ON THE LIPSCHITZNESS OF THE SOLUTION MAP FOR THE 2 D NAVIER-STOKES SYSTEM

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Abstract. We consider the Navier-Stokes system on $\mathbf{R}^2$. It is well-known that solutions with L^2 data become instantly smooth and persist globally. In this note, we show that the solution map is Lipschitz, when acting in $L^\infty \dot{H}^\sigma(\mathbf{R}^2)$ and $L^2_t \dot{H}^{\sigma+1}(\mathbf{R}^2)$, when $0 \leq \sigma < 1$. This generalizes an earlier result of Gallagher and Planchon [7], who showed the Lipschitzness in $L^2(\mathbf{R}^2)$. The question for the Lipschitzness of the map in $\dot{H}^\sigma(\mathbf{R}^2), \sigma \geq 1$ remains an interesting open question, which hinges upon the validity of an endpoint estimate for the trilinear form $(\phi, v, w) \to \int_{\mathbf{R}^2} (\frac{\partial \phi}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial v}{\partial x}) w dx$.

1. Introduction

In this work, we will be concerned with the Navier-Stokes problem d spatial dimensions

$$
(1) \qquad \left| \begin{array}{c} \partial_t u - \Delta u + u \cdot \nabla u = -\nabla p \qquad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^d \\ div(u) = 0 \\ u(0,x) = u_0(x). \end{array} \right.
$$

where the unknown functions are the velocity $u(t,x)$ and the pressure $p(t,x)$. This model (and its higher dimensional counterparts) have been subject of numerous investigations. We refer the interested reader to the excellent book by P G. Lemarié-Rieusset, [15] for detailed introduction into the field. We will restrict our exposition to the more recent publications and then only to the aspects that are directly relevant to our study.

To highlight the importance of the scaling in our problem, note that the solutions of (1) are invariant under the transformation $(u^\lambda(t,x), p^\lambda(t,x)) = (\lambda u(\lambda^2 t, \lambda x), \lambda p(\lambda^2 t, \lambda x))$, that is (u^λ, p^λ) is a solution for every $\lambda > 0$, whenever (u,p) is one. This of course dictates the importance of the "scale invariant spaces" $H^{d/2-1}(\mathbf{R}^d), L^d(\mathbf{R}^d), BMO^{-1}(\mathbf{R}^d)$ etc., which appear frequently in the literature.

In the classical works of Fujita-Kato, [4] and Kato, [16], it was shown that classical local solutions of (1) exist, whenever the initial data belongs to either $H^{d/2-1}(\mathbf{R}^d)$ or $L^d(\mathbf{R}^d)$, $d \geq 2$. The solutions were constructed by the usual iteration methods and this posed some problems regarding the uniqueness of these solutions. On the other hand, weak (or Leray) global solutions have been constructed by Leray and subsequently by many other authors, satisfying various additional properties. The uniqueness of such solutions however remains an open question, and it is indeed in the heart of the ubiquitous problem for persistence of regularity for (1), where $d \geq 3$.

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The classical result of Serrin, [23] implies uniqueness for Leray solutions up to time T, as long as $u \in L^\infty_t[0,T]L^2_x(\mathbf{R}^d)] \cap L^2_t[0,T]BMO_x(\mathbf{R}^d)]$. This however, holds automatically for the Leray solutions when $d=2$, since $\dot{H}^1(\mathbf{R}^2) \subset BMO(\mathbf{R}^2)$ and hence global solutions exist and moreover they are unique for the 2D Navier-Stokes system.

In dimensions $d \geq 3$, the situation is of course much more complicated and this problem remains one of the most challenging unresolved problems in mathematics. There are numerous works[1] regarding conditions on Leray solutions, which guarantee uniqueness, provided one starts with two solutions in the same class. A case of interest, which is not covered by Serrin's theorem, was whether uniqueness holds in the class $L^\infty([0,T], L^d(\mathbf{R}^d))$. That is, for every two Leray solutions $u, v \in L^\infty([0,T], L^d(\mathbf{R}^d))$ with identical initial data $u(0)=v(0)$, show that $u(t)=v(t)$ for $t \in (0,T)$. This was resolved relatively recently in the series of papers [5] (on $\mathbf{R}^3$), [20] (for $\mathbf{R}^d, d \geq 4$) and [19] (on domains $\Omega \subset \mathbf{R}^d, d \geq 4$). We should point out that these last results do not imply continuous dependence on initial data, they were rather concerned merely with uniqueness.

For the two dimensional case, the vorticity formulation plays an especially important role. Introduce the vorticity $\omega : \omega = \partial_2 u^1 - \partial_1 u^2$. Take an ∂_2 derivative for the first equation in (1), ∂_1 in the second one and subtract. We get the scalar vorticity equation

$$\partial_t \omega - \Delta \omega + u \cdot \nabla \omega = 0 \tag{2}$$

which has the added benefit of eliminating the pressure term. On the other hand, since $div(u)=0$, we get that there exists a function $\phi : u^1 = -\partial_2 \phi, u^2 = \partial_1 \phi$. Thus $\omega = \partial_2 u^1 - \partial_1 u^2 = -\Delta \phi$, whence $\phi = (-\Delta)^{-1}\omega$. As a consequence, one can retrieve the velocity u from the vorticity ω via the Biot-Savart law, which in two spatial dimensions takes the form

$$u = \nabla^\perp (-\Delta)^{-1} \omega = \nabla^\perp \left[\left(\frac{1}{2\pi} \ln|\cdot| \right) * \omega \right]. \tag{3}$$

Note that the result is a singular integral operator of order -1, i.e. u is one derivative smoother than ω. One can also rewrite (3) in terms of the Riesz projections in the form $u = R_2 |\nabla|^{-1} u^1 - R_1 |\nabla|^{-1} u^2$, where the Riesz projections are defined via $\widehat{R_j f}(\xi) = \xi_j / |\xi| \hat{f}(\xi)$. It is a standard fact, a consequence of the Calderón-Zygmund theory, that the Riesz transforms $R_j, j=1,\ldots,d$ are bounded on virtually all important function spaces, including $L^p, 1<p<\infty$, (and more generally Sobolev spaces $W^{s,p}, 1<p<\infty$), Besov spaces $B^s_{p,q}, 1 \leq p \leq \infty$, see Section 2 Going back to the relation between u and ω, we easily conclude that

$$\|u\|_{W^{s,p}} \sim \|\omega\|_{W^{s-1,p}} \quad \|u\|_{B^s_{p,q}} \sim \|\omega\|_{B^{s-1}_{p,q}} \qquad 1<p<\infty$$

Note that the natural energy space $L^2(\mathbf{R}^2)$ for the velocity corresponds to the space $\dot{H}^{-1}$ for the vorticity. This in particular would require $\int_{\mathbf{R}^2} \omega(x) dx = \hat{\omega}(0) = 0$. On the other hand, if one wishes to consider (2) with initial data, which is arbitrary measure, that clearly will force one to consider velocities outside the energy class $L^2(\mathbf{R}^2)$. There was a significant research interest in this question, namely to study solutions of the 2 D Navier-Stokes system with initial data given by a measure. We mention the works [13], [10] and the more recent ones [9], [11], [12], [8], in which the authors have addressed the question

[1]beside Serrin's result described above.

for existence and uniqueness for the vorticity equation with data in $\mathcal{M}(\mathbf{R}^2)$. Moreover, in [11], [12], [9] the authors completely describe the asymptotic behavior of such solutions by showing that all solutions tend to Oseen vortices[2].

In this work, our goal is to consider the problem for stability of the solutions to the Navier-Stokes system, when the initial data $u_0 \in L^2(\mathbf{R}^2)$. In particular, we aim at showing the global Lipschitzness of the solution map $U(t) : u_0 \to u(t)$ in appropriate function spaces. Such results do exist in the literature. In [6], the authors show a *Lipschitz results for the 3 D NS*, which essentially says that if $u_0 - v_0$ is small in a certain scale invariant Besov space say X, then $\sup_{t\geq 0} \|u(t) - v(t)\|_X \leq C(\|u\|_X)\|u_0 - v_0\|_X$. This implies in particular that the set of global solutions is open in the topology of X.

In a related work, [7], Gallagher and Planchon prove an *a priori* estimate

$$\|u(t) - v(t)\|_{L^2(\mathbf{R}^d)} + \int_0^T \|u(t) - v(t)\|^2_{\dot{H}^1} dt \leq exp(C\|u\|^q_{L^q_t \dot{B}^{d/r+2/q-1}_{r,q}})\|u_0 - v_0\|_{L^2(\mathbf{R}^d)}, \tag{4}$$

where $d \geq 2$, $2 \leq r < \infty, 2 < q < \infty : d/r + 2/q > 1$. Note that (4) *is valid without assuming smallness of* $\|u_0 - v_0\|_{L^2(\mathbf{R}^d)}$. In other words, Gallagher and Planchon's result is that the solution map is (globally) Lipschitz in $L^2(\mathbf{R}^d)$, provided $u \in L^q_t \dot{B}^{d/r+2/q-1}_{r,q}$. It is also interesting to point out in this regard, to the nonlinear instability result by Friedlander, Pavlović and Shvydkoy, [3]. In it the authors consider steady state solution U_0 to the inhomogeneous Navier-Stokes equation, which are linearly unstable in various L^p spaces. The results is that a solution nearby (to the inhomogeneous problem!) will exhibit exponential growth (or blow up in finite time in dimensions $d \geq 3$). This of course precludes the possibility of a Lipschitzness of the solution map for the inhomogeneous NS system.

Our main result however is for the homogeneous NS system and is in fact an extension of (4), when one considers the scale of Sobolev spaces $H^\sigma, \sigma \in (0, 1)$.

