Your Name: 

On this exam, you may use a calculator, but no books or notes.

It is not sufficient to just write down the answers. You must explain how you arrived at your answers and how you know they are correct.

1 (30) 
2 (30) 
3 (30) 
4 (30) 
5 (30) 
6 (30) 
7 (30) 
8 (30) 
9 (30) 
10 (30) 
Total (300) 

(1) Let \( A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \)

(a) Determine the eigenvalues and the eigenvectors.
(b) Use the eigenvectors to determine an invertible matrix \( P \), so that \( P^{-1}AP \) is a diagonal matrix.

(2) Let \( v_1 = (1, 1, -2), v_2 = (2, 2, -1), v_3 = (0, 2, 1), v_4 = (0, 1, 1) \).
Represent \( v_4 \) as a linear combination of \( v_1, v_2, v_3 \), that is find \( a_1, a_2, a_3 \), so that
\[ v_4 = a_1 v_1 + a_2 v_2 + a_3 v_3. \]
(3) Determine whether the matrix $A$ is diagonalizable by computing its eigenvalues.

$$A = \begin{pmatrix} 4 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$ 

(4) (a) Compute the eigenvalues for the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$ 

(b) Give an example of a $2 \times 2$ matrix, which is not diagonalizable.
(5) Find the volume of a tetrahedron, with vertices $ABC$, where $A = (2, 3, 1), B = (-2, 1, 1), C = (1, -3, 1), D = (0, 0, 0)$.

(6) For the matrix $A = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 \\ -1 & 1 & -1 & 0 & -1 \end{pmatrix}$, compute

(a) $\text{rank}(A)$
(b) a basis for the solutions of $Ax = 0$.
(c) a basis for the row space.
(7) Find the standard matrix for the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x, y, z) = (x + y + z, x - y, z)$. Find the standard matrix for the transformations $T^{-1}$ and $T^2$.

(8) The linear transformation $S : P_3 \to P_4$ is defined by $S[p(x)] = xp(x)$. Find its standard matrix.
(9) (a) Compute the dimension of the space of $3 \times 2$ matrices.
    (b) Find a basis for $\mathbb{R}^2$, that contains the vector $(2, 1)$.
    (c) Find the standard matrix for the transformation in the plane, which represents a rotation by $90^\circ$.

(10) Let $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. Find a matrix $P$, so that $P^{-1}AP$ is diagonal.