HOMEWORK 7

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Abstract. Please send me an email if you find mistakes. Thanks.

1. P239. # 29.1

Proof. (a). The mean value theorem holds because $x^2$ is continuous on $[-1, 2]$ and is differentiable on $(-1, 2)$. Let $f(x) = x^2$. Then

$$f(b) - f(a) = 2^2 - (-1)^2 = 3, b - a = 2 - (-1) = 3.$$ 

Then since $f'(x) = 2x$, $c$ can be taken as

$$c = \frac{1}{2}.$$ 

Then

$$f(b) - f(a) = f'(c)(b - a).$$

(c). The mean value theorem does not apply in this case because $|x|$ is not differentiable at 0.

(e). The mean value theorem holds because $\frac{1}{x}$ is continuous on $[1, 3]$ and is differentiable on $(1, 3)$. Let $f(x) = \frac{1}{x}$. Then

$$f(b) - f(a) = \frac{1}{3} - 1 = -\frac{2}{3}, b - a = 3 - 1 = 2.$$ 

Then since $f'(x) = -\frac{1}{x^2}$, $c$ can be taken as

$$c = \frac{1}{\sqrt{3}}.$$ 

Then

$$f(b) - f(a) = f'(c)(b - a).$$

□
2. P239. # 29.2

Proof. We know that \( \cos' x = -\sin x \) and \( |\sin x| \leq 1 \) for all \( x \in \mathbb{R} \). Then for \( x, y \in \mathbb{R} \), there exists \( c \in \mathbb{R} \) such that

\[
\cos x - \cos y = -\sin c (x - y).
\]

Then

\[
|\cos x - \cos y| = |\sin c||x - y| \leq |x - y|.
\]

This proves the claim. \( \Box \)

3. P239. # 29.3

Proof. (a). By the mean value theorem, there exists \( x \in (0, 2) \) such that

\[
f(2) - f(0) = f'(x)(2 - 0).
\]

So

\[
f'(x) = \frac{1}{2}.
\]

(b). For this one, we will have to invoke Theorem 29.8, the intermediate value theorem for derivatives. So we skip the proof. \( \Box \)

4. P239. #29.4

Proof. We follow the hint. For \( h(x) = f(x)e^{g(x)} \), \( h \) is continuous on \([a, b]\) and is differentiable on \((a, b)\). Moreover,

\[
h(a) = h(b) = 0.
\]

By the mean value theorem, there exists \( x \in (a, b) \) such that

\[
h'(x) = 0.
\]

That is to say

\[
h'(x) = f'(x)e^{g(x)} + f(x)e^{g(x)}g'(x) = (f'(x) + f(x)g'(x))e^{g(x)} = 0.
\]

Since \( e^{g(x)} \neq 0 \) for all \( x \), we obtain

\[
f'(x) + f(x)g'(x) = 0.
\]

This proves the claim. \( \Box \)
5. P239. # 29.7

Proof. (a). Since \( f'' = (f')' \) and \( f'' \equiv 0 \),

\[ f' = a \]

for some constant \( a \in \mathbb{R} \). We rewrite it,

\[ (f(x) - ax)' = 0. \]

Then \( f(x) - ax = b \), i.e., \( f(x) = ax + b \).

(b). By part (a),

\[ f'(x) = ax + b. \]

That is to say,

\[ \left( f(x) - \frac{a}{2}x^2 - bx \right)' = 0. \]

Then \( f(x) - \frac{a}{2}x^2 - bx = c \)

for some constant \( c \). Then

\[ f(x) = \frac{a}{2}x^2 + bx + c. \]

\[ \square \]

6. P240. # 29.9

Proof. This is obviously true for \( x \leq 0 \). We consider \( f(x) = e^x - x \) on \([0, \infty)\)

Then \( f'(x) = e^x - 1 \geq 0. \)

This holds for \( x \geq 0 \). So \( f \) is increasing on \([0, \infty)\):

\[ f(x) \geq f(0) = 1 \geq 0. \]

Then \( e^x \geq x. \)

\[ \square \]

7. P240. # 29.11

Proof. Consider \( f(x) = x - \sin x \). Then \( f'(x) = 1 - \cos x \geq 0 \) for all \( x \). So \( f \) is increasing on \([0, \infty)\).

\[ f(x) \geq f(0) = 0. \]

So

\[ x \geq \sin x. \]

\[ \square \]
Proof. (a). By mean value theorem,
\[ s_{n+1} - s_n = f(s_n) - f(s_{n-1}) = f'(c)(s_n - s_{n-1}). \]
Then
\[ |s_{n+1} - s_n| = |f'(c)(s_n - s_{n-1})| \leq a|s_n - s_{n-1}|. \]
Therefore,
\[ |s_{n+1} - s_n| \leq a^n|s_1 - s_0|. \]
So for \( m > n \),
\[ |s_m - s_n| \leq |s_m - s_{m-1}| + a|s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| \]
\[ \leq (a^{m-1} + a^{m-2} + \cdots + a^n)|s_1 - s_0| \]
\[ \leq \frac{a^n - a^m}{1 - a}|s_1 - s_0| \]
\[ = a^n \frac{1}{1 - a}|s_1 - s_0|. \]
We assume that \( |s_1 - s_0| \neq 0 \). Since \( \lim_{n \to \infty} a^n = 0 \), for any \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for \( n \geq N \),
\[ |a^n| \leq \frac{(1 - a)\epsilon}{|s_1 - s_0|}. \]
So
\[ |s_m - s_n| < \epsilon. \]
This proves that \( \{s_n\} \) is Cauchy. Hence \( s_n \) is a convergent sequence. This proves (a).

(b). Let \( s = \lim_{n \to \infty} s_n \). We take \( n \to \infty \) in \( s_n = f(s_{n-1}) \). Then because \( f \) is differentiable and so continuous on \( \mathbb{R} \), we obtain
\[ s = f(s). \]

\[ \Box \]

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