Theorem 1. *Let $u_0, v_0 \in L^2(\mathbf{R}^2)$ and let $0 \leq \sigma < 1$. Then the unique global Leray solutions u, v satisfy*

$$\sup_{0\leq t<\infty} \|u(t,\cdot) - v(t,\cdot)\|_{\dot{H}^\sigma(\mathbf{R}^2)} \leq \|u_0 - v_0\|_{\dot{H}^\sigma(\mathbf{R}^2)} exp(C_\sigma(\|u_0\|^2_{L^2} + \|v_0\|^2_{L^2})). \tag{5}$$

where C_σ is a constant, which depends on σ (and it may blow up as $\sigma \to 1$). In addition

$$\int_0^\infty \|u(t,\cdot) - v(t,\cdot)\|^2_{\dot{H}^{1+\sigma}} \leq C_\sigma \|u_0 - v_0\|^2_{\dot{H}^\sigma}(\|u_0\|^2_{L^2} + \|v_0\|^2_{L^2}) exp(C_\sigma(\|u_0\|^2_{L^2} + \|v_0\|^2_{L^2})). \tag{6}$$

Remarks:

- The case $\sigma = 0$ is a consequence of (4), although we provide a direct proof, see the beginning of Section 3. Indeed, take $d = 2, r = 2$ and any $2 < q < \infty$, say $q = 4$. Then by convexity, $\|u\|_{L^4_t B^{1/2}_{2,4}} \leq \|u\|^{1/2}_{L^2 \dot{H}^1} \|u\|^{1/2}_{L^\infty L^2} \leq C\|u_0\|_{L^2}$, see (11) below. According to (4)

$$\sup_{t\geq 0} \|u(t) - v(t)\|_{L^2(\mathbf{R}^d)} \leq exp(C\|u_0\|_{L^2})\|u_0 - v_0\|_{L^2}$$

[2]which are solutions to (2) with appropriately weighted delta functions as initial data

- The case $\sigma \geq 1$ and particularly the case $\sigma = 1$ remains an open question. We actually reduce the question to the boundedness the trilinear form
$$\Lambda(\vec{u}, v, w) = \int_{\mathbf{R}^2} (\vec{u} \cdot \nabla v) w dx \qquad div(u) = 0$$
in a product of appropriate Sobolev spaces. See Section 5, where we have some further discussions and conjectures regarding this issue.

In the next theorem, we extend the Lipschitz continuity results to the $\dot{B}^s_{p,p}$ setting for some $p > 2$.

Theorem 2. *Let $u_0, v_0 \in L^2(\mathbf{R}^2)$ be smooth and decaying functions. Fix $2 < p < 4$, $s : 1 - 2/p < s < 2/p$, so that $u_0 - v_0 \in \dot{B}^s_{p,p}(\mathbf{R}^2)$. Then the unique global Leray solutions u, v satisfy*

$$\sup_{0 \leq t < \infty} \|u(t, \cdot) - v(t, \cdot)\|_{\dot{B}^s_{p,p}(\mathbf{R}^2)} \leq \|u_0 - v_0\|_{\dot{B}^s_{p,p}(\mathbf{R}^2)} exp(C_{s,p}(\|u_0\|^p_{L^2} + \|v_0\|^p_{L^2})). \tag{7}$$

Moreover,

$$\int_0^\infty \|u(t) - v(t)\|^p_{\dot{B}^{s+2/p}_{p,p}} dt \leq \|u_0 - v_0\|^p_{\dot{B}^s_{p,p}} exp(C_{s,p}(\|u_0\|^p_{L^2} + \|v_0\|^p_{L^2})). \tag{8}$$

Regarding Theorem 2, we do not know how sharp the results are both in terms of the range of p $[2, 4)$ and in the interval $s \in (1 - 2/p, 2/p)$. This would be an interesting topic for a further investigation.

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2. Preliminaries

We will use the following formula for the Fourier transform and its inverse

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx,$$
$$f(x) = \int_{\mathbf{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

Introduce a function $\chi : \mathbf{R}^d \to \mathbf{R}^1_+$, so that $\chi \in C^\infty_0(\mathbf{R}^d)$ and $\chi(\xi) = 1$ for all $|\xi| < 1$, $\chi(\xi) = 0$ for all $|\xi| > 2$. Thus, one may form a partition of unity out of dilates of the function $\varphi(\xi) := \chi(\xi) - \chi(2\xi)$, that is

$$\chi(\xi) + \sum_{k=0}^\infty \varphi(2^{-k}\xi) = \sum_{k=-\infty}^\infty \varphi(2^{-k}\xi) = 1 \qquad \xi \neq 0.$$

Define the Littlewood-Paley operators via $\widehat{P_k g}(\xi) := \varphi(2^{-k}\xi)\hat{g}(\xi)$. We will also need $P_{\sim k} = P_{k-2} + \ldots + P_{k+2}$ and $P_{\leq k} := \sum_{m=-\infty}^k P_m g$, which may be defined via

$\widehat{P_{\leq k} g}(\xi) := \chi(2^{-k}\xi)\hat{g}(\xi)$. Note that P_k essentially restricts the Fourier transform to the set $\{\xi : |\xi| \sim 2^k\}$, whereas $P_{\leq k}$ to the set $\{\xi : |\xi| \lesssim 2^k\}$. In addition, these operators are uniformly bounded on any $L^p, 1 \leq p \leq \infty$, that is

$$\sup_k [\|P_k\|_{L^p \to L^p} + \|P_{\leq k}\|_{L^p \to L^p}] \lesssim \|\hat{\chi}\|_{L^1} < \infty.$$

We will often times denote $f_k := P_k f$. The Littlewood-Paley theorem states that for every $1 < p < \infty$, there exists c_p, C_p, so that

$$c_p \|(\sum_k |f_k|^2)^{1/2}\|_{L^p} \leq \|f\|_{L^p} \leq C_p \|(\sum_k |f_k|^2)^{1/2}\|_{L^p}$$

More generally, for every real s, one has the equivalence of the $\|f\|_{\dot{W}^{p,s}} := \||\nabla|^s f\|_{L^p}$ norm

$$c_p \|(\sum_k 2^{2ks}|f_k|^2)^{1/2}\|_{L^p} \leq \|f\|_{\dot{W}^{s,p}} \leq C_p \|(\sum_k 2^{2ks}|f_k|^2)^{1/2}\|_{L^p}$$

In particular, $\||\nabla|^s f_k\|_{L^p} \sim 2^{ks}\|f_k\|_{L^p}$.

Next, we state an useful inequality, which serves as an useful and sharper substitute of the Sobolev embedding theorem, namely the Bernstein inequality.

Lemma 1. *For every* $1 \leq p \leq q \leq \infty$,

$$\|P_k f\|_{L^q(\mathbf{R}^d)} \leq C_d 2^{kd(1/p-1/q)} \|f\|_{L^p(\mathbf{R}^d)}$$

The proof of this fact is very standard and in fact follows from the convolution inequality $\|f * g\|_{L^q} \leq \|f\|_{L^r}\|g\|_{L^p}$, whenever $1 \leq p, q, r \leq \infty : 1 + 1/q = 1/p + 1/r$. Indeed, by the kernel representation of $P_k f = 2^{kd}\hat{\chi}(2^k \cdot) * f$, we have

$$\|P_k f\|_{L^q(\mathbf{R}^d)} \leq 2^{kd}\|\hat{\chi}(2^k \cdot)\|_{L^r}\|f\|_{L^p(\mathbf{R}^d)} \sim C 2^{(kd-kd/r)}\|f\|_{L^p(\mathbf{R}^d)} = C 2^{kd(1/p-1/q)}\|f\|_{L^p(\mathbf{R}^d)}$$

Next, we have the following useful representation formula for the product P_k action on products. We have

$$P_k[fg] = P_k[f g_{\sim k}] + P_k[f_{\sim k} g] + P_k[P_{\nsim k} f P_{\nsim k} g].$$

We refer to these as low-high, high-low and high-high interactions. Note that in addition to this decomposition, we can further write $P_k(f g_{\sim k}) = P_k(f_{\leq k+3} g_{\sim k})$, since $P_k(f_{>k+3} g_{\sim k}) = 0$ by Fourier support considerations. For the high-high interactions, we can similarly write

$$P_k[S_{\nsim k} f, P_{\nsim k} g)] = \sum_{l > k+3} P_k[P_{\nsim k} f P_l g] = \sum_{l > k+3} P_k[P_{\sim l} f P_l g]$$

Summarize

$$P_k[fg] = P_k[f_{\leq k+3} g_{\sim k}] + P_k[f_{\sim k} g_{\leq k+3}] + \sum_{l > k+3} P_k[f_{\sim l} g_l]. \tag{9}$$

Sometimes, we will use the representation

$$P_k[f_{\leq k+3} g_{\sim k}] = f_{\leq k+3} g_k + [P_k, f_{\leq k+3}] g_{\sim k}.$$

To treat the resulting commutator term, one uses the Kato-Ponce estimate for commutators. We present a slightly different version of it, which will be useful for us in the sequel.

Lemma 2. *Let $1 \leq p, q, r \leq \infty$ and k is an integer. Then, there exists a constant $C = C_d$, so that for all Schwartz functions $f, g \in \mathcal{S}(\mathbf{R}^d)$*

$$\|[P_k, f]g\|_{L^r} \leq C_d 2^{-k} \|\nabla f\|_{L^p} \|g\|_{L^q}. \tag{10}$$

This is especially relevant and effective, when we have the low-high frequency interaction scenario outlined above.

For the Navier-Stokes equation, we record the well-known energy inequality, namely for Leray solutions of (1) one has

$$\|u(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla u(s,\cdot)\|^2 dxds \leq \|u_0\|_{L^2}^2. \tag{11}$$

3. Proof of Theorem 1

We start the section by providing a direct proof of Theorem 1 in the case $\sigma = 0$. While this proof is elementary and does not use any Littlewood-Paley theory (in contrast with the rest of the argument), it does show the main features of the proof.

