the duplication of a given straight line segment starting from a given point:

(a) Construct a line $\overline{AB}$ and a point p not on $\overline{AB}$.

(b) Construct an equilateral triangle ApC.

(c) Construct a circle centered at A with radius $m(\overline{AB})$.

(d) If C is outside the circle, let D be the point on $\overline{AC}$ on the circle. If C is inside the circle, extend $\overline{AC}$ until it meets the circle. Call this point of intersection D. Your will have two choices for D. Pick the one closest to C.

(e) Construct an equilateral triangle CDE.

(f) Measure $\overline{AB}$. Measure $\overline{pE}$.

(g) Now drag B. Drag p. What can you say about the measurements in (f)?

Send me the sketch under the name SECC. Don't hide anything! Be sure to include your answer to (g).

SA 13. *The Taylor circle*

An *altitude* of a triangle is the line from a vertex perpendicular to the opposite side. Note that the altitude need not intersect the side (because the side is a finite line segment); it does intersect the infinite line through the side (the extended side). The *foot* of the altitude is the intersection of the altitude and the extended side.

(a) Make an arbitrary triangle with extended sides and sketch the feet of the altitudes as blue points. Your sketch should show each altitude even when the triangle is deformed so it misses the side. [To make this triangle: sketch three points, highlight them, and instead of choosing *segment* choose *line* in the **Construct** menu.]

(b) From each foot, draw the perpendiculars to the other two sides, and sketch those intersections as green. (Again, even if one of these points misses a side, it still should show.)

(c) Choose any three green points and, using the technique of SA 6, sketch the circle that they lie on. Where are the other green points? State your conclusion as a conjecture. [Here's an example of the format, whose conclusion is false: Given a triangle, the feet of the altitudes and the intersections of the perpendiculars from each foot and the extended sides all lie on a straight line.]

Send me the sketch under the name TAYLOR. All I should see are the triangle, the red points, the green points, the circle, and your answer to (c).

SA 14. *The 9 point circle*

Again draw an arbitrary triangle. In red mark the midpoints of the sides. In green mark the feet of the altitudes. In blue mark the midpoints between the orthocenter (where the altitudes meet) and the vertices. Pick three of these points, and use the technique of SA 6 to sketch a circle they lie on. Where are the other marked points? Again, state your conclusion as a conjecture.

Send me the sketch under the name 9PT. All I should see is the triangle, the 9 points, the circle, and your answer.

SA15. *Simpson's line*

Draw a circle, and place a triangle on the circle (so the triangle is circumscribed by the circle). Place a point p on the circle that isn't on the triangle. In green mark the feet of the altitudes from p to the extended sides of the triangle. Draw an infinite line through two of these green points. Where is the third green point? State your conclusion as a conjecture.

Send me the sketch under the name SIMPSON. All I should see is the the circle, the green points, and the line through them.

SA16. *The Euler line*

Sketch a triangle. In green mark the circumcenter (where the perpendicular bisectors of the sides meet), the centroid (where the lines connecting each vertex to the midpoint of the opposite side meet), and the orthocenter (where the altitudes meet). Draw an infinite line through two of these green points. Where does the third point lie? State your conclusion as a conjecture.

Send me the sketch under the name EULER. All I should see is the triangle, the green points, and the line through them.

SA 17. *Ptolemy's theorem*

Create an arbitrary convex quadrilateral $ABCD$ inscribed in a circle. Measure $\overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC}$. Measure $\overline{AC} \cdot \overline{BD}$. What did you find? State your conclusion as a conjecture. Deform the quadrilateral to convince yourself this is a plausible hypothesis. Which deformations maximize the expression $\overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC}$? What kind of quadrilateral gives the maximum value?

Send me the sketch and your conclusionsunder the name PTOLEMY.

SA 18. Place a point p inside a circle, a point A on the circle, and sketch the ray from A through p. Let B be the point where this ray meets the circle. Measure $\overline{Ap} \cdot \overline{Bp}$.

(a) Leave p alone and move A around the circle. What happens to this product?

(b) Leave A alone and move p around the circle. Where is the product maximized?

Send me the sketch and your conclusions under the name ANON.

5.3 Transformational geometry

Note: When you are asked to prove something in this section, you are asked to prove it using transformational geometry techniques, not Euclidean axioms.

TG1. What is the inverse of $\rho_{p,\alpha}$? What is the inverse of $\tau_{\vec{v}}$?

TG 2. (a) Which isometries of the plane are symmetries of

- a line segment?

- an equilateral triangle
- an isosceles triangle that is not equilateral?
- a scalene triangle? (That is, a triangle with three unequal sides.)
- a circle?

