1 First Order Equations

1.1 Linear Equations

An equation is linear, if in the form

$$y' + p(t)y = q(t).$$

Introducing the integrating factor

$$\mu(t) = e^{\int p(t) dt}$$

the solutions is then in the form

$$y(t) = \frac{1}{\mu(t)} \left[ \int q(t)\mu(t) dt \right]$$

1.2 Separable Equations

Separable equations are in the form

$$y'(x) = F(x)G(y)$$

Their solution is found by writing \( \frac{y'}{G(y)} = F(x) \) and integrating in \( x \)

$$\int \frac{dy}{G(y)} = \int F(x) dx.$$
1.3 Exact Equations

Consider equations in the form

\[ M(x, y)dx + N(x, y)dy = 0, \]  

or equivalently \( y'(x) = \frac{-M(x, y)}{N(x, y)} \). We say that (1) is exact, if

\[ M_y = N_x. \]

In that case, we are looking for a function \( \psi \), so that

\[
\begin{cases}
\psi_x = M(x, y) \\
\psi_y = N(x, y)
\end{cases}
\]

The solutions to (1) are then given by

\[ \psi(x, y) = C. \]

2 Second Order Equations

2.1 Homogeneous equations with constant coefficients

These are in the form

\[ ay'' + by' + cy = 0 \]

We set up the characteristic equation as follows

\[ ar^2 + br + c = 0. \]  

Based on the type of solutions

\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

we get for (2), we distinguish several cases
2.1.1 Case I: Two real different roots

That is $r_1 \neq r_2$, both real (that is, if $b^2 - 4ac > 0$), we can write the solution to (1) as follows

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$  

2.1.2 Case II: Two complex conjugate solutions

If $r_{1,2} = \alpha \pm i\beta$, (that is, if $b^2 - 4ac < 0$), we write

$$y(t) = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t).$$

2.1.3 Case III: A double root

If $r_{1,2} = r$, (if $b^2 - 4ac = 0$)

$$y(t) = C_1 e^{rt} + C_2 te^{rt}.$$  

2.2 Inhomogeneous Equations

$$ay'' + by' + cy = g(t).$$

Rules for selecting a particular solution $Y_0$:

- If $g(t) = e^{\alpha t}$, then seek
  $$Y_0(t) = Ae^{\alpha t}.$$  
  This is unless $\alpha$ is a root of (2), in which case $Y_0(t) = Ate^{\alpha t}$. In the case, where $\alpha$ is a double root of (2), $Y_0(t) = At^2 e^{\alpha t}$.

- If $g(t) = e^{\alpha t} \cos(\beta t)$ or $g(t) = e^{\alpha t} \sin(\beta t)$, then seek
  $$Y_0(t) = Ae^{\alpha t} \cos(\beta t) + Be^{\alpha t} \sin(\beta t),$$  
  unless $\alpha + i\beta$ is a (complex) root of (2). In this case
  $$Y_0(t) = t(Ae^{\alpha t} \cos(\beta t) + Be^{\alpha t} \sin(\beta t)).$$
• If the right-hand sides discussed above come multiplied by polynomials (say of degree \( n \)), modify the corresponding \( Y_0 \) by multiplying by a polynomial of the same degree \( n \).

### 2.3 Mechanical vibrations

For a spring-mass system, with a weight \( w \) lbs, determine the mass from \( m = \frac{w}{g} = \frac{w}{32} \). If this mass extends the spring \( L \) ft., then its coefficient \( k = \frac{w}{L} \). Then, if \( \gamma \) is its viscosity coefficient, we write the equation of motion

\[
m u''(t) + \gamma u'(t) + k u(t) = 0.
\]  

(3)

If \( \gamma = 0 \), the solutions of (3) can then be written as

\[
u(t) = A \cos(\omega t) + B \sin(\omega t) = R \cos(\omega t - \delta),
\]

where \( R = \sqrt{A^2 + B^2} \) is called amplitude, \( \omega = \sqrt{\frac{k}{m}} \) is natural frequency and \( \delta = \tan^{-1}(B/A) \) is the phase.

If \( \gamma \neq 0 \), the solutions are in the form

\[
u(t) = Ae^{-\gamma t/2} \cos(\omega t) + Be^{-\gamma t/2} \sin(\omega t) = Re^{-\gamma t/2} \cos(\omega t - \delta)
\]

with

\[
R = \sqrt{A^2 + B^2}, \quad \delta = \tan^{-1}(B/A), \\
\omega = \sqrt{\frac{4km - \gamma^2}{2m}}.
\]
3 Laplace transform

For a function $f$ and $s$ sufficiently large, we define

$$\mathcal{L}[f](s) = \int_{0}^{\infty} e^{-st} f(t) dt.$$ 

Some examples are

$$\mathcal{L}[1](s) = \frac{1}{s}; \quad \mathcal{L}[t](s) = \frac{1}{s^2}; \quad \mathcal{L}[e^{at}](s) = \frac{1}{s-a},$$

see the list with formulas on page 317 for more info. The usefulness of this method is in the following

$$\mathcal{L}[f''](s) = s\mathcal{L}[f](s) - f(0)$$
$$\mathcal{L}[f'''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0)$$

This allows us to solve ODE’s in the form

$$ay''(t) + by'(t) + cy(t) = g(t),$$

Indeed, an application of the Laplace transform yields the following equation for $F(s) = \mathcal{L}[f](s)$,

$$a(s^2F(s) - sf(0) - f'(0)) + b(sF(s) - f(0)) + cF(s) = \mathcal{L}[g](s).$$

Solving for $F(s)$ yields the relation

$$F(s) = \frac{\mathcal{L}[g](s) + af'(0) + (as + b)f(0)}{as^2 + bs + c}.$$ 

Now, if we know how to invert this last explicit formula, we are done. This is done mostly with the methods of the table on page 317, namely formulas 1,2,3, 5,6, 9, 10, 12, 13, 14 come in handy.
4 Linear systems of ODE’s

For linear systems of ODE’s in the form

\[ x' = Ax, \]

where \( A = \begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix} \)

is a matrix, we solve as follows. First, determine the eigenvalues from the characteristic equation

\[ \det \begin{pmatrix}
  a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda
\end{pmatrix} = 0. \]

- If the eigenvalues \( \lambda_1, \ldots, \lambda_n \) are real, find the corresponding eigenvectors by solving the homogeneous system

\[ \begin{pmatrix}
  a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}.

The solution is in the form

\[ x(t) = C_1 e^{\lambda_1 t} \vec{x}^{(1)} + \ldots + C_n e^{\lambda_n t} \vec{x}^{(n)} \]

- For each pair of complex conjugate eigenvalues, take \( \lambda = \alpha + i\beta \), find the corresponding eigenvector \( \vec{\xi} = \vec{a} + i\vec{b} \). This will generate the following entries

\[ C_1 (e^{\alpha t} \cos(\beta t) \vec{a} - e^{\alpha t} \sin(\beta t) \vec{b}) + C_2 (e^{\alpha t} \cos(\beta t) \vec{b} + e^{\alpha t} \sin(\beta t) \vec{a}) \]

- For each repeated eigenvalue \( \lambda \) with a single eigenvector \( \vec{\xi} \), find the adjoint eigenvector \( \vec{\eta} : (A - \lambda I) \vec{\eta} = \vec{\xi} \) and then write the corresponding solution as

\[ x(t) = C_1 e^{\lambda t} \vec{\xi} + C_2 (t e^{\lambda t} \vec{\xi} + e^{\lambda t} \vec{\eta}). \]