3.1. **The solution map is Lipschitz in L^2.** Take u_0, v_0 to be a pair of smooth and decaying functions, so that $div(u_0) = div(v_0) = 0$. Denote the respective solutions by $u(t), v(t) : div(u) = div(v) = 0$. Then

$$\partial_t(u - v) - \Delta(u - v) + \mathbb{P}[u\nabla u - v\nabla v] = 0$$

where $\mathbb{P}$ is the Leray projection on the divregence free vector fields. Taking a dot product with $u - v$ yields

$$\frac{1}{2}\partial_t \|u(t) - v(t)\|_{L^2}^2 + \|\nabla[u(t) - v(t)]\|_{L^2}^2 + \int_{\mathbf{R}^2} \mathbb{P}[u\nabla u - v\nabla v] \cdot (u - v)dx = 0 \tag{12}$$

Clearly $\mathbb{P}$ above is redundant. We have

$$\int [u\nabla u - v\nabla v] \cdot (u - v)dx = \int [(u\nabla(u - v) + (u - v) \cdot \nabla v] \cdot (u - v)dx =$$
$$= \int [(u - v) \cdot \nabla v] \cdot (u - v)dx$$

To study this last expression, introduce the trilinear form (where we assume only $div(\xi) = 0$)

$$M(\vec{\xi}, \vec{\eta}, \vec{\zeta}) = \int [\xi \cdot \nabla \eta] \cdot \zeta dx = -\int [\xi \cdot \eta] \cdot \nabla \zeta dx$$

We see that from Hölder's inequality (and the representation above), we get

$$|\int [\xi \cdot \nabla \eta] \cdot \zeta dx| \leq \|\xi\|_{L^2} \|\nabla \eta\|_{L^4} \|\zeta\|_{L^4}$$
$$|\int [\xi \cdot \nabla \eta] \cdot \zeta dx| = |\int [\xi \cdot \eta] \cdot \nabla \zeta dx| \leq \|\xi\|_{L^2} \|\eta\|_{L^4} \|\nabla \zeta\|_{L^4}.$$

Interpolating between the last two estimates yields

$$|\int [\xi \cdot \nabla \eta] \cdot \zeta dx| \leq \|\xi\|_{L^2} \|\eta\|_{\dot{W}^{1/2,4}} \|\zeta\|_{\dot{W}^{1/2,4}} \leq C \|\xi\|_{L^2} \|\eta\|_{\dot{H}^1(\mathbf{R}^2)} \|\zeta\|_{\dot{H}^1(\mathbf{R}^2)}.$$

where in the last estimate we have applied the Sobolev embedding $\dot{H}^1(\mathbf{R}^2) \hookrightarrow \dot{W}^{1/2,4}(\mathbf{R}^2)$. All in all, applying the estimates for $M(u-v, v, u-v)$, we have shown

$$|\int [(u-v)\cdot\nabla v]\cdot(u-v)dx| \leq C\|u-v\|_{L^2}\|v\|_{\dot{H}^1}\|\nabla[u-v]\|_{L^2}$$

Thus, going back to (12), we obtain

$$\frac{1}{2}\partial_t\|u-v\|_{L^2}^2 + \|\nabla[u-v]\|_{L^2}^2 \leq C\|u-v\|_{L^2}\|v\|_{\dot{H}^1}\|\nabla[u-v]\|_{L^2}$$
$$\leq \frac{1}{2}\|\nabla[u-v]\|_{L^2}^2 + C^2\|u-v\|_{L^2}^2\|v\|_{\dot{H}^1}^2.$$

Hence

$$\partial_t\|u-v\|_{L^2}^2 \leq C_1\|u-v\|_{L^2}^2\|v\|_{\dot{H}^1}^2$$

and by Gronwall's lemma

$$\|u(t)-v(t)\|_{L^2}^2 \leq \|u_0-v_0\|_{L^2}^2 exp(C\int_0^t \|v(s)\|_{\dot{H}^1}^2 ds)$$

Since by (11), $\int_0^\infty \|v(s)\|_{\dot{H}^1}^2 ds \leq \|v_0\|_{L^2}^2$, we obtain

$$\sup_{t\geq 0}\|u(t)-v(t)\|_{L^2}^2 \leq \|u_0-v_0\|_{L^2}^2 exp(C\|v_0\|_{L^2}^2).$$

3.2. **The solution map is Lipschitz in $H^\sigma, 0<\sigma<1$.** Let u_0, v_0 be as above. By the classical existence and uniqueness results, there exists unique solutions $u(t)$ and $v(t)$ with corresponding vorticities $\rho(t), \gamma(t)$. Consider their respective equations and take a Littlewood-Paley projection. We have

$$\partial_t\rho_k - \Delta\rho_k + P_k[u\cdot\nabla\rho] = 0$$
$$\partial_t\gamma_k - \Delta\gamma_k + P_k[v\cdot\nabla\gamma] = 0.$$

Next, we write the nonlinearity in the form suggested by (9). We have

$$P_k[u\cdot\nabla\rho] = P_k[u_{\leq k+3}\cdot\nabla\rho_{\sim k}] + P_k[u_{\sim k}\cdot\nabla\rho_{\leq k+3}] + \sum_{l>k+3} P_k[u_{\sim l}\cdot\nabla\rho_l],$$
$$P_k[v\cdot\nabla\gamma] = P_k[v_{\leq k+3}\cdot\nabla\gamma_{\sim k}] + P_k[v_{\sim k}\cdot\nabla\gamma_{\leq k+3}] + \sum_{l>k+3} P_k[v_{\sim l}\cdot\nabla\gamma_l],$$

We further enhance this representation by

$$P_k[u_{\leq k+3}\cdot\nabla\rho_{\sim k}] = u_{\leq k+3}\cdot\nabla\rho_k + [P_k, u_{\leq k+3}]\cdot\nabla\rho_{\sim k}.$$

and similar for $P_k[v_{\leq k+3}\cdot\nabla\gamma_{\sim k}]$. Subtracting the equations for ρ_k and γ_k results in

$$\partial_t[\rho_k-\gamma_k] - \Delta[\rho_k-\gamma_k] + [u_{\leq k+3}\cdot\nabla\rho_k - v_{\leq k+3}\cdot\nabla\gamma_k] +$$
$$+[[P_k, u_{\leq k+3}]\cdot\nabla\rho_{\sim k} - [P_k, v_{\leq k+3}]\cdot\nabla\gamma_{\sim k}] + [P_k[u_{\sim k}\cdot\nabla\rho_{\leq k+3}] - P_k[v_{\sim k}\cdot\nabla\gamma_{\leq k+3}]]$$
$$+ \sum_{l>k+3}[P_k[u_{\sim l}\cdot\nabla\rho_l] - P_k[v_{\sim l}\cdot\nabla\gamma_l] = 0$$

This will be our basic equation for the difference $\rho_k - \gamma_k$. Multiply both sides by $\rho_k - \gamma_k$ and integrate in the spatial variables. We get

$$\frac{1}{2}\partial_t \int |\rho_k - \gamma_k|^2 dx + \|\nabla[\rho_k - \gamma_k]\|_{L^2}^2 + \int [u_{\leq k+3} \cdot \nabla \rho_k - v_{\leq k+3} \cdot \nabla \gamma_k][\rho_k - \gamma_k] dx =$$

$$= \int [[P_k, v_{\leq k+3}] \cdot \nabla \gamma_{\sim k} - [P_k, u_{\leq k+3}] \cdot \nabla \rho_{\sim k}][\rho_k - \gamma_k] dx +$$

$$+ \int [P_k[v_{\sim k} \cdot \nabla \gamma_{\leq k+3}] - P_k[u_{\sim k} \cdot \nabla \rho_{\leq k+3}]][\rho_k - \gamma_k] dx +$$

$$+ \sum_{l > k+3} \int [P_k[u_{\sim l} \cdot \nabla \rho_l] - P_k[v_{\sim l} \cdot \nabla \gamma_l][\gamma_k - \rho_k] dx.$$

This is our basic energy estimate that we work with. Next, we will need suitable estimates for the terms appearing therein.

3.3. **Estimates for** $\int [u_{\leq k+3} \cdot \nabla \rho_k - v_{\leq k+3} \cdot \nabla \gamma_k][\rho_k - \gamma_k] dx$**.** We have

$$\int [u_{\leq k+3} \cdot \nabla \rho_k - v_{\leq k+3} \cdot \nabla \gamma_k][\rho_k - \gamma_k] dx = \int (u_{\leq k+3} \cdot \nabla[\rho_k - \gamma_k])[\rho_k - \gamma_k] dx +$$

$$+ \int [(u_{\leq k+3} - v_{\leq k+3}) \cdot \nabla \gamma_k][\rho_k - \gamma_k] dx.$$

But $div(u_{\leq k+3}) = 0$, whence the first integral on the right hand side vanishes, whereas for the second one, we apply integration by parts (again use $div(u_{\leq k+3} - v_{\leq k+3}) = 0$) and estimate by

$$|\int [(u_{\leq k+3} - v_{\leq k+3}) \cdot \nabla \gamma_k][\rho_k - \gamma_k] dx| = |\int [(u_{\leq k+3} - v_{\leq k+3})\gamma_k] \cdot \nabla[\rho_k - \gamma_k] dx| \leq$$

$$\leq \frac{1}{16}\|\nabla[\rho_k - \gamma_k]\|_{L^2}^2 + 4\|(u_{\leq k+3} - v_{\leq k+3})\gamma_k\|_{L^2}^2 \leq$$

$$\leq \frac{1}{16}\|\nabla[\rho_k - \gamma_k]\|_{L^2}^2 + 4\|u_{\leq k+3} - v_{\leq k+3}\|_{L^\infty}^2 \|\gamma_k\|_{L^2}^2.$$

3.4. **Estimates for** $\int([P_k, v_{\leq k+3}] \cdot \nabla \gamma_{\sim k} - [P_k, u_{\leq k+3}] \cdot \nabla \rho_{\sim k})[\rho_k - \gamma_k] dx$**.** Write

$$\int([P_k, v_{\leq k+3}] \cdot \nabla \gamma_{\sim k} - [P_k, u_{\leq k+3}] \cdot \nabla \rho_{\sim k})[\rho_k - \gamma_k] dx =$$

$$= \int ([P_k, (v_{\leq k+3} - u_{\leq k+3})] \cdot \nabla \gamma_{\sim k})(\rho_k - \gamma_k) dx + \int ([P_k, u_{\leq k+3}] \cdot \nabla[\gamma_{\sim k} - \rho_{\sim k}])(\rho_k - \gamma_k) dx$$

To both terms, we may apply the Kato-Ponce commutator estimates in Lemma 2. We get for the first term

$$|\int [[P_k, (v_{\leq k+3} - u_{\leq k+3})] \cdot \nabla \gamma_{\sim k}][\rho_k - \gamma_k] dx| \leq \|[P_k, v_{\leq k+3} - u_{\leq k+3}] \cdot \nabla \gamma_{\sim k}\|_{L^2} \|\rho_k - \gamma_k\|_{L^2}$$

$$\leq C 2^{-k} \|\nabla[v_{\leq k+3} - u_{\leq k+3}]\|_{L^\infty} \|\nabla \gamma_{\sim k}\|_{L^2} \|\rho_k - \gamma_k\|_{L^2}.$$

Note that we may "move the derivatives" as follows

$$\|\nabla \gamma_{\sim k}\|_{L^2} \|\rho_k - \gamma_k\|_{L^2} \sim 2^k \|\gamma_{\sim k}\|_{L^2} \|\rho_k - \gamma_k\|_{L^2} \sim \|\gamma_{\sim k}\|_{L^2} \|\nabla[\rho_k - \gamma_k]\|_{L^2}$$