(b) Consider a line ℓ and two points p, q not on ℓ. Which isometries of the plane fix

- ℓ
- p
- p and q
- p and ℓ

For both (a) and (b), make your answers specific (for example, rather than saying "figure X has rotation symmetry," give the possible centers and angles of rotations that are symmetries of X)

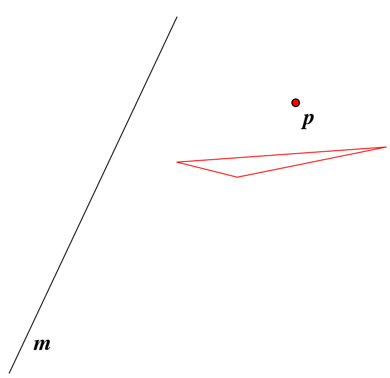
TG 3. Let's name the triangle Δ.

(a) Rotate Δ about p by $\frac{\pi}{4}$.

(b) Reflect Δ across m.

(c) Sketch $r_m \rho_{p,\pi}(\Delta)$.

(d) Sketch $\rho_{p,\pi} r_m(\Delta)$.



TG 4. Let ℓ be the x-axis, m the $y = x$, and p the origin (0,0).

(a) $r_\ell \circ r_m$ is a rotation. About what point by what angle?

(b) $r_m \circ r_\ell$ is also a rotation. About what point by what angle?

(c) What is $r_\ell \circ \rho_{p,\pi}$?

(d) What is $r_m \circ \rho_{p,\pi}$?

(e) What is $r_\ell \circ \rho_{p,\pi/2}$?

(f) What is $\rho_{p,\pi/2} \circ r_\ell$?

[For (c), (d), (e), and (f), your answer must be stated as exactly one rotation or one reflection or one translation, or one glide reflection.]

TG 5. (a) Does the set of all translations of the plane form a group? Why or why not?

(b) Does the set of all reflections of the plane form a group? Why or why not?

(c) The set of all rotations of the plane forms a group. What does this tell you about the composition of any two rotations, no matter what their centers or angles of rotation?

TG 6. Consider the transformation φ of $\mathbb{R}^2$ that takes (x, y) to $(2x, y)$.

(a) If ℓ is the x-axis, what is $\varphi[\ell]$?

(b) If m is the line $y = x$, what's $\varphi[m]$?

(c) If n is the line $y = ax + b$ what's $\varphi[m]$?

(d) Prove that φ is affine, but not a similarity and hence not an isometry.

TG 7. Let ℓ be a line and p a point on ℓ.

(a) Find a line m such that $r_\ell \circ \rho_{p,\pi/2} = r_m$. (Hint: Carry out the composition $r_\ell \circ \rho_{p,\pi/2}$ using *Sketchpad*. Use your example to make an educated guess about what m is. Then prove that you are correct using the Three-Point Theorem.)

(b) Do the same thing for these other three compositions (that is, express each one of them as reflection across an appropriate line): (i) $r_\ell \circ \rho_{p,3\pi/2}$; (ii) $\rho_{p,\pi/2} \circ r_\ell$; (iii) $\rho_{p,3\pi/2} \circ r_\ell$.

TG 8; Let ℓ be a line, let p be a point on ℓ, and let θ be any angle. Express $r_\ell \circ \rho_{p,\theta} \circ r_\ell$ as a single transformation. Express $\rho_{p\theta} \circ r_\ell \circ \rho_{p,\theta}$ as a single transformation.

TG 9. Let p be a point in the plane. Let G be the following set (actually, group) of transformations:

$$G = \{id, \rho_{p,2\pi/3}, \rho_{p,4\pi/3}\}.$$

Construct a figure whose symmetries are exactly the transformations in G. (Be careful—you need to make sure that the figure has no reflection symmetries. For example, an equilateral triangle will not work.)

TG 10. If p, q are vertices of a polygon P, we say p, q are adjacent iff $\overline{pq}$ is a side of P.

(a) Let P be a polygon. Show that if φ is a symmetry of P then for each vertex p, p and $\varphi(p)$ are adjacent.

(b) Show that for any polygon P there are at most $2n$ symmetries of P.

(c) Show that a polygon P has exactly $2n$ symmetries iff P is regular. [Hint: we already proved one direction — which one?]

TG 11. Prove that if ℓ is tangent to a circle at point p on the circle, then ℓ is perpendicular to the diameter through p. (Hint: consider reflection symmetry about the diameter through p.)

TG 12. Prove EP 15 using transformational geometry.

TG 13. Using the definition of parallelogram in the chapter 2 (definition 13):

(a) prove that the diagonals of a parallelogram bisect each other.

(b) prove that a the opposite sides of a parallelogram are parallel.

TG 14. Using the definition of a kite in the chapter 2 (definition 13)) prove that in a kite there are two disjoint pairs of sides with equal length.

TG 15. Using the definition of a rectangle in the chapter 2 (definition 13) prove that all the angles of a rectangle are right angles.

TG 16. Using the definition of a rhombus in the chapter 2 (definition 13) prove that all sides of a rhombus are equal.

TG 17. Prove that if a triangle has reflection symmetry, then at least one vertex must be on the line of reflection, and that the triangle is isosceles.

TG 18. Let $p = (0, 0), q = (1, 1)$ and Δ the triangle whose vertices are (-1,0), (1,0), (0,1).