Similarly for the second term, we use Lemma 2 and we may also move a derivative to get

$$
\begin{aligned}
&|\int [[P_k, u_{\leq k+3}] \cdot \nabla[\gamma_{\sim k} - \rho_{\sim k}]][\rho_k - \gamma_k] dx| \leq \\
&\leq C2^{-k} \|\nabla u_{\leq k+3}\|_{L^\infty} \|\rho_k - \gamma_k\|_{L^2} \|\nabla[\gamma_{\sim k} - \rho_{\sim k}]\|_{L^2} \leq \\
&\leq C2^{-k} \|\nabla u_{\leq k+3}\|_{L^\infty} \|\nabla[\rho_k - \gamma_k]\|_{L^2} \|[\gamma_{\sim k} - \rho_{\sim k}]\|_{L^2} \leq \\
&\leq \frac{1}{16} \|\nabla[\rho_k - \gamma_k]\|_{L^2}^2 + C2^{-2k} \|\nabla u_{\leq k+3}\|_{L^\infty}^2 \|[\gamma_{\sim k} - \rho_{\sim k}]\|_{L^2}^2.
\end{aligned}
$$

3.5. **Estimates for** $\int [P_k[v_{\sim k} \cdot \nabla \gamma_{\leq k+3}] - P_k[u_{\sim k} \cdot \nabla \rho_{\leq k+3}]][\rho_k - \gamma_k] dx$**.** Integration by parts (recall $div(u_{\sim k}) = div(v_{\sim k}) = 0$) and Cauchy's inequality yield

$$
\begin{aligned}
&|\int [P_k[v_{\sim k} \cdot \nabla \gamma_{\leq k+3}] - P_k[u_{\sim k} \cdot \nabla \rho_{\leq k+3}]][\rho_k - \gamma_k] dx| = \\
&= |\int [P_k[v_{\sim k} \gamma_{\leq k+3}] - P_k[u_{\sim k} \rho_{\leq k+3}]] \cdot \nabla[\rho_k - \gamma_k] dx| \leq \\
&\leq \frac{1}{16} \|\nabla[\rho_k - \gamma_k]\|_{L^2}^2 + C\|v_{\sim k} \gamma_{\leq k+3} - u_{\sim k} \rho_{\leq k+3}\|_{L^2}^2 \leq \\
&\leq \frac{1}{16} \|\nabla[\rho_k - \gamma_k]\|_{L^2}^2 + C(\|v_{\sim k} - u_{\sim k}\|_{L^2}^2 \|\gamma_{\leq k+3}\|_{L^\infty}^2 + \|(\rho - \gamma)_{\leq k+3}\|_{L^\infty}^2 \|u_{\sim k}\|_{L^2}^2).
\end{aligned}
$$

3.6. **Estimates for** $\sum_{l \geq k+3} \int [P_k[u_{\sim l} \cdot \nabla \rho_l] - P_k[v_{\sim l} \cdot \nabla \gamma_l]][\gamma_k - \rho_k] dx$**.** We estimate

$$
\begin{aligned}
&|\sum_{l \geq k+3} \int [P_k[u_{\sim l} \cdot \nabla \rho_l] - P_k[v_{\sim l} \cdot \nabla \gamma_l][\gamma_k - \rho_k] dx| = \\
&= |\int (\sum_{l \geq k+3} P_k[u_{\sim l} \rho_l - v_{\sim l} \gamma_l]) \cdot \nabla[\gamma_k - \rho_k] dx| \leq \\
&\leq \frac{1}{16} \|\nabla[\rho_k - \gamma_k]\|_{L^2}^2 + C(\sum_{l \geq k+3} \|P_k[u_{\sim l} \rho_l - v_{\sim l} \gamma_l]\|_{L^2})^2
\end{aligned}
$$

Since $l > k+3$, it is beneficial to apply the Bernstein inequality from Lemma 1 to the second term.

$$
\begin{aligned}
&\|P_k[u_{\sim l} \rho_l - v_{\sim l} \gamma_l]\|_{L^2(\mathbf{R}^2)} \leq C2^k \|u_{\sim l} \rho_l - v_{\sim l} \gamma_l\|_{L^1(\mathbf{R}^2)} \leq \\
&\leq C2^k (\|u_{\sim l} - v_{\sim l}\|_{L^2} \|\rho_l\|_{L^2} + \|\rho_l - \gamma_l\|_{L^2} \|v_{\sim l}\|_{L^2}).
\end{aligned}
$$

3.7. **Putting all energy estimates together.** Using all the estimates obtained for the various terms, we arrive at

$$
\begin{aligned}
&\partial_t \|\rho_k - \gamma_k\|_{L^2}^2 + \frac{1}{32} \|\nabla[\rho_k - \gamma_k]\|_{L^2}^2 \leq C\|u_{\leq k+3} - v_{\leq k+3}\|_{L^\infty}^2 \|\gamma_k\|_{L^2}^2 + \\
&+ C2^{-2k} \|\nabla u_{\leq k+3}\|_{L^\infty}^2 \|[\gamma_{\sim k} - \rho_{\sim k}]\|_{L^2}^2 + \\
&+ C\|v_{\sim k} - u_{\sim k}\|_{L^2}^2 \|\gamma_{\leq k+3}\|_{L^\infty}^2 + C\|(\rho - \gamma)_{\leq k+3}\|_{L^\infty}^2 \|u_{\sim k}\|_{L^2}^2 + \\
&+ C2^{2k} (\sum_{l > k+3} \|u_{\sim l} - v_{\sim l}\|_{L^2} \|\rho_l\|_{L^2} + \|\rho_l - \gamma_l\|_{L^2} \|v_{\sim l}\|_{L^2})^2
\end{aligned}
$$

Multiply both sides by 2^{-2ks} for $0 < s < 1$ and sum in $k \in \mathcal{Z}$. We obtain

$$\partial_t \|\rho-\gamma\|^2_{\dot{H}^{-s}} + \frac{1}{32}\|\rho-\gamma\|^2_{\dot{H}^{1-s}} \leq C\sum_k 2^{-2ks}\|u_{\leq k+3}-v_{\leq k+3}\|^2_{L^\infty}\|\gamma_k\|^2_{L^2} +$$
$$+C\sum_k 2^{-2k(1+s)}\|\nabla u_{\leq k+3}\|^2_{L^\infty}\|\gamma_{\sim k}-\rho_{\sim k}\|^2_{L^2} +$$
$$+C\sum_k 2^{-2ks}(\|v_{\sim k}-u_{\sim k}\|^2_{L^2}\|\gamma_{\leq k+3}\|^2_{L^\infty} + \|(\rho-\gamma)_{\leq k+3}\|^2_{L^\infty}\|u_{\sim k}\|^2_{L^2}) +$$
$$+C\sum_k 2^{-2ks}2^{2k}(\sum_{l>k+3}\|u_{\sim l}-v_{\sim l}\|_{L^2}\|\rho_l\|_{L^2} + \|\rho_l-\gamma_l\|_{L^2}\|v_{\sim l}\|_{L^2})^2$$

For the first term, we apply the Bernstein inequality

$$\sum_k 2^{-2ks}\|u_{\leq k+3}-v_{\leq k+3}\|^2_{L^\infty}\|\gamma_k\|^2_{L^2} \leq C\|\gamma\|^2_{L^2}\sup_k 2^{-2ks}\|u_{\leq k+3}-v_{\leq k+3}\|^2_{L^\infty} \leq$$
$$\leq C\|\gamma\|^2_{L^2}(\sup_k 2^{-ks}\sum_{l\leq k+3} 2^l\|(u-v)_l\|_{L^2})^2 \leq C\|\gamma\|^2_{L^2}\|u-v\|^2_{\dot{H}^{1-s}}.$$

The second term gets estimated by a very similar argument

$$\sum_k 2^{-2k(1+s)}\|\nabla u_{\leq k+3}\|^2_{L^\infty}\|\gamma_{\sim k}-\rho_{\sim k}\|^2_{L^2} \leq C\|\gamma-\rho\|^2_{\dot{H}^{-s}}(\sup_k 2^{-k}\|\nabla u_{\leq k+3}\|_{L^\infty})^2 \leq$$
$$\leq C\|\gamma-\rho\|^2_{\dot{H}^{-s}}(\sup_k 2^{-k}\sum_{l\leq k+3} 2^l\|\nabla u_l\|_{L^2})^2 \leq C\|\gamma-\rho\|^2_{\dot{H}^{-s}}\|\nabla u\|^2_{L^2}.$$

For the third term

$$\sum_k 2^{-2ks}\|v_{\sim k}-u_{\sim k}\|^2_{L^2}\|\gamma_{\leq k+3}\|^2_{L^\infty} \leq C\|u-v\|^2_{\dot{H}^{1-s}}(\sup_k 2^{-k}\|\gamma_{\leq k+3}\|_{L^\infty})^2 \leq$$
$$\leq C\|u-v\|^2_{\dot{H}^{1-s}}(\sup_k 2^{-k}\sum_{l\leq k+3} 2^l\|\gamma_l\|_{L^2})^2 \leq C\|u-v\|^2_{\dot{H}^{1-s}}\|\gamma\|^2_{L^2}$$

and similarly

$$\sum_k 2^{-2ks}\|(\rho-\gamma)_{\leq k+3}\|^2_{L^\infty}\|u_{\sim k}\|^2_{L^2} \leq C\|\nabla u\|^2_{L^2}(\sup_k 2^{-k(1+s)}\|(\rho-\gamma)_{\leq k+3}\|_{L^\infty})^2 \leq$$
$$\leq C\|\nabla u\|^2_{L^2}(\sup_k 2^{-k(1+s)}\sum_{l\leq k+3} 2^l\|(\rho-\gamma)_l\|_{L^2})^2 \leq C\|\nabla u\|^2_{L^2}\|\gamma-\rho\|^2_{\dot{H}^{-s}}.$$

Finally, for the high-high frequency term, we have

$$\sum_k 2^{2k(1-s)}(\sum_{l>k+3}\|u_{\sim l}-v_{\sim l}\|_{L^2}\|\rho_l\|_{L^2} + \|\rho_l-\gamma_l\|_{L^2}\|v_{\sim l}\|_{L^2})^2 =$$
$$= \sum_{l_1,l_2}(\|u_{\sim l_1}-v_{\sim l_1}\|_{L^2}\|\rho_{l_1}\|_{L^2} + \|\rho_{l_1}-\gamma_{l_1}\|_{L^2}\|v_{\sim l_1}\|_{L^2}) \times$$
$$\times(\|u_{\sim l_2}-v_{\sim l_2}\|_{L^2}\|\rho_{l_2}\|_{L^2} + \|\rho_{l_2}-\gamma_{l_2}\|_{L^2}\|v_{\sim l_2}\|_{L^2}) \sum_{k<\min(l_1,l_2)-3} 2^{2k(1-s)}.$$

We now observe

$$\sum_{k<\min(l_1,l_2)-3} 2^{2k(1-s)} \leq \frac{C}{1-s} 2^{2\min(l_1,l_2)(1-s)} \leq \frac{C}{1-s} 2^{l_1(1-s)} 2^{l_2(1-s)}$$

It is clear that we can bound the contribution of the high-high frequency terms by

$$\begin{aligned}
&\lesssim (\sum_l 2^{l(1-s)}(\|u_{\sim l} - v_{\sim l}\|_{L^2}\|\rho_l\|_{L^2} + \|\rho_l - \gamma_l\|_{L^2}\|v_{\sim l}\|_{L^2})^2 \leq \\
&\lesssim (\sum_l 2^{2l(1-s)}\|u_{\sim l} - v_{\sim l}\|^2_{L^2})(\sum_l \|\rho_l\|^2_{L^2}) + (\sum_l 2^{-2sl}\|\rho_l - \gamma_l\|^2_{L^2})(\sum_l 2^{2l}\|v_{\sim l}\|_{L^2})^2 \leq \\
&\leq C_s(\|u-v\|^2_{\dot{H}^{1-s}}\|\rho\|^2_{L^2} + \|\rho-\gamma\|^2_{\dot{H}^{-s}}\|\nabla v\|^2_{L^2}).
\end{aligned}$$

where we have found that $C_s = C/(1-s)$. We obtain

$$\partial_t\|\rho-\gamma\|^2_{\dot{H}^{-s}} + \frac{\|\rho-\gamma\|^2_{\dot{H}^{1-s}}}{32} \leq C_s[\|\rho-\gamma\|^2_{\dot{H}^{-s}}(\|\nabla u\|^2_{L^2} + \|\nabla v\|^2_{L^2}) + \|u-v\|^2_{\dot{H}^{1-s}}(\|\rho\|^2_{L^2} + \|\gamma\|^2_{L^2})]$$

Recall however that for every $1 < p < \infty, \alpha \in \mathbf{R}^1$, $\|\rho\|_{\dot{W}^{\alpha,p}} \sim \|u\|_{\dot{W}^{\alpha+1,p}}$ and similar for γ. We can drop for a second the term $\|\rho-\gamma\|^2_{\dot{H}^{1-s}}/32$ and rewrite the last estimate as

$$\partial_t\|u-v\|^2_{\dot{H}^{1-s}} \leq C_s\|u-v\|^2_{\dot{H}^{1-s}}(\|\nabla u\|^2_{L^2} + \|\nabla v\|^2_{L^2})$$

An application of the Gronwall's inequality yields for every $T > 0$,

$$\|u(T,\cdot) - v(T,\cdot)\|^2_{\dot{H}^{1-s}} \leq \|u_0 - v_0\|^2_{\dot{H}^{1-s}} exp\left(C_s \int_0^T (\|\nabla u(z,\cdot)\|^2_{L^2} + \|\nabla v(z,\cdot)\|^2_{L^2})dz\right)$$

By the energy estimate (11), we can estimate the expression inside the *exp* by $C(\|u_0\|^2_{L^2} + \|v_0\|^2_{L^2})$ and therefore

$$\sup_{0\leq T<\infty} \|u(T,\cdot) - v(T,\cdot)\|^2_{\dot{H}^{1-s}} \leq \|u_0 - v_0\|^2_{\dot{H}^{1-s}} exp\left(C_s(\|u_0\|^2_{L^2} + \|v_0\|^2_{L^2})\right),$$

which is (5) with $\gamma = 1-s$. Note that the constant C_s appearing in the estimate may, as a consequence of this argument, blow up in both $s \to 0$ or $s \to 1$. As we saw in Section 3.1, this does not happen as $s \to 0$.

Bringing back the term $\|\rho-\gamma\|^2_{\dot{H}^{1-s}}/32 \sim \|u-v\|^2_{\dot{H}^{2-s}}$ into play, we realize that after integration in $[0,T]$

$$\begin{aligned}
&\|u(T,\cdot) - v(T,\cdot)\|^2_{\dot{H}^{1-s}} + \int_0^T \|u(z) - v(z)\|^2_{\dot{H}^{2-s}}dz \leq \\
&\leq C_s \int_0^T \|u(z) - v(z)\|^2_{\dot{H}^{1-s}}(\|\nabla u(z)\|^2_{L^2} + \|\nabla v(z)\|^2_{L^2})dz \leq \\
&\leq C_s \sup_{0\leq s\leq T} \|u(z) - v(z)\|^2_{\dot{H}^{1-s}} \int_0^T (\|\nabla u(z)\|^2_{L^2} + \|\nabla v(z)\|^2_{L^2})dz.
\end{aligned}$$

We now ignore the term $\|u(T,\cdot) - v(T,\cdot)\|^2_{\dot{H}^{1-s}}$ and use (5) to bound $\sup_{0\leq z\leq T}\|u(z) - v(z)\|^2_{\dot{H}^{1-s}}$ and the energy estimate (11) for $\int_0^T (\|\nabla u(z)\|^2_{L^2} + \|\nabla v(z)\|^2_{L^2})dz$.

We obtain,

$$\int_0^T \|u(z)-v(z)\|^2_{\dot{H}^{2-s}}dzC_s \leq \|u_0-v_0\|^2_{\dot{H}^{1-s}}(\|u_0\|^2_{L^2}+\|v_0\|^2_{L^2})exp\left(C_s(\|u_0\|^2_{L^2}+\|v_0\|^2_{L^2})\right)$$

This last inequality gives (6) and hence the complete proof of Theorem 1.

4. Proof of Theorem 2

A basic tool in our approach will be the norm equivalence $2^k\|u_k\|^p_{L^p} \sim \|\nabla[|u_k|^{p/2}]\|^2_{L^2}$. More precisely, for every $2<p<\infty$, there are two constants $c_{p,d}, C_{p,d}$, so that

$$c_{p,d}2^{2k}\|u_k\|^p_{L^p(\mathbf{R}^d)} \leq \|\nabla[|u_k|^{p/2}]\|^2_{L^2(\mathbf{R}^d)} \leq C_{p,d}2^{2k}\|u_k\|^p_{L^p(\mathbf{R}^d)} \tag{13}$$

for all Schwartz functions u. This is shown in [22], with the improvement to the full range $1<p<\infty$ in [2]. This will allow us to perform energy estimates in L^p spaces, which we will now do. There are some technical details before we start. For every smooth function u , we have $\nabla|u| = \nabla\sqrt{u^2} = \frac{u}{|u|}\nabla u = sgn(u)\nabla u$. So, for a function $u(t,x)$, we have for every $p>1$

$$\partial_t|u|^p = \partial_t\sqrt{u^{2p}} = \frac{p|u|^p}{u}\partial_t u.$$

Applying this to the smooth function $\rho_k-\gamma_k$, we obtain from the equation for the difference $\rho_k-\gamma_k$,

$$\begin{aligned}
\frac{1}{p}\partial_t\int|\rho_k-\gamma_k|^p dx \quad &= \int\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}\partial_t(\rho_k-\gamma_k)dx = \\
&= \int\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}[\Delta[\rho_k-\gamma_k]+[u_{\leq k+3}\cdot\nabla\rho_k - v_{\leq k+3}\cdot\nabla\gamma_k]] + \\
&+ \int\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}[[P_k,u_{\leq k+3}]\cdot\nabla\rho_{\sim k}-[P_k,v_{\leq k+3}]\cdot\nabla\gamma_{\sim k}] + \\
&+ \int\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}[P_k[u_{\sim k}\cdot\nabla\rho_{\leq k+3}]-P_k[v_{\sim k}\cdot\nabla\gamma_{\leq k+3}]] + \\
&+ \int\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}[\sum_{l>k+3}[P_k[u_{\sim l}\cdot\nabla\rho_l]-P_k[v_{\sim l}\cdot\nabla\gamma_l]]
\end{aligned}$$

The terms, which we get to estimate, are similar to those appearing in the computations in Section 3.

4.1. Rewriting $\int\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}\Delta[\rho_k-\gamma_k]dx$.
We have

$$\begin{aligned}
&\int\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}\Delta[\rho_k-\gamma_k]dx = -\int\nabla\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}\cdot\nabla[\rho_k-\gamma_k]dx = \\
&= -\int\nabla|\rho_k-\gamma_k|^{p-1}\nabla|\rho_k-\gamma_k|dx = -\frac{4(p-1)}{p^2}\int|\nabla[|\rho_k-\gamma_k|^{p/2}]|^2dx = \\
&= -\frac{4(p-1)}{p^2}\|\nabla[|\rho_k-\gamma_k|^{p/2}]\|^2_{L^2(\mathbf{R}^d)}.
\end{aligned}$$

Note that the expression $\|\nabla[|\rho_k - \gamma_k|^{p/2}]\|^2_{L^2(\mathbf{R}^d)}$ is exactly what sits in the middle of (13) for $u_k = \rho_k - \gamma_k$.

4.2. **Estimates for** $\int [u_{\leq k+3} \cdot \nabla \rho_k - v_{\leq k+3} \cdot \nabla \gamma_k] \frac{|\rho_k - \gamma_k|^p}{\rho_k - \gamma_k} dx$**.** We perform an integration by parts, which due to the property $div(u_{\leq k+3}) = div(v_{\leq k+3}) = 0$ reads

$$
\begin{aligned}
&|\int [u_{\leq k+3} \cdot \nabla \rho_k - v_{\leq k+3} \cdot \nabla \gamma_k] \frac{|\rho_k - \gamma_k|^p}{\rho_k - \gamma_k} dx| = \\
&= |\int [u_{\leq k+3} \rho_k - v_{\leq k+3} \gamma_k] \cdot \nabla \left[\frac{|\rho_k - \gamma_k|^p}{\rho_k - \gamma_k} \right] dx|.
\end{aligned}
$$