(a) What are the vertices of $\delta_{p,3}(\Delta)$?

(b) What are the vertices of $\delta_{q,3}(\Delta)$?

TG 19. If two lines are parallel, then their images under a dilation are also parallel. Why? [Hint: you might find a Euclidean axiom useful.]

TG 20. Find the symmetries of the shapes in the handout.

TG 21. Find the symmetries of the tilings in the handout.

TG 22. Find the symmetries of the frieze patterns in the handout.

5.4 Polyhedra

PH 1. List all symmetries of the cube.

PH 2. What are the symmetries of (a) a parallelpiped which is not a cube? (b) a sphere? (c) a tetrahedon?

PH 3. Which of the following are ruled out by Euler's formula?

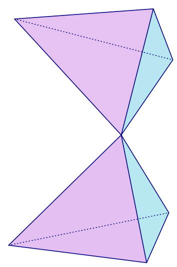
(a) a convex polyhedron with 5 vertices, 10 faces and 13 edges.

(b) a convex polyhedron with 5 vertices, 13 faces and 10 edges.

PH 4. Actually, neither of the polyhedra in PH 3 exists. This is because a convex polyhedron with 5 vertices can only have 5 or 6 faces. Why is this?

PH 5. Determine v, e and f (the numbers of vertices, edges and faces) for (a) a pyramid whose base is an n-sided polygon ("n-gon"); (b) a bipyramid whose base is an n-gon; (c) a prism whose base is an n-gon. Verify that each of these polyhedra satisfies Euler's formula.

PH 6. (a) Consider the double tetrahedral "cone":

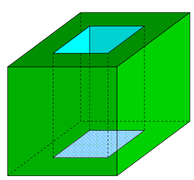


Determine v, e, and f. What is $v+e-f$? Does this make you doubt Euler's formula? Why or why not?

(b) If you glue two polyhedra so they touch only at a single vertex (as we did in (a)), what is $v+e-f$?[57] Briefly explain how you know this.

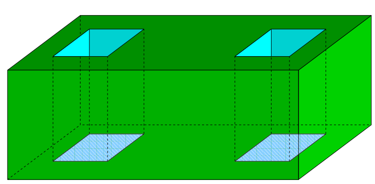
PH 7. In this problem we consider solids which are not, strictly speaking, polyhedra, because not all faces are polygons — some faces have holes.

(a) Consider a square prism with a hole drilled through the middle (a "big square donut"), as pictured.



Determine v, e and f. What is $v+f-e$?

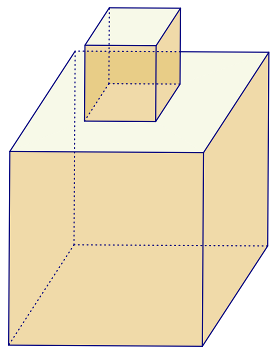
(b) Suppose that you drill two holes, as in the figure below. Determine v, e, f. What is $v+f-e$?



(c) Now suppose that you drill h holes as we did in (a) and (b). What is $v+f-e$? Briefly explain how you know this.

(d) Now consider the "cube above cube" below:

[57] The polyhedra do not have to be the same kind.



Determine v, e, f. What is $v + f - e$?

(e) What if you stack n cubes above each other, each one smaller than the previous one, as in (d).[58] What is $v + f - e$? Briefly explain how you know this.

5.5 Logic

L1. Let P be the statement: Abraham Lincoln was born in Kentucky. Let Q be the statement: George Washington chopped down a cherry tree when he was a little boy. Let R be the statement: John F. Kennedy was born in Massachusetts. P and R are true, Q is false. Knowing that, which of the following statements are true and which are false?

(a) $P \Rightarrow Q$.

(b) $Q \Rightarrow P$.

(c) P and Q.

(d) P or Q.

(e) $\neg P$

(f) $\neg Q$.

(g) $(P \Rightarrow Q)$ or R.

(h) $(Q \Rightarrow P)$ or R.

(i) $(Q \Rightarrow P)$ and R.

L2. For each of the following statements, state the conclusion and state the hypothesis.

(a) Every dog has its day.

(b) Cats do not like baths.

(c) Horses are bigger than dogs.

(d) All horses have the same color.

L3. Make the quantifiers and logical connectives explicit for the following statements:

[58] so in (d), $n = 2$

(a) Every person is married to somebody.

(b) Some person is married to everyone.

(c) No person is married to everyone.

(d) No person is married to anyone.

L4. In which of the following sets is "Every number is smaller than another number" true, and in which is it false?

(a) the real line

(b) the integers

(c) the positive integers

(d) the negative integers

L5. In the text we said that "Every number is bigger than another number" can be expressed as: $\forall x(x$ is a number $\Rightarrow \exists y\; y$ is a number and $x > y)$. But it cannot be expressed as follows: $\forall x\; \exists y(x, y$ are numbers and $x > y)$. Why not?[59]

[59] Hint: this is a trick question about context.