Furthermore, write $u_{\leq k+3} \rho_k - v_{\leq k+3} \gamma_k = u_{\leq k+3}(\rho_k - \gamma_k) + (u_{\leq k+3} - v_{\leq k+3}) \gamma_k$ and observe that

$$
\int (\rho_k - \gamma_k) u_{\leq k+3} \cdot \nabla \left[\frac{|\rho_k - \gamma_k|^p}{\rho_k - \gamma_k} \right] dx = \int |\rho_k - \gamma_k| u_{\leq k+3} \cdot \nabla [|\rho_k - \gamma_k|^{p-1}] dx = 0.
$$

since $div(u_{\leq k+3}) = 0$. Next, due to the formula $\nabla |u| = sgn(u) \nabla u$, we have the pointwise bound

$$
|\nabla \left[\frac{|\rho_k - \gamma_k|^p}{\rho_k - \gamma_k} \right] | \leq Cp |\rho_k - \gamma_k|^{p/2-1} |\nabla [|\rho_k - \gamma_k|^{p/2}]|. \tag{14}
$$

for some constant C. Inserting this in the remaining nonzero terms and applying the Hölder's inequality yields the bound

$$
\begin{aligned}
&|\int [u_{\leq k+3} \cdot \nabla \rho_k - v_{\leq k+3} \cdot \nabla \gamma_k] \frac{|\rho_k - \gamma_k|^p}{\rho_k - \gamma_k} dx| \leq \\
&\leq C_p \int |u_{\leq k+3} - v_{\leq k+3}| |\gamma_k| |\rho_k - \gamma_k|^{p/2-1} |\nabla [|\rho_k - \gamma_k|^{p/2}]| dx \leq \\
&\leq Cp \|\nabla [|\rho_k - \gamma_k|^{p/2}]\|_{L^2} \|\rho_k - \gamma_k\|_{L^p}^{(p-2)/2} \|(u_{\leq k+3} - v_{\leq k+3}) \gamma_k\|_{L^p} \leq \\
&\leq Cp \|\nabla [|\rho_k - \gamma_k|^{p/2}]\|_{L^2} \|\rho_k - \gamma_k\|_{L^p}^{(p-2)/2} \|u_{\leq k+3} - v_{\leq k+3}\|_{L^\infty} \|\gamma_k\|_{L^p}.
\end{aligned}
$$

We further estimate the right hand side by Cauchy-Schwartz and then Young's inequality

$$
\begin{aligned}
&\varepsilon \|\nabla [|\rho_k - \gamma_k|^{p/2}]\|_{L^2}^2 + D_{p,\varepsilon} \|\rho_k - \gamma_k\|_{L^p}^{(p-2)} \|u_{\leq k+3} - v_{\leq k+3}\|_{L^\infty}^2 \|\gamma_k\|_{L^p}^2 \leq \\
&\leq \varepsilon \|\nabla [|\rho_k - \gamma_k|^{p/2}]\|_{L^2}^2 + \varepsilon 2^{2k} \|\rho_k - \gamma_k\|_{L^p}^p + D_{p,\varepsilon} 2^{-k(p-2)} \|u_{\leq k+3} - v_{\leq k+3}\|_{L^\infty}^p \|\gamma_k\|_{L^p}^p.
\end{aligned}
$$

where we require that $\varepsilon = \varepsilon_p$ is so small, so that $100 C_p \varepsilon < (p-1)/p^2$, where $C_p = C_{p,2}$ is the constant appearing in (13). This of course ensures that one would be able to hide the term $\varepsilon(\|\nabla[|\rho_k - \gamma_k|^{p/2}]\|_{L^2}^2 + 2^{2k} \|\rho_k - \gamma_k\|_{L^p}^p)$ behind the positive term $4(p-1)p^{-2} \|\nabla [|\rho_k - \gamma_k|^{p/2}]\|_{L^2}^2$ arising in the energy estimate.

4.3. **Estimates for** $\int [[P_k, v_{\leq k+3}] \cdot \nabla \gamma_{\sim k} - [P_k, u_{\leq k+3}] \cdot \nabla \rho_{\sim k}] \frac{|\rho_k - \gamma_k|^p}{\rho_k - \gamma_k} dx$**.** Similar to Section 3.4, represent

$$
[P_k, v_{\leq k+3}] \cdot \nabla \gamma_{\sim k} - [P_k, u_{\leq k+3}] \cdot \nabla \rho_{\sim k}] = [P_k, (v_{\leq k+3} - u_{\leq k+3})] \cdot \nabla \gamma_{\sim k}] + [P_k, u_{\leq k+3}] \cdot \nabla [\gamma_{\sim k} - \rho_{\sim k}]
$$

whence by Lemma 2

$$
\begin{aligned}
&|\int [[P_k, v_{\leq k+3}]\cdot \nabla \gamma_{\sim k} - [P_k, u_{\leq k+3}]\cdot \nabla \rho_{\sim k}]\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}dx| \leq \\
&\leq C2^{-k}\|\nabla[v_{\leq k+3}-u_{\leq k+3}]\|_{L^\infty}\|\nabla \gamma_{\sim k}\|_{L^p}\|\rho_k-\gamma_k\|_{L^p}^{p-1}+ \\
&+C2^{-k}\|\nabla u_{\leq k+3}\|_{L^\infty}\|\nabla[\rho_{\sim k}-\gamma_{\sim k}]\|_{L^p}\|\rho_k-\gamma_k\|_{L^p}^{p-1}
\end{aligned}
$$

Again, using the equivalence $\|\nabla f_k\|_{L^p}\sim 2^k\|f_k\|_{L^p}$ and Young's inequality, we may estimate the last expression by

$$
\varepsilon 2^{2k}\|\rho_k-\gamma_k\|_{L^p}^p + C_\varepsilon 2^{-2k(p-1)}(\|\nabla[v_{\leq k+3}-u_{\leq k+3}]\|_{L^\infty}^p\|\gamma_{\sim k}\|_{L^p}^p + \|\nabla u_{\leq k+3}\|_{L^\infty}^p\|\rho_{\sim k}-\gamma_{\sim k}\|_{L^p}^p)
$$

4.4. **Estimates for $\int [P_k[v_{\sim k}\cdot \nabla \gamma_{\leq k+3}] - P_k[u_{\sim k}\cdot \nabla \rho_{\leq k+3}]]\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}dx$.** Write

$$
P_k[v_{\sim k}\cdot \nabla \gamma_{\leq k+3}] - P_k[u_{\sim k}\cdot \nabla \rho_{\leq k+3}] = P_k[(v_{\sim k}-u_{\sim k})\cdot \nabla \gamma_{\leq k+3}] + P_k[u_{\sim k}\cdot \nabla(\gamma_{\leq k+3}-\rho_{\leq k+3})].
$$

whence

$$
\begin{aligned}
&|\int [P_k[v_{\sim k}\cdot \nabla \gamma_{\leq k+3}] - P_k[u_{\sim k}\cdot \nabla \rho_{\leq k+3}]]\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}dx| \leq \\
&\leq \|\rho_k-\gamma_k\|_{L^p}^{p-1}(\|v_{\sim k}-u_{\sim k}\|_{L^p}\|\nabla \gamma_{\leq k+3}\|_{L^\infty} + \|u_{\sim k}\|_{L^p}\|\nabla(\gamma_{\leq k+3}-\rho_{\leq k+3})\|_{L^\infty}) \leq \\
&\leq \varepsilon 2^{2k}\|\rho_k-\gamma_k\|_{L^p}^p + C_\varepsilon 2^{-2k(p-1)}\|v_{\sim k}-u_{\sim k}\|_{L^p}^p\|\nabla \gamma_{\leq k+3}\|_{L^\infty}^p + \\
&+C_\varepsilon 2^{-2k(p-1)}\|u_{\sim k}\|_{L^p}^p\|\nabla(\gamma_{\leq k+3}-\rho_{\leq k+3})\|_{L^\infty}^p.
\end{aligned}
$$

4.5. **Estimates for $\sum_{l\geq k+3}\int [P_k[u_{\sim l}\cdot \nabla \rho_l] - P_k[v_{\sim l}\cdot \nabla \gamma_l]]\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}dx$.** It is again beneficial to integrate by parts. We obtain

$$
\begin{aligned}
&\int [P_k[u_{\sim l}\cdot \nabla \rho_l] - P_k[v_{\sim l}\cdot \nabla \gamma_l]]\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}dx = \\
&= -\int [P_k[u_{\sim l}\rho_l] - P_k[v_{\sim l}\gamma_l]]\cdot \nabla\left[\frac{|\rho_k-\gamma_k|^p}{\rho_k-\gamma_k}\right]dx
\end{aligned}
$$

From the pointwise control (14) and Hölder's inequality, we estimate by

$$
\begin{aligned}
&(\sum_{l\geq k+3}\|P_k[u_{\sim l}\rho_l - v_{\sim l}\gamma_l]\|_{L^p})\|\rho_k-\gamma_k\|_{L^p}^{(p-2)/2}\|\nabla[|\rho_k-\gamma_k|^{p/2}]\|_{L^2} \leq \\
&\leq \varepsilon\|\nabla[|\rho_k-\gamma_k|^{p/2}]\|_{L^2}^2 + C_\varepsilon(\sum_{l\geq k+3}\|P_k[u_{\sim l}\rho_l - v_{\sim l}\gamma_l]\|_{L^p})^2\|\rho_k-\gamma_k\|_{L^p}^{(p-2)} \leq \\
&\leq \varepsilon\|\nabla[|\rho_k-\gamma_k|^{p/2}]\|_{L^2}^2 + \varepsilon 2^{2k}\|\rho_k-\gamma_k\|_{L^p}^p + D_{p,\varepsilon}2^{-k(p-2)}(\sum_{l\geq k+3}\|P_k[u_{\sim l}\rho_l - v_{\sim l}\gamma_l]\|_{L^p})^p.
\end{aligned}
$$

We further estimate (similar to Section 3.6) via the Bernstein inequality

$$
\begin{aligned}
&\|P_k[u_{\sim l}\rho_l - v_{\sim l}\gamma_l]\|_{L^p(\mathbf{R}^2)} \leq C2^k\|P_k[u_{\sim l}\rho_l - v_{\sim l}\gamma_l]\|_{L^{2p/(p+2)}(\mathbf{R}^2)} \leq \\
&\leq C2^k\|u_{\sim l}-v_{\sim l}\|_{L^p}\|\rho_l\|_{L^2} + C2^k\|v_{\sim l}\|_{L^2}\|\rho_l-\gamma_l\|_{L^p}.
\end{aligned}
$$

whence we obtain an estimate for the contribution of the high-high term in the form

$$\varepsilon\|\nabla[|\rho_k-\gamma_k|^{p/2}]\|_{L^2}^2+\varepsilon 2^{2k}\|\rho_k-\gamma_k\|_{L^p}^p+$$
$$+D_{p,\varepsilon}2^{2k}(\sum_{l\geq k+3}\|u_{\sim l}-v_{\sim l}\|_{L^p}\|\rho_l\|_{L^2}+\|v_{\sim l}\|_{L^2}\|\rho_l-\gamma_l\|_{L^p})^p.$$

4.6. **Putting all energy estimates together.** The energy estimate for $\|\rho_k-\gamma_k\|_{L^p}^p$ may be rewritten by using the estimates in the previous sections. Since we are interested in the quantity $\|u-v\|_{\dot B^s_{p,p}}\sim\|\rho-\gamma\|_{\dot B^{s-1}_{p,p}}$, we multiply by $2^{spk}2^{-kp}$ and sum in k. We obtain[3]

$$\partial_t\sum_k 2^{spk}2^{-kp}\|\rho_k-\gamma_k\|_{L^p}^p+\frac{2}{(p-1)}\sum_k 2^{spk}2^{-kp}\|\nabla[|\rho_k-\gamma_k|^{p/2}]\|_{L^2}^2\leq$$
$$\leq D_{p,\varepsilon}\sum_k 2^{k(sp-2p+2)}\|u_{\leq k+3}-v_{\leq k+3}\|_{L^\infty}^p\|\gamma_k\|_{L^p}^p+$$
$$+D_{p,\varepsilon}\sum_k 2^{k(sp-3p+2)}(\|\nabla[v_{\leq k+3}-u_{\leq k+3}]\|_{L^\infty}^p\|\gamma_{\sim k}\|_{L^p}^p+\|\nabla u_{\leq k+3}\|_{L^\infty}^p\|\rho_{\sim k}-\gamma_{\sim k}\|_{L^p}^p)+$$
$$+D_{p,\varepsilon}\sum_k 2^{k(sp-3p+2)}\|u_{\sim k}\|_{L^p}^p\|\nabla[\gamma_{\leq k+3}-\rho_{\leq k+3}]\|_{L^\infty}^p+$$
$$+D_{p,\varepsilon}\sum_k 2^{k(sp-p+2)}(\sum_{l\geq k+3}[\|u_{\sim l}-v_{\sim l}\|_{L^p}\|\rho_l\|_{L^2}+\|v_{\sim l}\|_{L^2}\|\rho_l-\gamma_l\|_{L^p})^p.$$

We now observe that by (13) the left hand side of the estimate is bigger than $\partial_t\|\rho-\gamma\|_{\dot B^{s-1}_{p,p}}^p+\frac{c}{p-1}\sum_k 2^{k(sp-p+2)}\|\rho_k-\gamma_k\|_{L^p}^p$. It remains to suitably estimate the right hand side and use the Gronwall's inequality. We have by Bernstein inequality (Lemma 1)

$$\sum_k 2^{k(sp-2p+2)}\|u_{\leq k+3}-v_{\leq k+3}\|_{L^\infty}^p\|\gamma_k\|_{L^p}^p\leq\sum_k 2^{k(sp-2p+2)}(\sum_{l\leq k+3}2^{2l/p}\|u_l-v_l\|_{L^p})^p\|\gamma_k\|_{L^p}^p$$

For $s<2/p$, we have

$$(\sum_{l\leq k+3}2^{2l/p}\|u_l-v_l\|_{L^p})^p\leq(\sup_l 2^{sl}\|u_l-v_l\|_{L^p})^p(\sum_{l\leq k+3}2^{l(2/p-s)})^p\leq C_s\|u-v\|_{\dot B^s_{p,p}}^p 2^{k(2-ps)},$$

whence we derive the bound $C_s\|u-v\|_{\dot B^s_{p,p}}^p\|\gamma\|_{\dot B^{4/p-2}_{p,p}}^p$ for the first term on the right hand side. The considerations in the other high-low or low-high terms are similar and we obtain

$$\sum_k 2^{k(sp-3p+2)}(\|\nabla[v_{\leq k+3}-u_{\leq k+3}]\|_{L^\infty}^p\|\gamma_{\sim k}\|_{L^p}^p\leq C_p\|u-v\|_{\dot B^s_{p,p}}^p\|\gamma\|_{\dot B^{4/p-2}_{p,p}}^p$$
$$\sum_k 2^{k(sp-3p+2)}\|\nabla u_{\leq k+3}\|_{L^\infty}^p\|\rho_{\sim k}-\gamma_{\sim k}\|_{L^p}^p\leq C_p\|u\|_{\dot B^{4/p-1}_{p,p}}^p\|\rho-\gamma\|_{\dot B^{s-1}_{p,p}}^p$$
$$\sum_k 2^{k(sp-3p+2)}\|u_{\sim k}\|_{L^p}^p\|\nabla[\gamma_{\leq k+3}-\rho_{\leq k+3}]\|_{L^\infty}^p\leq C_p\|u\|_{\dot B^{4/p-1}_{p,p}}^p\|\rho-\gamma\|_{\dot B^{s-1}_{p,p}}^p$$

Note that in the last three estimates, we needed only $s<2/p+1$ and this is why the constants are denoted by C_p. Next, for the high-high frequency interaction, we have that

[3]due to our choice of ε, all the terms containing $\|\rho_k-\gamma_k\|_{L^p}^p$ can be hidden behind the left hand side

$s > 1-2/p$ and hence $sp-p+2>0$. Let $0<\kappa<sp-p+2$, say $\kappa=(sp-p+2)/2$. We estimate by Cauchy-Schwartz as a constant times

$$\sum_k 2^{k(sp-p+2)}(\sum_{l\geq k+3}[\|u_{\sim l}-v_{\sim l}\|_{L^p}\|\rho_l\|_{L^2}+\|v_{\sim l}\|_{L^2}\|\rho_l-\gamma_l\|_{L^p})^p \leq$$
$$\leq \sum_k 2^{k(sp-p+2)}(\sum_{l\geq k+3}2^{-l\kappa p'/p})^{p/p'}\sum_{l\geq k+3}2^{l\kappa}[\|u_{\sim l}-v_{\sim l}\|_{L^p}^p\|\rho_l\|_{L^2}^p+\|v_{\sim l}\|_{L^2}^p\|\rho_l-\gamma_l\|_{L^p}^p] \leq$$
$$\lesssim \sum_k 2^{k(sp-p+2-\kappa)}\sum_{l\geq k+3}2^{l\kappa}[\|u_{\sim l}-v_{\sim l}\|_{L^p}^p\|\rho_l\|_{L^2}^p+\|v_{\sim l}\|_{L^2}^p\|\rho_l-\gamma_l\|_{L^p}^p].$$

Now, since $sp-p+2-\kappa>0$, we may interchange the k and l summations and conclude that $\sum_{k\leq l-3}2^{k(sp-p+2-\kappa)}\sim 2^{l(sp-p+2-\kappa)}$. We obtain the bound

$$\sum_l 2^{l(sp-p+2)}[\|u_{\sim l}-v_{\sim l}\|_{L^p}^p\|\rho_l\|_{L^2}^p+\|v_{\sim l}\|_{L^2}^p\|\rho_l-\gamma_l\|_{L^p}^p] \leq$$
$$\lesssim \sum_l 2^{lsp}[\|u_{\sim l}-v_{\sim l}\|_{L^p}^p(\sup_l 2^{l(2/p-1)}\|\rho_l\|_{L^2})^p+\sum_l 2^{lp(s-1)}\|\rho_l-\gamma_l\|_{L^p}^p(\sup_l 2^{2l/p}\|v_{\sim l}\|_{L^2})^p$$
$$\leq C_s[\|u-v\|_{\dot{B}^s_{p,p}}^p\|\rho\|_{\dot{H}^{2/p-1}}^p+\|\rho-\gamma\|_{\dot{B}^{s-1}_{p,p}}^p\|v\|_{\dot{H}^{2/p}}^p].$$

To summarize, for all $s: 1-2/p<s<2/p$, and by taking into account $\|\rho\|_{\dot{B}^\alpha_{p,q}}\sim\|u\|_{\dot{B}^{\alpha+1}_{p,q}}$, we obtain (after integration in time) the energy inequality

$$\begin{aligned}&\|u(t)-v(t)\|_{\dot{B}^s_{p,p}}^p+\tfrac{c}{p-1}\int_0^t\|u-v\|_{\dot{B}^{s+2/p}_{p,p}}^p dz \leq\\ &\leq C_s\int_0^t\|u-v\|_{\dot{B}^s_{p,p}}^p(\|u\|_{\dot{B}^{4/p-1}_{p,p}}^p+\|u\|_{\dot{H}^{2/p}}^p+\|v\|_{\dot{H}^{2/p}}^p)dz,\end{aligned} \tag{15}$$

where C_s depends on s and may blow up as $s\to 2/p$ or $s\to 1-2/p$. Ignoring the positive term $\int_0^t\|u-v\|_{\dot{B}^{s+2/p}_{p,p}}^p dz$ and applying the Gronwall's inequality yields

$$\|u(t)-v(t)\|_{\dot{B}^s_{p,p}}^p\leq\|u_0-v_0\|_{\dot{B}^s_{p,p}}^p exp(C_s\int_0^t(\|u\|_{\dot{B}^{4/p-1}_{p,p}}^p+\|u\|_{\dot{H}^{2/p}}^p+\|v\|_{\dot{H}^{2/p}}^p)dz) \tag{16}$$

However, the integrals under the exp are controlled uniformly in time. Indeed, by the Gagliardo-Nirenberg's inequality and since $L^2(\mathbf{R}^2)\hookrightarrow\dot{B}^{-1}_{\infty,\infty}(\mathbf{R}^2)$

$$\int_0^t\|u\|_{\dot{B}^{4/p-1}_{p,p}}^p dz\leq\|u\|_{L^p\dot{B}^{4/p-1}_{p,p}}^p\leq\|u\|_{L^\infty\dot{B}^{-1}_{\infty,\infty}}^{p-2}\|u\|_{L^2\dot{B}^1_{2,2}}^2\leq\|u\|_{L^\infty L^2}^{p-2}\|u\|_{L^2\dot{H}^1}^2\leq\|u_0\|_{L^2}^p,$$

where in the last line, we have used the energy dissipation (11). Similarly, we control

$$\begin{aligned}&\int_0^t(\|u\|_{\dot{H}^{2/p}}^p+\|v\|_{\dot{H}^{2/p}}^p)dz\leq\|u\|_{L^p\dot{H}^{2/p}}^p+\|v\|_{L^p\dot{H}^{2/p}}^p\leq\\ &\leq C\|u\|_{L^2\dot{H}^1}^{p-2}\|u\|_{L^\infty L^2}^2+C\|v\|_{L^2\dot{H}^1}^{p-2}\|v\|_{L^\infty L^2}^2\leq C(\|u_0\|_{L^2}^p+\|v_0\|_{L^2}^p)\end{aligned}$$

and hence, inserting this in (16) yields

$$\|u(t)-v(t)\|_{\dot{B}^s_{p,p}}\leq\|u_0-v_0\|_{\dot{B}^s_{p,p}}exp(C_{s,p}(\|u_0\|_{L^2}^p+\|v_0\|_{L^2}^p)),$$

which is the claim in (7). If we go back to our energy estimate (15) and ignore instead the positive term $\|u(t)-v(t)\|^p_{\dot{B}^s_{p,p}}$, we obtain the bound

$$\int_0^t \|u-v\|^p_{\dot{B}^{s+2/p}_{p,p}} dz \le \sup_{0\le z<\infty} \|u(z)-v(z)\|^p_{\dot{B}^s_{p,p}} \int_0^t (\|u\|^p_{\dot{B}^{4/p-1}_{p,p}} + \|u\|^p_{\dot{H}^{2/p}} + \|v\|^p_{\dot{H}^{2/p}}) dz \le$$
$$\le \|u_0-v_0\|^p_{\dot{B}^s_{p,p}} exp(C_{s,p}(\|u_0\|^p_{L^2} + \|v_0\|^p_{L^2})),$$

hence (8).

5. Discussion and open problems

In order to motivate the discussion, we outline (following our previous argument) an approach for showing Lipschitzness of the solution map in $\dot{H}^1$. Taking two solutions u, v with their corresponding vorticities ρ, γ, we have the scalar equation for the difference $\rho-\gamma$

$$\partial_t(\rho-\gamma) - \Delta(\rho-\gamma) + u\cdot\nabla\rho - v\cdot\nabla\gamma = 0$$

Taking a dot product with $\rho-\gamma$, we obtain the energy equation

$$\frac{1}{2}\partial_t\|\rho-\gamma\|^2_{L^2} + \|\nabla[\rho-\gamma]\|^2_{L^2} \le |\int_{\mathbf{R}^2} (u\cdot\nabla\rho - v\cdot\nabla\gamma)(\rho-\gamma)| = |\int_{\mathbf{R}^2} [(u-v)\cdot\nabla\gamma][\rho-\gamma]|.$$

Introduce, as suggested in the abstract

$$\Lambda(\vec{u}, v, w) := \int_{\mathbf{R}^2} (\vec{u}\cdot\nabla v) w dx.$$

where $div(u) = 0$. If one can show the estimate

$$|\Lambda(\vec{u}, v, w)| \le C\|\vec{u}\|_{\dot{H}^1}\|v\|_{L^2}\|w\|_{\dot{H}^1}, \tag{17}$$

then the problem for Lipschitzness in $\dot{H}^1$ will be resolved. Indeed, assuming (17), we have

$$\frac{1}{2}\partial_t\|\rho-\gamma\|^2_{L^2} + \|\nabla[\rho-\gamma]\|^2_{L^2} \le C\|u-v\|_{\dot{H}^1}\|\gamma\|_{L^2}\|\rho-\gamma\|_{\dot{H}^1}.$$

Since $\|u-v\|_{\dot{H}^1} \sim \|\rho-\gamma\|_{L^2}$, we obtain (after hiding certain terms behind $\|\nabla[\rho-\gamma]\|^2_{L^2}$,

$$\partial_t\|\rho-\gamma\|^2_{L^2} \le C\|\rho-\gamma\|^2_{L^2}\|\gamma\|^2_{L^2},$$

whence

$$\|\rho(t)-\gamma(t)\|^2_{L^2} \le \|\rho_0-\gamma_0\|^2_{L^2} exp(\int_0^t \|\gamma(s)\|^2_{L^2} ds).$$

Note however that $\int_0^\infty \|\gamma(s)\|^2_{L^2} ds \sim \int_0^\infty \|\nabla u(s)\|^2_{L^2} \le \|u_0\|^2_{L^2}$. Thus

$$\|u(t)-v(t)\|^2_{\dot{H}^1} \sim \|\rho(t)-\gamma(t)\|^2_{L^2} \le \|\rho_0-\gamma_0\|^2_{L^2} exp(C\|u_0\|^2_{L^2}) \sim \|u_0-v_0\|^2_{\dot{H}^1} exp(C\|u_0\|^2_{L^2}).$$

This shows that Lipschitzness in $\dot{H}^1$ is pretty much contingent upon the validity of (17), at least if one follows the proof proposed here.

Let us show that (17) is almost correct in some sense. Indeed, noting that since $div(u) = 0$,

$$\Lambda(\vec{u}, v, w) = -\int_{\mathbf{R}^2} (\vec{u}\cdot v)\nabla w dx$$

and an application of the Hölder's inequality yields

$$|\Lambda(\vec{u}, v, w)| \leq \|u\|_{L^\infty}\|v\|_{L^2}\|w\|_{\dot{H}^1}.$$

However, since $\dot{H}^1(\mathbf{R}^2)$ just fails to embed into $L^\infty(\mathbf{R}^2)$, we cannot claim (17) from this argument.

Interestingly, an enhanced version of this argument fails as well. In order to explain that, we introduce a related trilinear form, this one acting only on scalar functions

$$\Gamma(\phi, v, w) = \int_{\mathbf{R}^2} Q(\phi, v) w dx = \int_{\mathbf{R}^2} [\partial_2\phi\partial_1 v - \partial_1\phi\partial_2 v] w dx$$

Since $div(u) = 0$, it is clear that it can be written as $u = curl\phi$, whence $\Lambda(\vec{u}, v, w) = \Gamma(\phi, v, w)$. Moreover $\|u\|_{\dot{H}^1} \sim \|\phi\|_{\dot{H}^2}$. Thus, (17) is equivalent to the estimate

$$|\Gamma(\phi, v, w)\| \leq C\|\phi\|_{\dot{H}^2}\|v\|_{L^2}\|w\|_{\dot{H}^1} \tag{18}$$

Note that a simple integration by parts yields $\Gamma(\phi, v, w) = -\int \phi Q(v, w) dx$. Denote by $\mathcal{H}^1$ the Hardy space, which the predual of BMO, $(\mathcal{H}^1)^* = BMO$. We have

$$\begin{aligned}&|\Gamma(\phi, v, w)| = |\int_{\mathbf{R}^2}[|\nabla|\phi][|\nabla|^{-1}Q(v, w)]dx| \leq \\ &\leq \||\nabla|\phi\|_{BMO(\mathbf{R}^2)}\||\nabla|^{-1}Q(v, w)\|_{\mathcal{H}^1} \leq C\|\phi\|_{\dot{H}^2(\mathbf{R}^2)}\||\nabla|^{-1}Q(v, w)\|_{\mathcal{H}^1}.\end{aligned}$$

where in the last inequality, we have used that $\dot{H}^1(\mathbf{R}^2) \hookrightarrow BMO(\mathbf{R}^2)$. But now again, the estimate needed to close the argument, namely

$$\||\nabla|^{-1}Q(v, w)\|_{\mathcal{H}^1} \leq C\|v\|_{L^2}\|w\|_{\dot{H}^1},$$

fails. This is not so straightforward, see Theorem 2, [24] (or the details in its proof on p. 458, [24]). In fact, this is a *double endpoint failure as explained in* [24]. This shows that an $\mathcal{H}^1 - BMO$ duality argument is likely to be insufficient to establish (17) (or the equivalent (18)). Thus, we propose the following open problems for consideration.

Problem 1: Prove or disprove (17) (or the equivalent (18)).

An affirmative answer to that will of course imply the H^1 Lipshitzness of the solution map. In case the answer to Problem 1 is negative, it does not automatically imply that the solution map is not Lipshcitz. Thus, one is faced with

Problem 2: Prove or disprove the Lipshitzness of the solution map in H^1.

5.1. **Postscript.** After the paper was submitted for publication, I have asked several colleagues for input on the validity of (18), basically asking for a counterexample. Terry Tao, [25] has provided me with the following

Proposition 1. *(Tao,* [25]*) The estimate*

$$\|f_x g_y - f_y g_x\|_{L^2(\mathbf{R}^2)} \leq C\|\nabla f\|_{L^2(\mathbf{R}^2)}\|\nabla^2 g\|_{L^2(\mathbf{R}^2)} \tag{19}$$

fails. Since (19) *is equivalent to* (18)*, we conclude that* (18) *fails as well.*

Proof. Here, we briefly indicate Tao's argument. The idea is that one can deduce from (19) the false Sobolev embedding estimate $\dot{H}^1(\mathbf{R}^2) \hookrightarrow L^\infty(\mathbf{R}^2)$.

Indeed, assume (19). Fix x_0, y_0, N, ε and $\psi \in C_0^\infty(\mathbf{R}^2)$. Test (19) with $f = \varepsilon^{-1} e^{iNx/\varepsilon}\psi(x/\varepsilon, y/\varepsilon)$, $g(x+x_0, y+y_0)$. Here $N >> 1$, $0 < \varepsilon << 1$.

Observe that $\|\nabla f\|_{L^2} \lesssim N/\varepsilon$ and

$$\|f_x g_y - f_y g_x\|_{L^2(\mathbf{R}^2)} = \frac{N}{\varepsilon^2}\|\psi(x/\varepsilon, y/\varepsilon) g_y(x+x_0, y+y_0)\|_{L^2(\mathbf{R}^2)} + O(1/\varepsilon^2).$$

It follows

$$\frac{N}{\varepsilon^2}\|\psi(x/\varepsilon, y/\varepsilon) g_y(x+x_0, y+y_0)\|_{L^2(\mathbf{R}^2)} + O(1/\varepsilon^2) \le \frac{CN}{\varepsilon}\|\nabla^2 g\|_{L^2(\mathbf{R}^2)}$$

Taking $N \to \infty$ yields

$$\frac{1}{\varepsilon}\|\psi(x/\varepsilon, y/\varepsilon) g_y(x+x_0, y+y_0)\|_{L^2(\mathbf{R}^2)} \le C\|\nabla^2 g\|_{L^2(\mathbf{R}^2)} \tag{20}$$

But

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\mathbf{R}^2}|\psi(x/\varepsilon, y/\varepsilon)|^2|g_y(x+x_0, y+y_0)|^2 dxdy = |g_y(x_0, y_0)|^2$$

Thus, a limt $\varepsilon \to 0$ in (20) implies

$$|g_y(x_0, y_0)| \le C\|\nabla^2 g\|_{L^2(\mathbf{R}^2)}$$

Since the roles of x, y are reversible, this implies the false Sobolev embedding formula $\|\nabla g\|_{L^\infty(\mathbf{R}^2)} \le C\|\nabla^2 g\|_{L^2(\mathbf{R}^2)}$. Therefore, (19) and hence (18) fail.

□

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